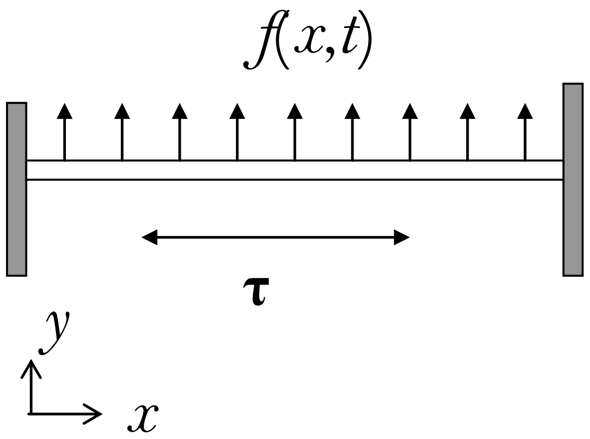
Chapter 6 Distributed Parameter Systems

Extending the first 5 chapters (particularly Chapter 4) to systems with distributed mass and stiffness properties:

Strings, rods and beams

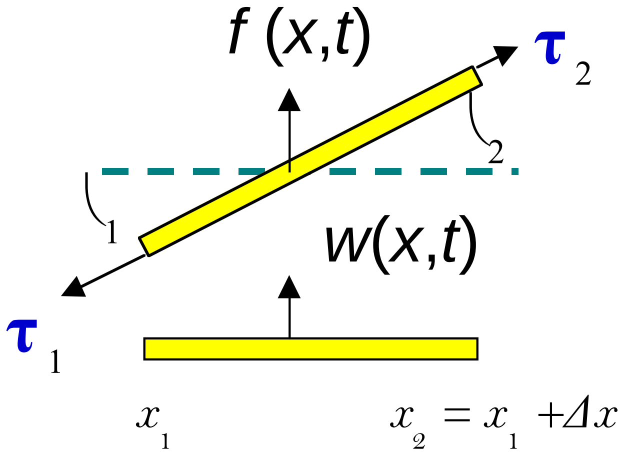
Tacoma Narrows Bridge

The string/cable equation



- **Start by considering a uniform string stretched between two fixed boundaries**
- **Assume constant, axial tension τ in string**
- **Let a distributed force $f(x,t)$ act along the string**

Examine a small element of the string



$$\sum F_y = \rho A \Delta x \frac{\partial^2 w(x,t)}{\partial t^2} =$$

$$-\tau_1 \sin\theta_1 + \tau_2 \sin\theta_2 + f(x,t)\Delta x$$

- **Force balance on an infinitesimal element**
- **Now *linearize* the sine with the small angle approximate sin*x* = tan*x* =slope of the string**
- **Writing the slope as the derivative:** $\tan x = \dfrac{\partial w}{\partial x}$

Substitute into *F=ma* and use a Taylor expansion:

$$\left.\left(\tau \frac{\partial w(x,t)}{\partial x}\right)\right|_{x_2} - \left.\left(\tau \frac{\partial w(x,t)}{\partial x}\right)\right|_{x_1} + f(x,t)\Delta x = \rho A \frac{\partial^2 w(x,t)}{\partial t^2}\Delta x$$

Recall the Taylor series of $\tau \partial w / \partial x$ about x_1:

$$\left.\left(\tau \frac{\partial w}{\partial x}\right)\right|_{x_2} = \left.\left(\tau \frac{\partial w}{\partial x}\right)\right|_{x_1} + \Delta x \frac{\partial}{\partial x}\left.\left(\tau \frac{\partial w}{\partial x}\right)\right|_{x_1} + O(\Delta x^2) + \ldots$$

$$\frac{\partial}{\partial x}\left.\left(\tau \frac{\partial w(x,t)}{\partial x}\right)\right|_{x_1} \Delta x + f(x,t)\Delta x = \rho A \frac{\partial^2 w(x,t)}{\partial t^2}\Delta x \Rightarrow$$

$$\frac{\partial}{\partial x}\left(\tau \frac{\partial w(x,t)}{\partial x}\right) + f(x,t) = \rho A \frac{\partial^2 w(x,t)}{\partial t^2}$$

Since τ is constant, and for no external force, the equation of motion becomes:

$$c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2}, \qquad c = \sqrt{\frac{\tau}{\rho A}} \tag{6.8}$$

wave speed

Second order in time and second order in space, therefore 4 constants of integration. Two from initial conditions:

$$w(x,0) = w_0(x), \quad w_t(x,0) = \dot{w}_0(x) \ \text{ at } t = 0$$

And two from boundary conditions (e.g. a fixed-fixed string):

$$w(0,t) = w(\ell,t) = 0, \qquad t > 0$$

Each of these terms has a physical interpretation:

- **Deflection is $w(x,t)$ in the y-direction**
- **The slop of the string is $w_x(x,t)$**
- **The restoring force is $\tau w_{xx}(x,t)$**
- **The velocity is $w_t(x,t)$**
- **The acceleration is $w_{tt}(x,t)$**

  at any point x along the string at time t

Note that the above applies to cables as well as strings

There are Two Solution Types for Two Situations

- **This is called the wave equation and if there are no boundaries, or they are sufficiently far away it is solved as a wave phenomena**
  - **Disturbance results in propagating waves***
- **If the boundaries are finite, relatively close together, then we solve it as a vibration phenomena (focus of *Vibrations*)**
  - **Disturbance results in vibration**

*focus of courses in *Acoustics* and in *Wave Propagation*

Solution of the Wave Equation:

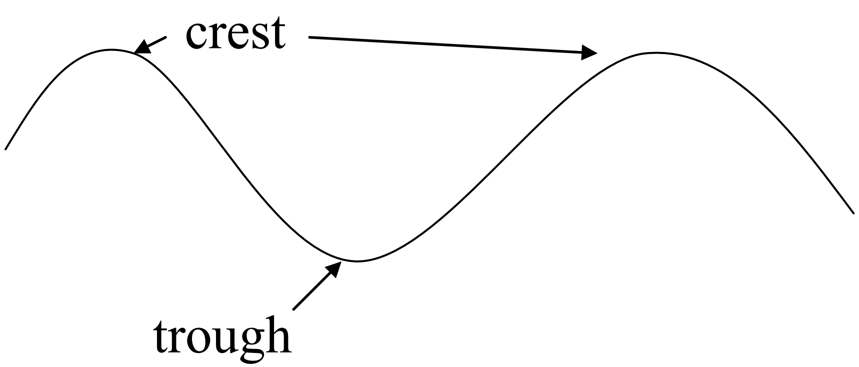
- **Interpret $w(x,t)$ as a stress, particle velocity, or displacement to examine the propagation of waves in elastic media**
  - Called *wave propagation*
- **Interpret $w(x,t)$ as a pressure to examine the propagation of sound in a fluid**
  - Called *acoustics*

Solution of the "string equation" as a Wav e

A solution is of the form:

$$w(x,t) = w_1(x - ct) + w_2(x + ct)$$

This describes on wave traveling forward and one wave traveling backwards (called traveling waves as the form of the wave moves along the media)



The wave speed is *c*

Think of waves in a pool of water

Example 6.1.1 what are the boundary conditions for this system?

A force balance at ℓ yields

$$\sum_y F_{x=\ell} = \tau \sin\theta + kw(\ell, t) = 0$$

$$\Rightarrow \tau \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\ell} = -k\, w(x,t)\big|_{x=\ell}$$

At $x = 0$, the BC is $w(x,t)\big|_{x=0} = 0$

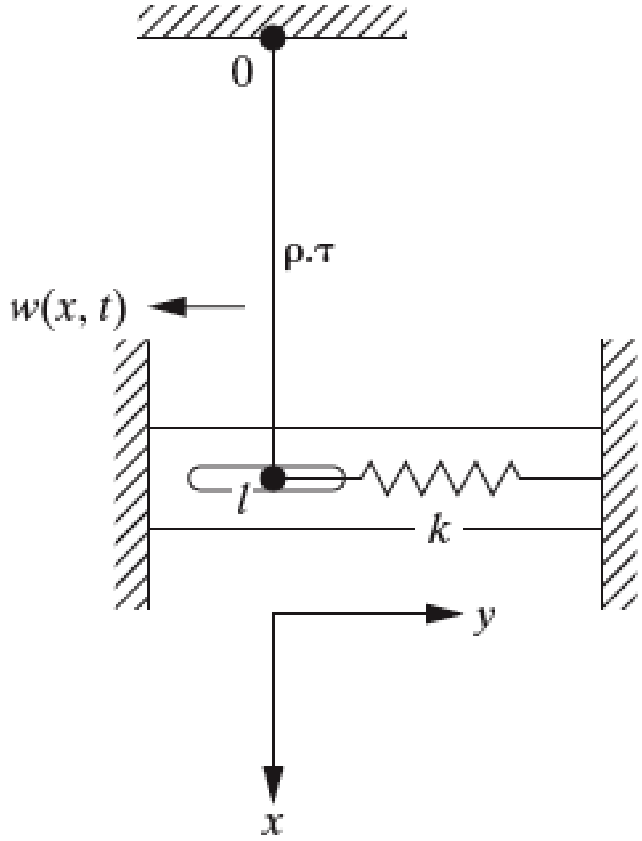


Fig 6.2

6.2 Modes and Natural Frequencies

$$w(x,t) = X(x)T(t) \Rightarrow$$

$$c^2 X''(x)T(t) = X(x)\ddot{T}(t) \quad \text{where } '' = \frac{d^2}{dx^2} \text{ and } \ddot{} = \frac{d^2}{dt^2}$$

$$\Rightarrow \frac{c^2 X''(x)}{X(x)} = \frac{\ddot{T}(t)}{T(t)}, \qquad \Rightarrow \frac{\ddot{T}(t)}{T(t)} = -\omega^2 \quad \Rightarrow \ddot{T}(t) + \omega^2 T(t) = 0 \qquad (1)$$

$$\text{and} \quad \frac{d}{dx}\left(\frac{X''(x)}{X(x)}\right) = 0 \Rightarrow \frac{c^2 X''(x)}{X(x)} = -\omega^2$$

$$\Rightarrow X''(x) + \frac{\omega^2}{c^2} X(x) = 0 \Rightarrow X''(x) + \beta^2 X(x) = 0 \qquad (2)$$

$$\beta = \frac{\omega}{c} \Rightarrow \omega = c\beta$$

Solving the Time Equation (1)

$$\ddot{T}(t) + \omega^2 T(t) = 0 \Rightarrow T(t) = A\sin\omega t + B\cos\omega t$$

This implies that *T* is oscillating with frequency ω

Solving the Spatial Equation (2)

$$X''(x) + \beta^2 X(x) = 0 \Rightarrow X(x) = a_1 \sin\beta x + a_2 \cos\beta x$$

At this point we have 4 unknowns: A, B, a_1, and a_2

Use the Boundary Conditions to determine a_1 and a_2

$$w(0,t)=0 \Rightarrow X(0)T(t)=0 \Rightarrow X(0)=0$$

$$\Rightarrow a_1\sin(0)+a_2\cos(0)=0 \Rightarrow a_2=0$$

$$w(\ell,t)=0 \Rightarrow X(\ell)T(t)=0 \Rightarrow X(\ell)=0$$

$$\Rightarrow a_1\sin(\beta\ell)=0$$

$$\Rightarrow \beta\ell = n\pi, \quad \Rightarrow \beta_n=\frac{n\pi}{\ell}, \quad n=1,2,3,\cdots$$

(But n cannot be zero, why?)

$$X(x)=a_1\sin(\beta_n x) \Rightarrow X_n(x)=a_n\sin(\beta_n x), \quad n=1,2,3,\cdots$$

The $X_n(x)$ are called *eigenfunctions*

Constructing the total solution by recombining *X*(*x*) and *T*(*t*)

$$\omega = c\beta \Rightarrow \omega_n = \frac{n\pi c}{\ell} \Rightarrow T_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t)$$

$$\Rightarrow w_n(x,t) = X_n(x)T_n(t) = c_n \sin\omega_n t \sin\beta_n x + d_n \cos\omega_n t \sin\beta_n x$$

$$= c_n \sin(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x) + d_n \cos(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x)$$

So there are n solutions, since the system is linear we add them up:

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x) + d_n \cos(\frac{n\pi}{\ell}ct)\sin(\frac{n\pi}{\ell}x)$$

We still do not know the constants c_n and d_n
but we have yet to use the initial conditions

Orthogonality is used to evaluate the remaining constants from the initial conditions

$$\int_0^{\ell} \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx = \begin{cases} \ell/2, & n = m \\ 0, & n \neq m \end{cases} = \frac{\ell}{2}\delta_{nm} \quad (6.28)$$

(proving this looks a lot like homework!)

From the initial poistion:

$$w(x,0) = w_0(x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi}{\ell}x\right)\cos(0) \Rightarrow$$

$$\int_0^{\ell} w_0(x)\sin\left(\frac{m\pi}{\ell}x\right)dx = \sum_{n=1}^{\infty} d_n \int_0^{\ell} \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx \Rightarrow$$

$$d_m = \frac{2}{\ell}\int_0^{\ell} w_0(x)\sin(\frac{m\pi}{\ell}x)dx, \quad m = 1, 2, 3\ldots \qquad (6.31)$$

$$m \to n \Rightarrow$$

$$d_n = \frac{2}{\ell}\int_0^{\ell} w_0(x)\sin(\frac{n\pi}{\ell}x)dx, \quad n = 1, 2, 3\ldots$$

$$\dot{w}_0(x) = \sum_{n=1}^{\infty} c_n \sigma_n \mathrm{c}\sin(\frac{n\pi}{\ell}x)\cos(0) \qquad (6.32)$$

$$c_n = \frac{2}{n\pi c}\int_0^{\ell} \dot{w}_0(x)\sin(\frac{n\pi}{\ell}x)dx, \quad n = 1, 2, 3\ldots \qquad (6.33)$$

The Eigenfunctions become the vibration mode shapes

$$w_0(x) = \sin\frac{\pi}{\ell}x,$$ which is the first eigenfunction (n=1)

$$\dot{w}_0(x) = 0, \Rightarrow c_n = 0, \quad \forall n$$

$$d_n = \frac{2}{\ell}\int_0^\ell \sin(\frac{\pi}{\ell}x)\sin(\frac{n\pi}{\ell}x)dx = 0, \quad n = 2, 3\ldots$$

$$d_1 = 1 \Rightarrow$$

$$w(x,t) = \sin(\frac{\pi}{\ell}x)\cos\left(\frac{\pi c}{\ell}t\right)$$

Causes vibration in the first mode shape

A more systematic way to generate the *characteristic* equation is write the boundary conditions (6.20) in matrix form

$$a_1 \sin\beta\ell = 0 \quad \text{and} \quad a_2 = 0$$

$$\Rightarrow \begin{bmatrix} \sin\beta\ell & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \det\left(\begin{bmatrix} \sin\beta\ell & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 \Rightarrow \underline{\sin\beta\ell = 0}$$

This seemingly longer approach works in general and will be used to compute the characteristic equation in more complicated situations

The the solution of the boundary value problem results in an eigenvalue problem

The spatial solution becomes:

For $n=1,2,3\ldots,$ $\boxed{X_n(x) = a_n \sin\left(\frac{n\pi}{\ell}x\right)}$ (6.22)

Here the index n results because of the indexed value of σ

The spatial problem also can be written as:

$$\boxed{-\frac{\partial^2}{\partial x^2}(X_n) = \lambda_n X_n; \quad X_n(x) \neq 0, \quad X_n(0) = X_n(\ell) = 0} \qquad (6.23)$$

Which is also an eigenvalue, eigenfunction problem where $\lambda = \beta^2$ is the eigenvalue and X_n is the eigenfunction.

This is Analogous to Matrix Eigenvalue Problem

$$A \rightarrow -\tau \frac{\partial^2}{\partial x^2}, \text{ plus boundary conditions, matrix} \rightarrow \text{ operator}$$

$$\mathbf{u}_i \rightarrow X_n(x), \text{ eigenvector} \rightarrow \text{eigenfunction}$$

$\alpha X_n(x)$ also an eigenfunction

orthoganality also results and the condition of normalizing

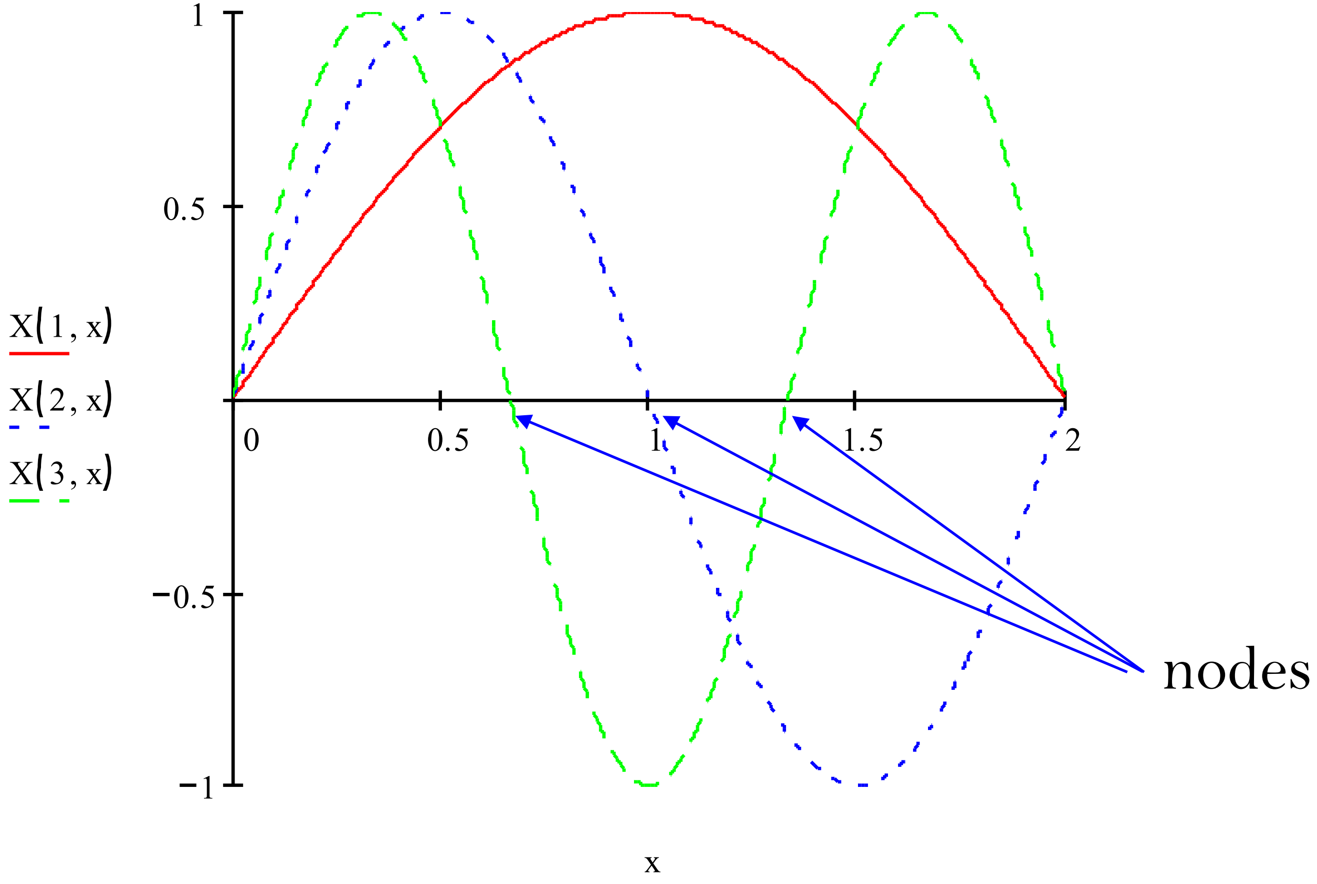
plays the same role

eigenvectors become mode shapes, eigenvalues frequencies

modal expansion will also happen

Plots of mode shapes (fig 6.3)

$$\sin\left(\frac{n\pi}{2}x\right)$$



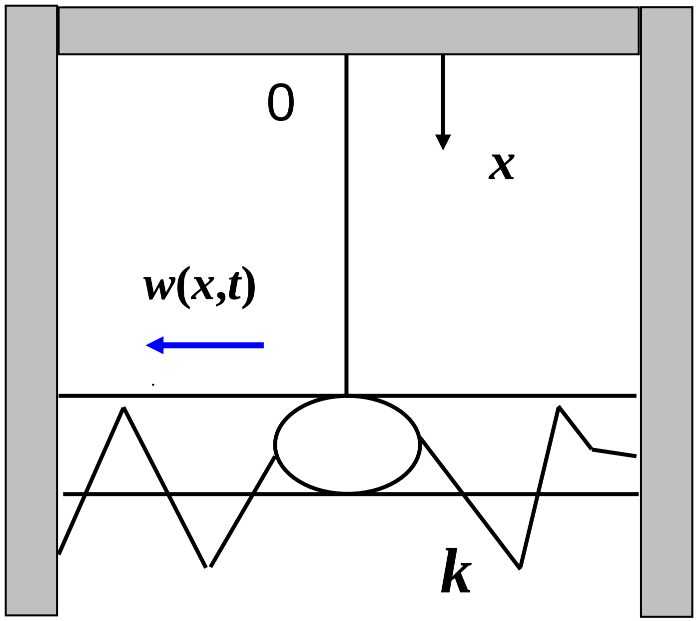
Example 6.2.2: Piano wire: *l* =1.4 m, τ =11.1x10^4 N, *m*=110 g. Compute the first natural frequency.

$$\rho A = 110 \text{ g per } 1.4 \text{ m} = 0.0786 \text{ kg/m}$$

$$\omega_1 = \frac{\pi c}{\ell} = \frac{\pi}{1.4}\sqrt{\frac{\tau}{\rho A}} = \frac{\pi}{1.4}\sqrt{\frac{11.1\times 10^4 \text{ N}}{0.0786 \text{ kg/m}}}$$

$$= 2666.69 \text{ rad/s} \quad \text{or } \underline{424 \text{ Hz}}$$

Example 6.2.3: Compute the mode shapes and natural frequencies for the following system:



A cable hanging from the top and attached to a spring of tension τ and density ρ

In this case the characteristic equation must be solved numerically

$$\sum F_y\Big|_{x=\ell} = 0 \Rightarrow \tau \sin\theta + kw(\ell,t) = 0 \Rightarrow$$

$$\tau \frac{\partial w(x,t)}{\partial x}\bigg|_{x=\ell} = -kw(\ell,t)$$

$$X(x) = a_1 \sin\beta x + a_2 \cos\beta x,$$

$$X(0) = 0 \Rightarrow a_2 = 0 \text{ and } X(x) = a_1 \sin\beta x$$

$$\Rightarrow \tau\beta \cos\beta\ell = -k \sin\beta\ell$$

$$\Rightarrow \tan\beta\ell = -\frac{\tau\beta}{k}$$

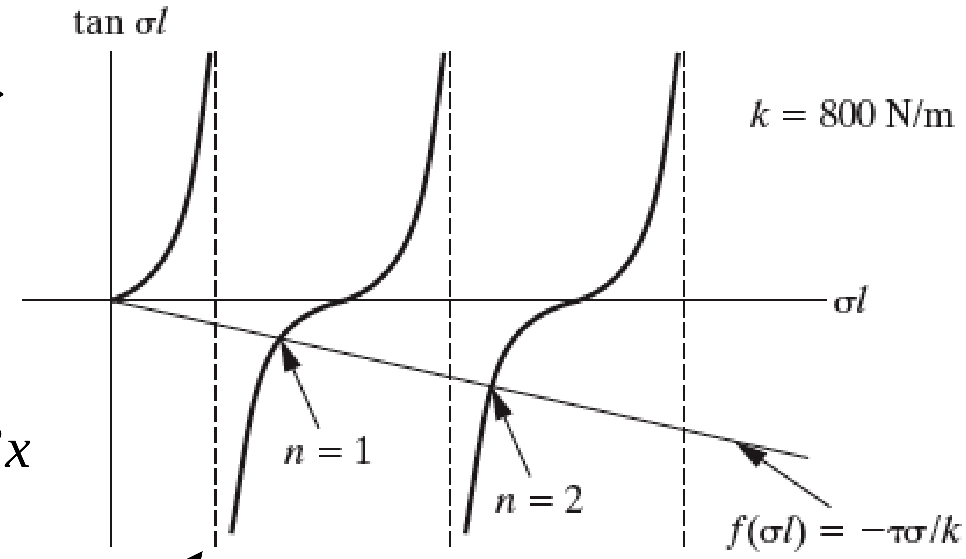


Fig 6.4

The characteristic equation must be solved numerically for β_n

Note text uses σ not β

The Mode Shapes and Natural Frequencies are

$$k = 1000, \tau = 10, \ell = 2$$

$$\beta_n = 1.497, 2.996, 4.501$$

$$6.013 \ldots \frac{(2n-1)\pi}{2\ell}$$

$$X_n = a_n \sin(\beta_n x)$$

- The values of β_n must be found numerically
- The eigenfunctions are again sinusoids
- The value shown for β_n is for large n
- See Window 6.3 for a summary of method

Example: Compute the response of the piano wire to :

$$\text{initial conditions: } w_0(x) = \sin\frac{3\pi x}{\ell}, \dot{w}_0(x) = 0$$

$$\text{Solution: } w(x,t) = \sum_{i=1}^{\infty}(c_n \sin\frac{n\pi}{\ell}x \sin\frac{n\pi c}{\ell}t + d_n \sin\frac{n\pi}{\ell}x \cos\frac{n\pi c}{\ell}t)$$

$$\dot{w}_0(x) = 0 = \sum_{i=1}^{\infty}(c_n \frac{n\pi c}{\ell}\sin\frac{n\pi}{\ell}x\cos(0)) \Rightarrow c_n = 0, \forall n$$

$$w(x,t) = \sum_{i=1}^{\infty} d_n \sin\frac{n\pi}{\ell}x\cos\frac{n\pi c}{\ell}t, \text{ at } t = 0, \text{ Eq.(6.31)} \Rightarrow$$

$$d_n = \frac{2}{\ell}\int_0^{\ell}\sin\frac{3\pi}{\ell}x\sin\frac{m\pi}{\ell}x dx, \quad m = 1,2,3\ldots$$

$= 0$, for all m except $m=3$, $d_3 = 1$

$$w(x,t) = \sin\frac{3\pi}{\ell}x\sin\frac{3\pi c}{\ell}t$$

Some calculation details:

$$\int_0^\ell w_0(x)\sin\frac{m\pi}{\ell}x\,dx$$

$$=\sum_{n=1}^{\infty} d_n \int_0^\ell \sin\frac{n\pi}{\ell}x\sin\frac{m\pi}{\ell}x\,dx$$

$$=d_m\frac{\ell}{2}$$

$$w(\frac{\ell}{2},t)=\sin 6\pi\sin\frac{3\pi c}{\ell}t$$

$$=0$$

$$w(\frac{\ell}{4},t)=\sin\frac{3\pi}{4}\sin\frac{3\pi c}{\ell}t$$

$$=0.707\sin\frac{3\pi c}{\ell}t$$

$$\ell=1.4\text{ m},\ c=11.89\Rightarrow$$

$$w(\frac{1.4}{4},t)=0.707\sin 80.4t$$

Summary of Separation of Variables

- **Substitute $w(x,t)=X(x)T(t)$ into equation of motion and boundary conditions**
- **Manipulate all x dependence onto one side and set equal to a constant ($-\beta^2$)**
- **Solve this spatial equation which results in eigenvalues β_n and eigenfunctions X_n**
- **Next solve the temporal equation to get $T_n(t)$ in terms of β_n and two constants of integration**

- **Re form the product $w_n(x,t)=X_n(x)T_n(t)$ in terms of the $2n$ constants of integration**

- **Form the sum**

$$w(x,t)=\sum_{n=1}^{\infty} X_n(x)(A_n \sin \omega_n t + B_n \cos \omega_n t)$$

- **Use the initial displacement, initial velocity and the orthogonality of $X_n(x)$ to compute A_n and B_n.**

6.3 Vibration of Rods and Bars

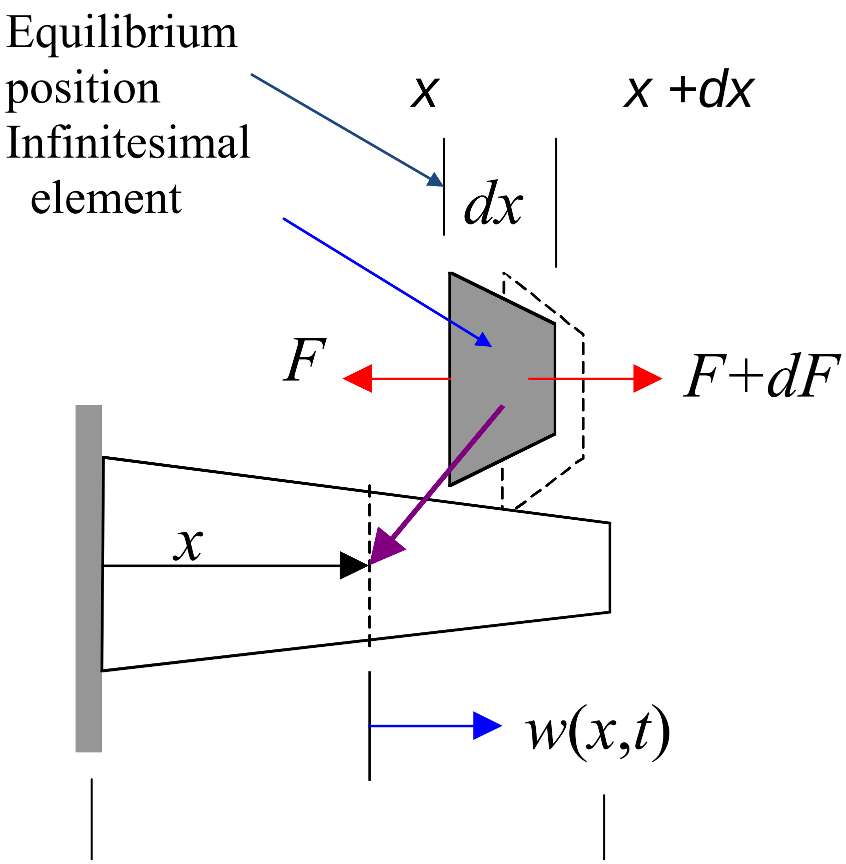


Figure 6.5

- **Consider a small element of the bar**
- **Deflection is now along *x* (called longitudinal vibration)**
- ***F= ma* on small element yields the following:**

Force balance:

$$F + dF - F = \rho A(x) dx \frac{\partial^2 w(x,t)}{\partial t^2} \qquad (6.53)$$

Constitutive relation:

$$F = EA(x)\frac{\partial w(x,t)}{\partial x} \Rightarrow dF = \frac{\partial}{\partial x}\left(EA(x)\frac{\partial w(x,t)}{\partial x}\right)dx$$

$$\frac{\partial}{\partial x}\left(EA(x)\frac{\partial w(x,t)}{\partial x}\right) = \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} \qquad (6.55)$$

$$A(x) = \text{constant} \Rightarrow \quad \frac{E}{\rho}\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \qquad (6.56)$$

At the clamped end: $w(0,t) = 0,$ (6.57)

At the free end: $EA\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\ell} = 0$ (6.58)

Example 6.3.1: compute the mode shapes and natural frequencies of a cantilevered bar with uniform cross section.

$$w(x,t) = X(x)T(t) \rightarrow c^2 w_{xx}(x,t) = w_{tt}(x,t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = -\sigma^2 \qquad (6.59)$$

$$\Rightarrow \begin{cases} X''(x) + \sigma^2 X(x) = 0, \quad X(0) = 0, \quad AEX'(\ell) = 0 & \text{(i)} \\ \ddot{T}(t) + c^2\sigma^2 T(t) = 0, \quad \text{initial conditions} & \text{(ii)} \end{cases}$$

From equation (i) the form of the spatial solution is

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

Next apply the boundary conditions in (i) to get the characteristic equation and the form of the eigenfunction:

Apply the boundary conditions to the spatial solution to get

$$a\sin(0)+b\cos(0)=0$$

$$a\cos(\sigma\ell)-b\sin(\sigma\ell)=0$$

$$\Rightarrow b=0 \text{ and } \det\begin{bmatrix} 0 & 1 \\ \cos(\sigma\ell) & -\sin(\sigma\ell) \end{bmatrix}=0$$

$$\Rightarrow \begin{cases} \cos\sigma\ell=0 \Rightarrow \sigma_n=\dfrac{2n-1}{2\ell}\pi, & n=1,2,3,\cdots \\ X_n(x)=a_n\sin(\dfrac{(2n-1)\pi x}{2\ell}), & n=1,2,3,\cdots \end{cases}$$

Next consider the time response equation (ii):

$$\ddot{T}_n(t) + c^2\left(\frac{2n-1}{2\ell}\pi\right)^2 T(t) = 0$$

$$\Rightarrow T_n(t) = A_n \sin\frac{(2n-1)c\pi}{2\ell}t + B_n \cos\frac{(2n-1)c\pi}{2\ell}t$$

$$\omega_n = \frac{(2n-1)c\pi}{2\ell} = \frac{(2n-1)\pi}{2\ell}\sqrt{\frac{E}{\rho}}, \qquad n = 1, 2, 3\cdots \qquad (6.63)$$

Example 6.3.2 Given $v_0(l)$=3 cm/s, ρ =8x10^3 kg/m^3 and E=20x10^{10} N/m^2, compute the response.

$$w(x,t)=\sum_{n=1}^{\infty}(c_n\sin\sigma_n ct+d_n\cos\sigma_n ct)\sin\frac{(2n-1)}{2\ell}\pi x$$

$$d_n=\frac{2}{\ell}\int_0^{\ell} w_0(x)\sin\frac{(2n-1)}{2\ell}\pi x dx=0\Rightarrow$$

$$w_t(x,0)=0.03\delta(x-\ell)=\sum_{n=1}^{\infty}c_n\sigma_n c\cos(0)\sin\frac{(2n-1)}{2\ell}\pi x$$

Multiply by the mode shape indexed m and integrate:

$$\Rightarrow 0.03\int_0^{\ell}(\sin\frac{(2m-1)}{2\ell}\pi x)\delta(x-\ell)dx$$

$$=\sum_{n=1}^{\infty}\int_0^{\ell} c_n\sigma_n c\sin\frac{(2m-1)}{2\ell}\pi x\sin\frac{(2n-1)}{2\ell}\pi x dx$$

$$\Rightarrow 0.03\sin\frac{(2m-1)}{2}\pi=\frac{c\sigma_m\ell}{2}c_m \Rightarrow c_m=\frac{1}{\pi}\sqrt{\frac{\rho}{E}}\frac{0.06(-1)^{m+1}}{(2m-1)}$$

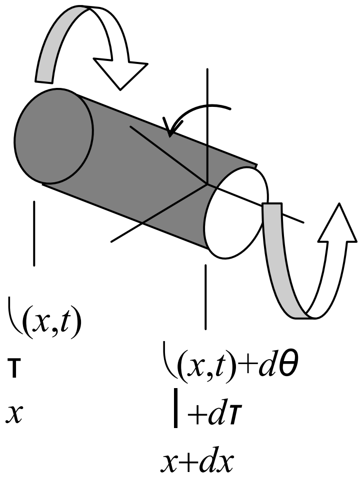
$$c_n=\sqrt{\frac{8\times10^3}{210\times10^9}}\frac{0.12(-1)^{n+1}}{\pi(2n-1)}=7.455\times10^{-6}\frac{(-1)^{n+1}}{(2n-1)}\ \text{m}$$

$$w(x,t)=7.455\times10^{-6}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\sin\left(\frac{2n-1}{10}\pi x\right)\sin\left(512.348(2n-1)t\right)\ \text{m}$$

Various Examples Follow From Other Boundary Conditions

- **Table 6.1 page 524 gives a variety of different boundary conditions. The resulting frequencies and mode shapes are given in Table 6.2 page 525.**
- **Various problems consist of computing these values**
- **Once modes/frequencies are determined, use *mode summation* to compute the response from IC**
- **See the book by Blevins:*Mode Shapes and Natural Frequencies* for more boundary conditions**

6.4 Torsional Vibrations



$$d\tau = \frac{\partial \tau}{\partial x} dx, \text{ from calculus}$$

$$\tau = GJ \frac{\partial \theta(x,t)}{\partial x}, \quad \text{from solid mechanics}$$

G=shear modulus

J=polar moment of area cross section

Summing moments on the element dx

$$\tau + \frac{\partial \tau}{\partial x} dx - \tau = \rho J \frac{\partial^2 \theta(x,t)}{\partial t^2} dx$$

Where ρ is the shaft's mass density

Combining these expressions yields;

$$\frac{\partial}{\partial x}\left(GJ \frac{\partial \theta(x,t)}{\partial x} \right) = \rho J \frac{\partial^2 \theta(x,t)}{\partial t^2}, GJ \text{ constant} \Rightarrow$$

$$\frac{\partial^2 \theta(x,t)}{\partial t^2} = \frac{G}{\rho} \frac{\partial^2 \theta(x,t)}{\partial x^2}$$

The initial and boundary conditions for torsional vibration problems are:

- Two spatial conditions (boundary conditions)
- Two time conditions (initial conditions)
- See Table 6.4 for a list of conditions and Equation (6.67) and Table 6.3 for odd cross section
- Clamped-free rod:

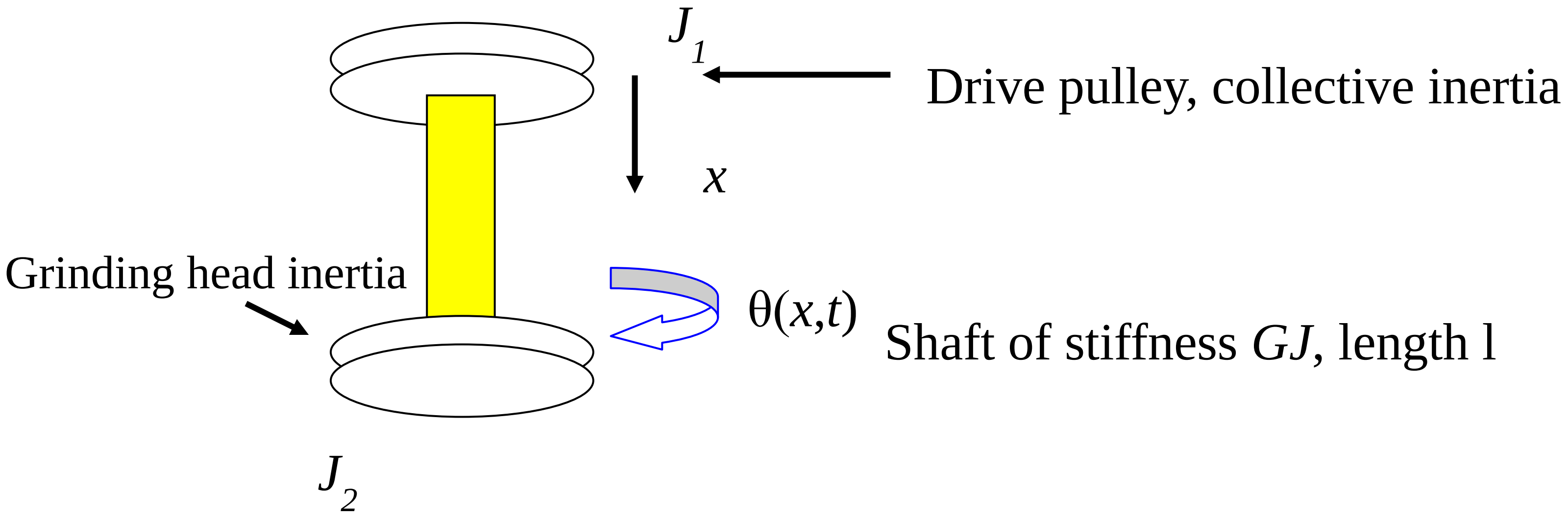
$$\theta(0,t) = 0 \quad \text{Clamped boundary (0 deflection)}$$

$$G\theta_x(\ell,t) = 0 \ \text{Free boundary (0 torque)}$$

$$\theta(x,0)=\theta_0(x) \quad \text{and} \quad \theta_t(x,0)=\dot{\theta}_0(x)$$

Example 6.4.1: grinding shaft vibrations

- **Top end of shaft is connected to pulley (x = 0)**
- **J_1 includes collective inertia of drive belt, pulley and motor**



Use torque balance at top and bottom to get the Boundary Conditions:

$$GJ\,\frac{\partial\theta(x,t)}{\partial x}\bigg|_{x=0} = J_1\,\frac{\partial^2\theta(x,t)}{\partial t^2}\bigg|_{x=0} \quad \text{at top}$$

$$GJ\,\frac{\partial\theta(x,t)}{\partial x}\bigg|_{x=\ell} = -J_2\,\frac{\partial^2\theta(x,t)}{\partial t^2}\bigg|_{x=\ell} \quad \text{at bottom}$$

The minus sign follows from right hand rule.

Again use separation of variables to attempt a solution

$$\theta(x,t) = \Theta(x)T(t) \Rightarrow$$

$$\frac{\Theta''(x)}{\Theta(x)} = \left(\frac{\rho}{G}\right)\frac{\ddot{T}(t)}{T(t)} = -\sigma^2$$

$$c^2 = \left(\frac{G}{\rho}\right)$$

$$\Theta''(x) + \sigma^2\Theta(x) = 0, \quad \ddot{T}(t) + \omega^2 T(t) = 0$$

$$\omega = \sigma c = \sigma\sqrt{\frac{G}{\rho}}$$

The next step is to use the boundary conditions:

$$\text{Boundary Condition at } x = 0 \Rightarrow$$

$$GJ\Theta'(0)T(t) = J_1\Theta(0)\ddot{T}(t) \Rightarrow$$

$$\frac{GJ\Theta'(0)}{J_1\Theta(0)} = \frac{\ddot{T}(t)}{T(t)} = -c^2\sigma^2 \Rightarrow$$

$$\Theta'(0) = -\frac{\sigma^2 J_1}{\rho J}\Theta(0)$$

Similarly the boundary condition at l yields:

$$\Theta'(\ell) = \frac{\sigma^2 J_2}{\rho J}\Theta(\ell)$$

The Boundary Conditions reveal the *Characteristic Equation*

$$\Theta(x) = a_1 \sin\sigma x + a_2 \cos\sigma x \Rightarrow \Theta(0) = a_2$$

$$\Theta'(x) = a_1\sigma \cos\sigma x - a_2\sigma \sin\sigma x \Rightarrow \Theta'(0) = a_1\sigma$$

$$x = 0 \Rightarrow$$

$$\Theta'(0) = -\frac{\sigma^2 J_1}{\rho J}\Theta(0) \Rightarrow \underline{a_1 = -\frac{\sigma J_1}{\rho J} a_2}$$

$$x = \ell \Rightarrow$$

$$\Theta'(\ell) = \frac{\sigma^2 J_1}{\rho J}\Theta(\ell) \Rightarrow a_1\sigma \cos\sigma\ell - a_2\sigma \sin\sigma\ell = \frac{\sigma^2 J_1}{\rho J} a_1 \sin\sigma\ell + a_2 \cos\sigma\ell$$

$$\Rightarrow \underline{\tan(\sigma\ell) = \frac{\rho J\ell(J_1 + J_2)(\sigma\ell)}{J_1 J_2(\sigma\ell)^2 - (\rho J\ell)^2}} \quad \longleftarrow \quad \text{THE CHARACTERISTIC EQUATION} \quad (6.82)$$

Solving the for the first mode shape

$$\tan(\sigma\ell) = \frac{\rho J\ell(J_1 + J_2)(\sigma\ell)}{J_1 J_2(\sigma\ell)^2 - (\rho J\ell)^2}$$ has 0 as its first solution:

Numerically solve for $\sigma_n\ell, \quad n = 1, 2, 3, \ldots,$ and $\omega_n = \sigma_n\sqrt{\frac{G}{\rho}}$

Note for $n = 1, \sigma_1 = 0 \Rightarrow \omega_1 = 0 \Rightarrow \ddot{T}(t) = 0 \Rightarrow$

$T(t) = a + bt$ the rigid body mode of the shaft turning

$\Rightarrow \Theta_1''(x) = 0, \Rightarrow \Theta_1(x) = a_1 + b_1 x \Rightarrow$

$x = 0 \Rightarrow b_1 = 0 \Rightarrow \Theta_1(x) = a_1$ the first mode shape

Solutions of the Characteristic Equation involve solving a transcendental equation

$$(bx^2 - a)\tan x = x$$

$$x = \sigma\ell, \quad a = \frac{\rho J \ell}{J_1 + J_2}, \quad b = \frac{J_1 J_2}{(J_1 + J_2)\rho J \ell}$$

$$J_1 = J_2 = 10 \text{ kg}\cdot\text{m}^2/\text{rad}, \ \rho = 2700 \text{ kg/m}^3,$$

$$J = 5 \text{ kg}\cdot\text{m}^2/\text{rad}, \ \ \ell = 0.25 \text{ m}$$

$$G = 25\times 10^9 \text{ Pa}$$

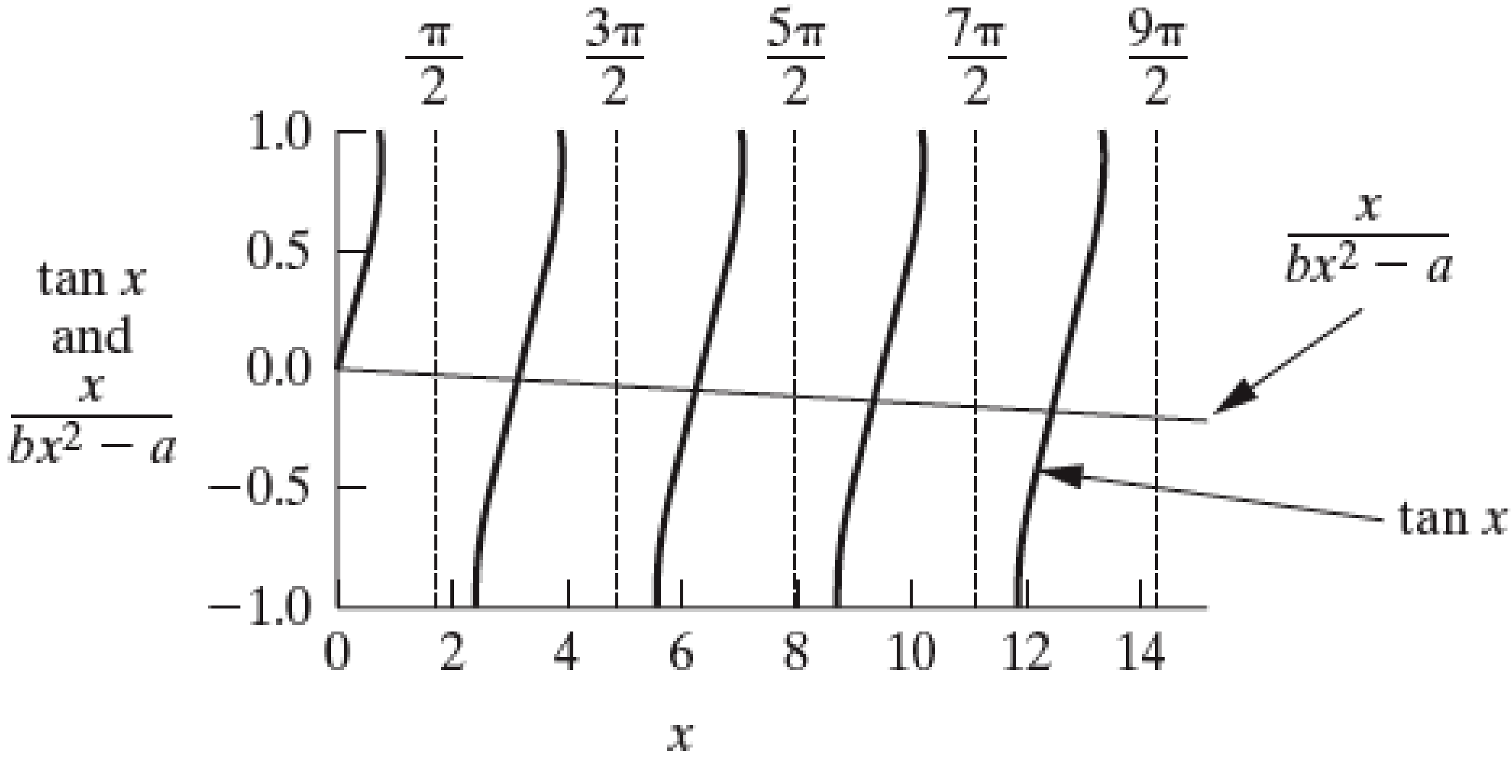
$$\Rightarrow$$

$$f_1 = 0 \text{ Hz}, \ \ f_2 = 38,013 \text{ Hz},$$

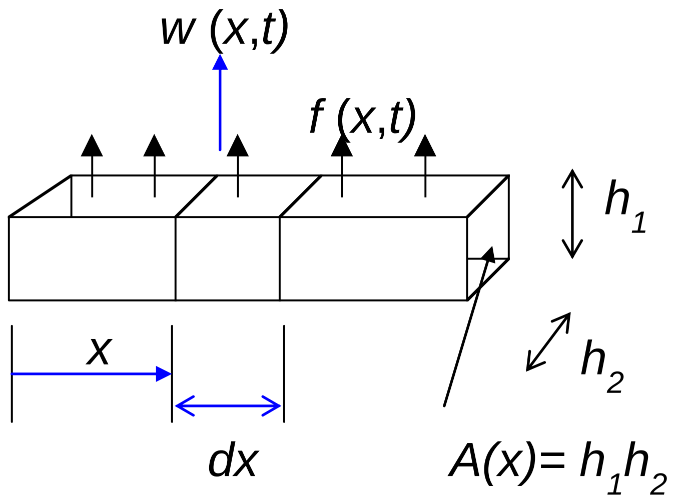
$$f_3 = 76,026 \text{ Hz}, \quad f_4 = 114,039 \text{ Hz},$$

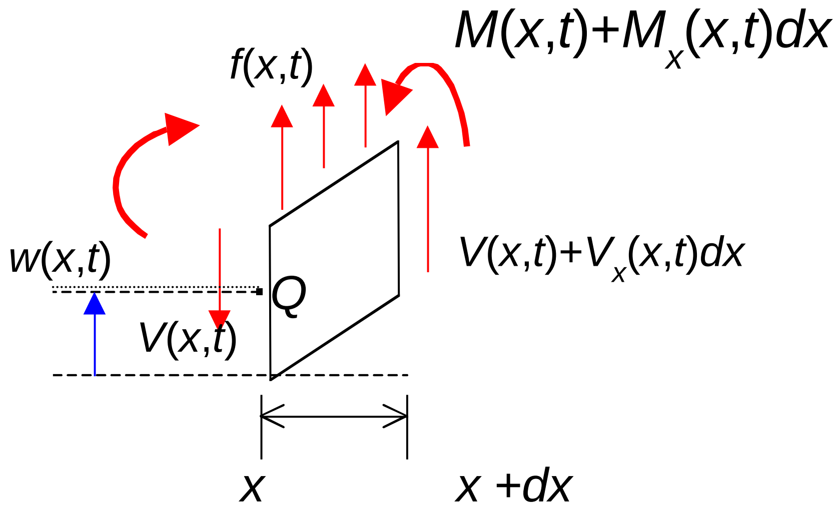
Fig 6.9 Plots of each side of eq (6.82)

to assist in find initial guess for numerical routines used to compute the roots.



6.5 Bending vibrations of a beam





$$\text{bending stiffness } = EI(x)$$

$$E = \text{ Youngs modulus}$$

$$I(x) = \text{cross-sect. area moment of inertia about } z$$

$$M(x,t) = EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}$$

Next sum forces in the *y*-direction (up, down)

Sum moments about the point *Q*

Use the moment given from stenght of materials

Assume sides do not bend (no shear deformation)

Summing forces and moments yields:

$$\left(V(x,t) + \frac{\partial V(x,t)}{\partial x}dx\right) - V(x,t) + f(x,t)dx = \rho A(x)dx\,\frac{\partial^2 w(x,t)}{\partial t^2}$$

$$\left(M(x,t) + \frac{\partial M(x,t)}{\partial x}dx\right) - M(x,t) + \left[V(x,t) + \frac{\partial V(x,t)}{\partial x}dx\right]dx$$

$$+ f(x,t)dx\,\frac{dx}{2} = 0$$

$$\Rightarrow \left[\frac{\partial M(x,t)}{\partial x} + V(x,t)\right]dx + \left[\frac{\partial V(x,t)}{\partial x} + \frac{f(x,t)}{2}\right](dx)^2 = 0$$

$$\Rightarrow V(x,t) = -\frac{\partial M(x,t)}{\partial x}$$

Substitute into force balance equation yields:

$$-\frac{\partial^2 M(x,t)}{\partial x^2}dx + f(x,t)dx = \rho A(x)dx\frac{\partial^2 w(x,t)}{\partial t^2}$$

Dividing by dx and substituting for M yields

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] = f(x,t)$$

Assume constant stiffness to get:

$$\frac{\partial^2 w(x,t)}{\partial t^2} + c^2\frac{\partial^4 w(x,t)}{\partial x^4} = 0, \quad c = \sqrt{\frac{EI}{\rho A}}$$

The possible boundary conditions are (choose 4):

Free end

$$\text{bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0$$

$$\text{shear force} = \frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0$$

Clamped (or fixed) end

$$\text{deflection} = w = 0$$

$$\text{slope} = \frac{\partial w}{\partial x} = 0$$

Pinned (or simply supported) end

$$\text{deflection} = w = 0$$

$$\text{bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0$$

Sliding end

$$\text{slope} = \frac{\partial w}{\partial x} = 0$$

$$\text{shear force} = \frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0$$

Solution of the time equation yields the oscillatory nature:

$$c^2 \frac{X''''(x)}{X(x)} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2 \Rightarrow$$

$$\ddot{T}(t) + \omega^2 T(t) = 0 \Rightarrow$$

$$T(t) = A\sin\omega t + B\cos\omega t$$

Two initial conditions:

$$w(x,0) = w_0(x), w_t(x,0) = \dot{w}_0(x)$$

Spatial equation results in a boundary value problem (BVP)

$$X''''(x) - \left(\frac{\omega}{c}\right)^2 X(x) = 0.$$

Define $\beta^4 = \left(\frac{\omega}{c}\right)^2 = \frac{\rho A \omega^2}{EI}$

Let $X(x) = Ae^{\sigma x}$ to get:

$$X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x$$

Apply boundary conditions to get 3 constants and the characteristic equation

Example 6.5.1: compute the mode shapes and natural frequencies for a clamped-pinned beam.

At fixed end $x = 0$ and

$$X(0) = 0 \Rightarrow a_2 + a_4 = 0$$

$$X'(0) = 0 \Rightarrow \beta(a_1 + a_3) = 0$$

At the pinned end, $x = \ell$ and

$$X(\ell) = 0 \Rightarrow$$

$$a_1 \sin\beta\ell + a_2 \cos\beta\ell + a_3 \sinh\beta\ell + a_4 \cosh\beta\ell = 0$$

$$EIX''(\ell) = 0 \Rightarrow$$

$$\beta^2(-a_1 \sin\beta\ell - a_2 \cos\beta\ell + a_3 \sinh\beta\ell + a_4 \cosh\beta\ell) = 0$$

The 4 boundary conditions in the 4 constants can be written as the matrix equation:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ \sin\beta\ell & \cos\beta\ell & \sinh\beta\ell & \cosh\beta\ell \\ -\beta^2\sin\beta\ell & -\beta^2\cos\beta\ell & \beta^2\sinh\beta\ell & \beta^2\cosh\beta\ell \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\mathbf{a}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B\mathbf{a} = \mathbf{0}, \mathbf{a} \neq \mathbf{0} \Rightarrow \det(B) = 0 \Rightarrow$$

$$\underline{\tan\beta\ell = \tanh\beta\ell}$$

The characteristic equation

Solve numerically (fsolve) to obtain solution to transcendental (characteristic) equation

$$\beta_1\ell = 3.926602 \qquad \beta_2\ell = 7.068583 \qquad \beta_3\ell = 10.210176$$

$$\beta_4\ell = 13.351768 \qquad \beta_5\ell = 16.493361 \qquad \ldots$$

$$n > 5 \Rightarrow$$

$$\beta_n\ell = \frac{(4n+1)\pi}{4}$$

Next solve *B***a**=0 for 3 of the constants:

With the eigenvalues known, now solve for the eigenfunctions:

$B\mathbf{a} = \mathbf{0}$ yields 3 constants in terms of the 4th:

$a_1 = -a_3$ from the first equation

$a_2 = -a_4$ from the second equation

$(\sinh \beta_n \ell - \sin \beta_n \ell) a_3 + (\cosh \beta_n \ell - \cos \beta_n \ell) a_4 = 0$

from the third (or fourth) equation

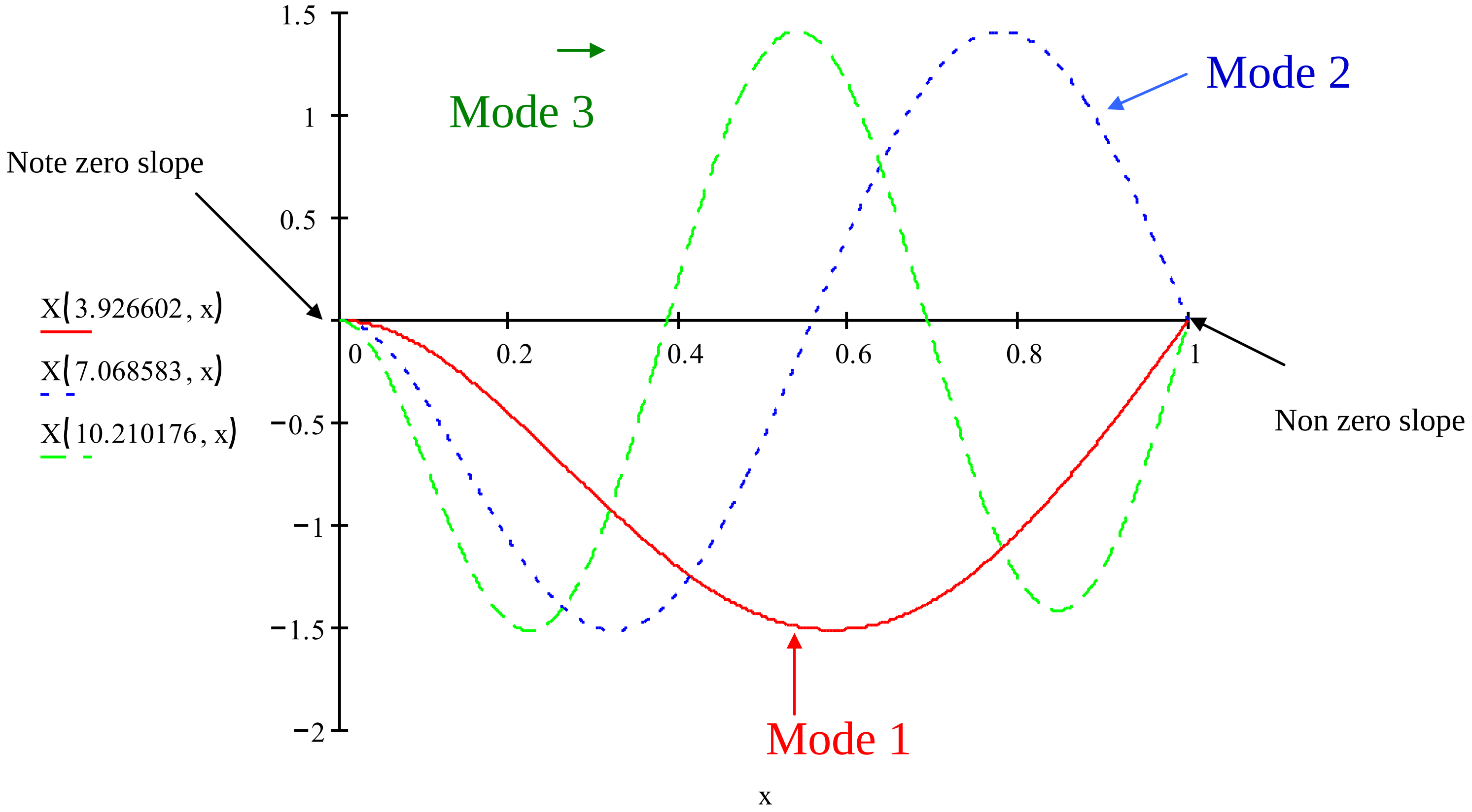
Solving yields:

$$a_3 = -\frac{\cosh \beta_n \ell - \cos \beta_n \ell}{\sinh \beta_n \ell - \sin \beta_n \ell} a_4$$

$$\Rightarrow X_n(x) = (a_4)_n \left[\frac{\cosh \beta_n \ell - \cos \beta_n \ell}{\sinh \beta_n \ell - \sin \beta_n \ell} (\sinh \beta_n \ell x - \sin \beta_n \ell x) - \cosh \beta_n \ell x + \cos \beta_n \ell x \right]$$

Plot the mode shapes to help understand the system response

$$X(n,x) := \frac{\cosh(n) - \cos(n)}{\sinh(n) - \sin(n)} \cdot (\sinh(n \cdot x) - \sin(n \cdot x)) - \cosh(n \cdot x) + \cos(n \cdot x)$$



Again, the modeshape orthogonality becomes important and is computed as follows:

Write the eigenvalue problem twice, once for *n* and once for *m*.

$$X_n''''(x) = \beta_n^4 X_n(x) \quad \text{and} \quad X_m''''(x) = \beta_m^4 X_m(x)$$

Multiply by $X_m(x)$ and $X_n(x)$ respectively, integrate and subtact to get:

$$\int_0^\ell X_n''''(x) X_m(x) dx - \int_0^\ell X_m''''(x) X_n(x) dx = (\beta_n^4 - \beta_m^4) \int_0^\ell X_n(x) X_m(x) dx$$

Then integrate the left hand side twice by parts to get:

Use integration by parts to evaluate the integrals in the orthogonality condition.

apply $\int udv = uv - \int vdu$ twice:

$$\int_0^\ell \underbrace{X_m(x)}_{u}\,\underbrace{X_n''''(x)dx}_{dv} = \underbrace{X_m}_{u}\,\underbrace{X_n'''}_{v}\Big|_0^\ell - \int_0^\ell \underbrace{X_n'''}_{v}\,\underbrace{X_m'dx}_{du}$$

$$= \cancelto{0}{X_m(\ell)}\,X_n'''(\ell) - \cancelto{0}{X_m(0)}\,X_n'''(0) - \int_0^\ell \underbrace{X_m'}_{u}\,\underbrace{X_n'''dx}_{dv}$$

$$-\int_0^{\ell} \underset{u}{X_m'}\ \underset{dv}{X_n'''}dx = -X_m'(x)\, X_n''(x)\Big|_0^{\ell} + \int_0^{\ell} X_n''(x) X_m''(x)dx$$

$$= -X_m'(\ell)\cancelto{0}{X_n''(\ell)} + X_m'(0)\cancelto{0}{X_n''(0)} + \int_0^{\ell} X_n''(x) X_m''(x)dx$$

Thus

$$\int_0^{\ell} X_n''''(x) X_m(x)dx - \int_0^{\ell} X_m''''(x) X_n(x)dx = (\beta_n^4 - \beta_m^4)\int_0^{\ell} X_n(x) X_m(x)dx$$

$$\Rightarrow \underbrace{\int_0^{\ell} X_n''(x) X_m''(x)dx - \int_0^{\ell} X_n''(x) X_m''(x)dx}_{0} = \underbrace{(\beta_n^4 - \beta_m^4)}_{\neq 0}\int_0^{\ell} X_n(x) X_m(x)dx$$

$$\Rightarrow \int_0^{\ell} X_n(x) X_m(x)dx = 0, \forall n, m, n \neq m$$

The solution can be computed via modal expansion based on orthogonality of the modes.

$$\int_0^\ell X_n(x) X_m(x) dx = \delta_{nm}$$

$$w(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) X_n(x)$$

$$w(x,0) = w_0(x) = \sum_{n=1}^{\infty} B_n X_n(x) \Rightarrow \underline{B_n = \int_0^\ell w_0(x) X_n(x) dx}$$

$$w_t(x,0) = \dot{w}_0(x) = \sum_{n=1}^{\infty} \omega_n A_n X_n(x) \Rightarrow \underline{A_n = \frac{1}{\omega_n} \int_0^\ell \dot{w}_0(x) X_n(x) dx}$$

Summary of the Euler-Bournoulli Beam Assumptions

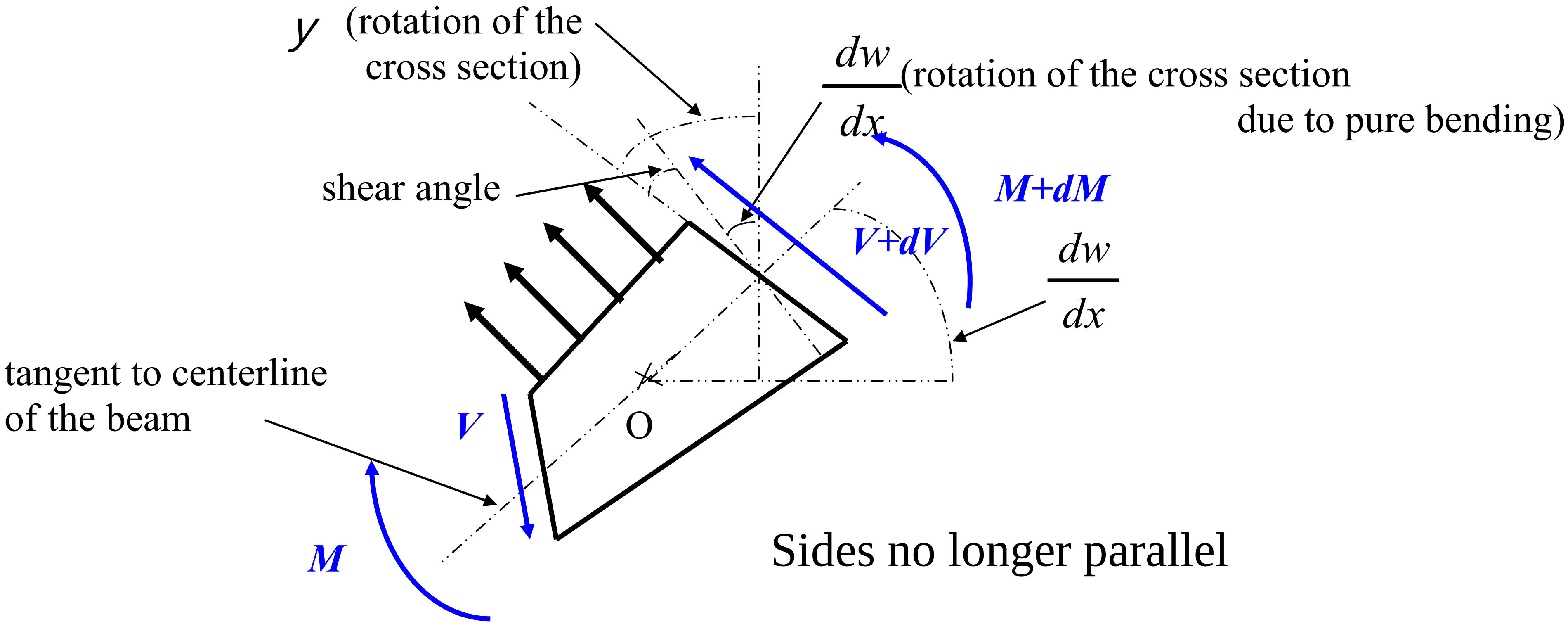
- Uniform along its span and slender
- Linear, homogenous, isotropic elastic material without axial loads
- Plane sections remain plane
- Plane of symmetry is plane of vibration so that rotation & translation decoupled
- Rotary inertia and shear deformation neglected
- Tables 6.4 and 6.5 give eigensolutions for several configurations (see Blevins for more)

Continuing with the Timoshenko beam

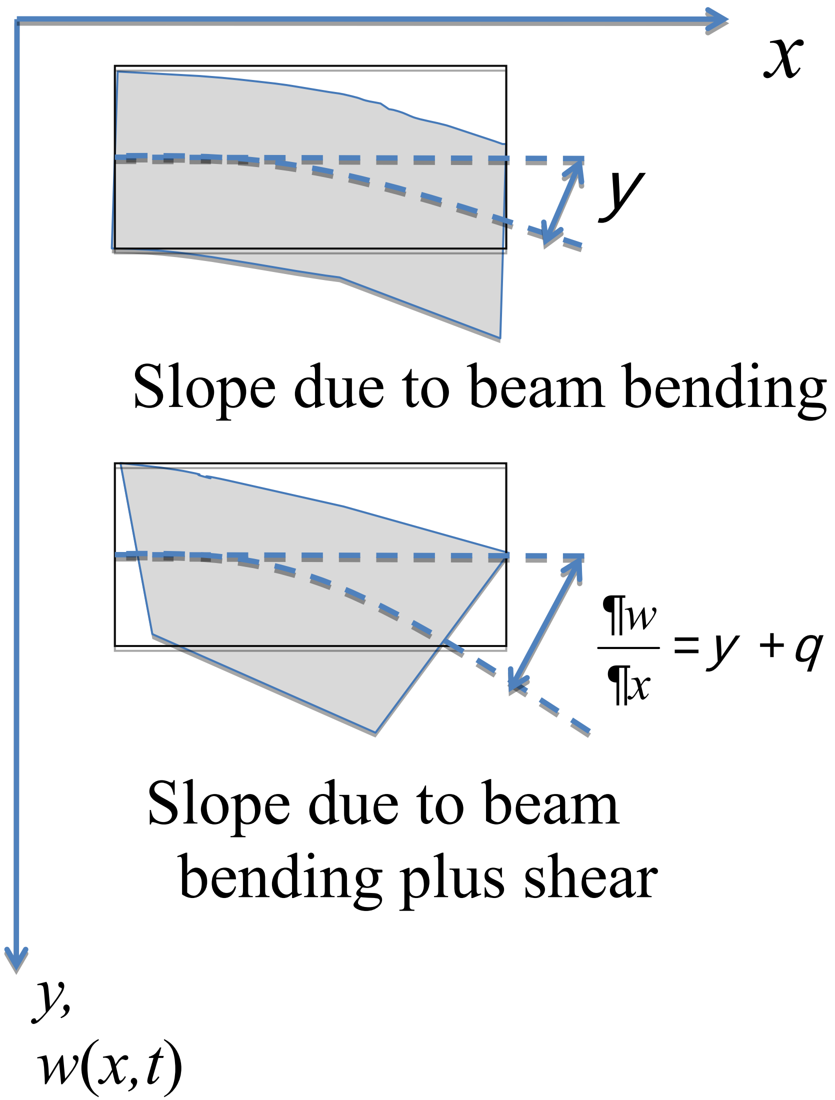
Including the effects of shear deformation and rotational inertia

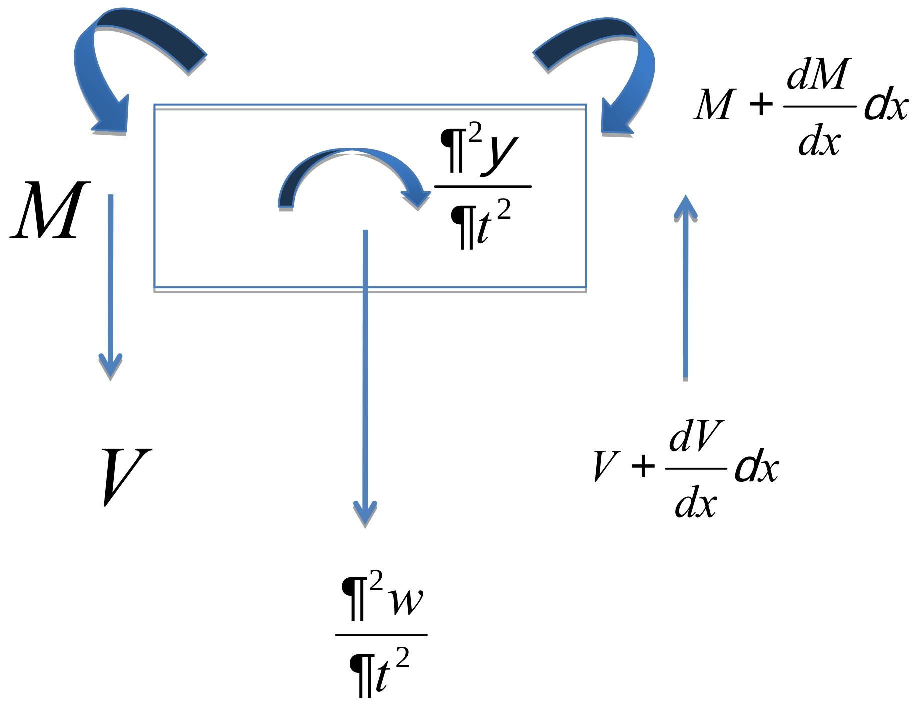
Timoshenko beam equation
short fat beams

- Add in the effect of rotary inertia, and
- Shear deformation



Thick Beam FBD





Dynamics this time with rotary inertia

The bending moment becomes:

$$EI\frac{d\psi(x,t)}{dx} = M(x,t)$$

Shear force equation becomes (from deforms):

$$\kappa^2 AG\left[\psi(x,t) - \frac{dw(x,t)}{dx}\right] = V(x,t)$$

A dynamic force balance gives

$$\rho A(x)dx\frac{\partial^2 w(x,t)}{\partial t^2} = -\left[V(x,t) + \frac{\partial V(x,t)}{\partial x}dx\right] + V(x,t) + f(x,t)dx$$

G is the shear modulus and κ^2 is the shear coefficient (just κ in some treatments, 5/6 for rectangular cross section and 9/10 for circular).

Including rotational inertia, the moment balance becomes:

$$\rho I(x)dx\frac{\partial^2\psi}{\partial t^2}=\left[M+\frac{\partial M}{\partial x}dx\right]-M+\left[V+\frac{\partial V}{\partial x}dx\right]dx+f\,\frac{dx^2}{2}$$

Combining these last few expressions yields:

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\left[EI\,\frac{\partial\psi}{\partial x}\right]+\kappa^2AG\left(\frac{\partial w}{\partial x}-\psi\right)=\rho I\,\frac{\partial^2\psi}{\partial t^2}\\&\frac{\partial}{\partial x}\left[\kappa^2AG\left(\frac{\partial w}{\partial x}-\psi\right)\right]+f=\rho A\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\quad\text{equations of motion}$$

If all coefficients constant, and no external force $\Rightarrow$

$$EI\,\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{\kappa^2G}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\rho^2 I}{\kappa^2G}\,\frac{\partial^4 w}{\partial t^4}=0$$

Four Boundary Conditions

(2 at each end)

$$\text{Clamped Boundary (say at } x = 0)$$

$$\psi(0,t) = 0, \quad w(0,t) = 0$$

deflections

$$\text{Simply Supported Boundary (say at } x = 0)$$

Bending moment

$$EI \frac{\partial \psi(0,t)}{\partial x} = 0, \quad w(0,t) = 0$$

deflection

$$\text{Free Boundary (say at } x = 0)$$

$$\kappa^2 AG\left(\frac{\partial w}{\partial x} - \psi\right) = 0, \quad EI \frac{\partial \psi(0,t)}{\partial x} = 0$$

Shear force

Bending moment

4 initial conditions

$$\psi(x,0) = \psi_0(x)$$

$$\psi_t(x,0) = \dot{\psi}_0(x)$$

$$w(x,0) = w_0(x)$$

$$w_t(x,0) = \dot{w}_0(x)$$

Rotation from neutral axis

Example: Compute the frequencies of a pinned-pinned beam for both beam models*

$w(x,t) = X(x)T(t)$ does not work, so try the more restricted form

$$w_n(x,t) = \sin\left(\frac{n\pi x}{\ell}\right)\cos\omega_n t$$

Harmonic time response

Pinned modes of EB

$$EI\left(\frac{n\pi}{\ell}\right)^4 \sin\left(\frac{n\pi x}{\ell}\right)\cos\omega_n t - \rho I\left(1+\frac{E}{\kappa^2 G}\right)\left(\frac{n\pi}{\ell}\right)^2 \omega_n^2 \sin\left(\frac{n\pi x}{\ell}\right)\cos\omega_n t$$

$$= -\frac{\rho^2 I}{\kappa^2 G}\omega_n^4 \sin\left(\frac{n\pi x}{\ell}\right)\cos\omega_n t + \rho A\omega_n^2 \sin\left(\frac{n\pi x}{\ell}\right)\cos\omega_n t$$

This yields the characteristic equation directly in terms of the natural frequencies of oscillation:

$$\omega_n^4\left(\frac{\rho r^2}{\kappa^2 G}\right)-\left(1+\frac{n^2\pi^2 r^2}{\ell^2}+\frac{n^2\pi^2 r^2}{\ell^2}\frac{E}{\kappa^2 G}\right)\omega_n^2+\frac{n^2\alpha^2\pi^2}{\ell^4}=0$$

where $\alpha^2=\dfrac{EI}{\rho A}, \qquad r^2=\dfrac{I}{A}$

This is quadratic in omega squared and has roots as follows:

Analysis of frequencies

Euler Bernoulli freq:

$$\omega_n^2 = \frac{\alpha^2 n^2 \pi^4}{\ell^4}$$

Euler Bernoulli plus effect of rotary inertia:

$$\omega_n^2 = \frac{\alpha^2 n^2 \pi^4}{\ell^4 \left[1 + n^2 \pi^2 r^2 / \ell^2\right]}$$

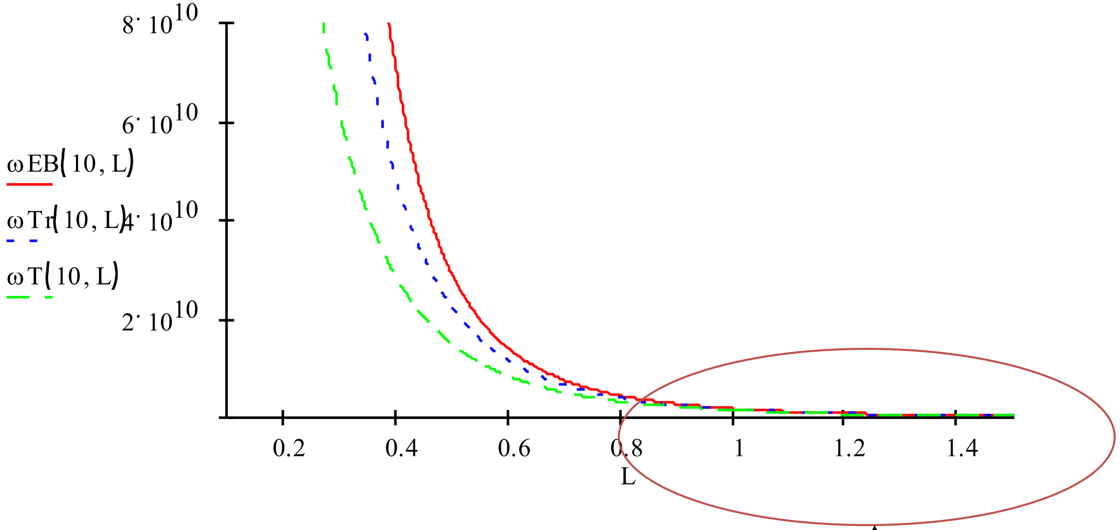
Euler Bernoulli plus effect of rotary inertia and shear deformation:

$$\omega_n^2 = \frac{\alpha^2 n^2 \pi^4}{\ell^4 \left[1 + \left(n^2 \pi^2 r^2 / \ell^2\right) E / \kappa G\right]}$$

Reduces frequency, more prominent effect for larger frequencies.

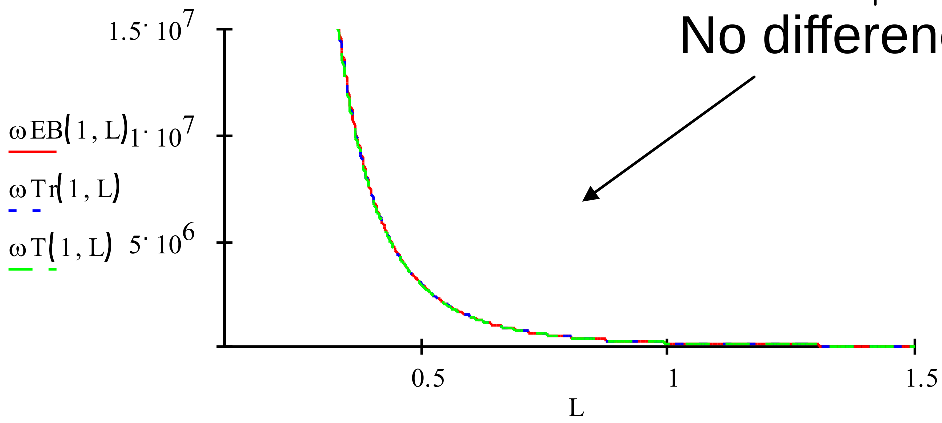
Comparison of frequencies versus length

n=10

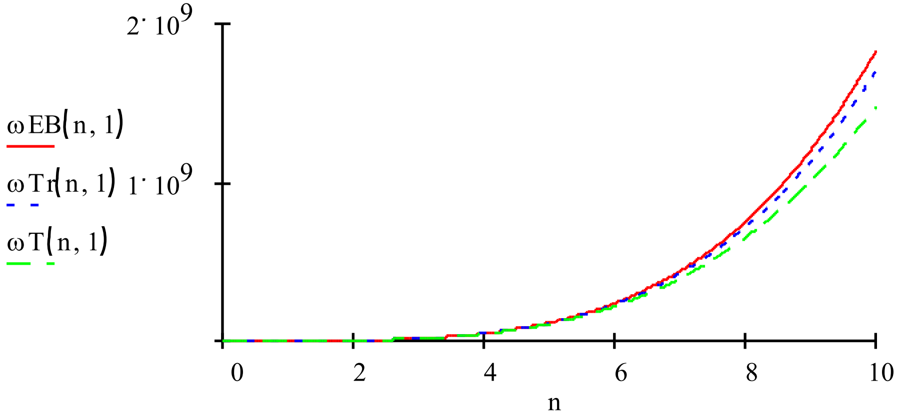


No difference

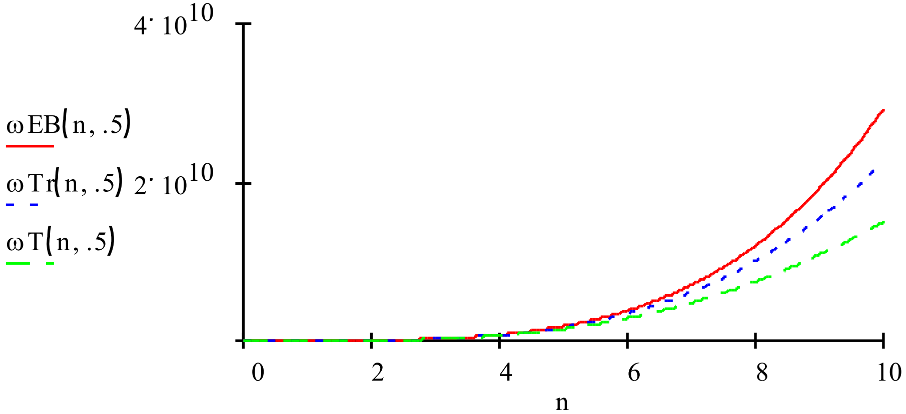
n=1



Comparison of frequencies versus mode number



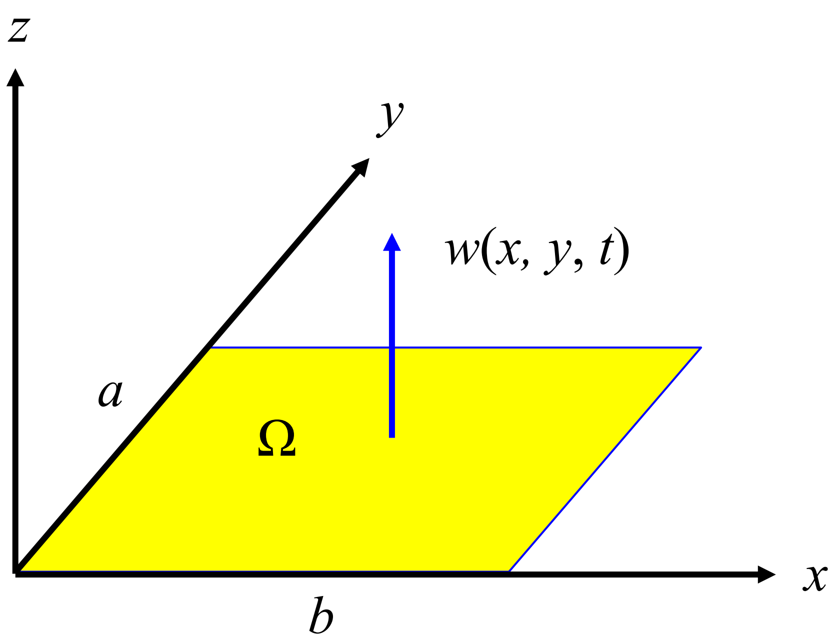
L=1 m



L=0.5 m

Rule of thumb is $L > 10\ h$

6.6 Vibration of membranes and plates



- The domain (Ω) is now a plane rather then a line: Two dimensional
- Membrane is a two dimensional string
- Plate is a two dimensional beam

Figure 6.13

The membrane equation:

$$\tau \nabla^2 w(x, y, t) = \rho w_{tt}(x, y, t), \quad x, y \in \Omega$$

Tension per unit length density(mass/area)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The *Laplace* operator

$$\frac{\partial^2 w(x, y, t)}{\partial x^2} + \frac{\partial^2 w(x, y, t)}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 w(x, y, t)}{\partial t^2}$$

$$c = \sqrt{\tau / \rho}$$

Boundary and initial conditions

$w(x, y, t) = 0$ for some part of the boundary $\partial\Omega$

$\dfrac{\partial w(x, y, t)}{\partial n} = 0$ for some other part of the boundary $\partial\Omega$

where this derivative donotes the derivative normal to the plane of the membrane

plus the usual initial conditions

Example 6.6.1 Compute the natural frequencies of a square membrane of 1 m side.

$$c^2\left[\frac{\partial^2 w}{\partial^2 x}+\frac{\partial^2 w}{\partial^2 y}\right]=\frac{\partial^2 w}{\partial^2 t},\quad x,y\in\Omega$$

$$w(x,y,t)=X(x)Y(y)T(t)\Rightarrow$$

$$X''YT+XY''T=\frac{1}{c^2}XY\ddot{T}\Rightarrow\frac{X''Y+XY''}{XY}=\overbrace{\frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)}=-\omega^2}^{\text{Temporal equation}}$$

$$\frac{X''Y+XY''}{XY}=\frac{X''}{X}+\frac{Y''}{Y}=-\omega^2\Rightarrow\frac{X''}{X}=-\frac{Y''}{Y}-\omega^2=-\alpha^2$$

$$\Rightarrow\begin{cases}\dfrac{X''}{X}=-\alpha^2\\[2ex]\dfrac{Y''}{Y}=-\gamma^2\end{cases}$$

where $\omega^2=\alpha^2+\gamma^2$

$$X''(x) + \alpha^2 X(x) = 0 \Rightarrow A\sin\alpha x + B\cos\alpha x$$

$$Y''(y) + \gamma^2 Y(y) = 0 \Rightarrow C\sin\gamma y + D\cos\gamma y$$

$$X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y$$

$$+ A_3 \cos\alpha x \sin\gamma y + A_4 \cos\alpha x \cos\gamma y$$

Now apply the boundary conditions:

along $x = 0$:

$$T(t)X(0)Y(y) = T(t)(A_3 \sin\gamma y + A_4 \cos\gamma y) = 0 \Rightarrow$$

$$A_3 \sin\gamma y + A_4 \cos\gamma y = 0 \quad \text{which must hold for any } y$$

$$\Rightarrow \underline{A_3 = A_4 = 0}$$

Now we have:

$$X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y$$

Along $x = 1$, $w(1, y, t) = 0$:

$$A_1 \sin\alpha \sin\gamma y + A_2 \sin\alpha \cos\gamma y = 0 \Rightarrow$$

$$\sin\alpha(A_1 \sin\gamma y + A_2 \cos\gamma y) = 0 \Rightarrow$$

$$\text{either } \sin\alpha = 0 \quad \text{or } A_1 = A_2 = 0$$

$$\Rightarrow \sin\alpha = 0 \Rightarrow \underline{\alpha_n = n\pi,\ n = 1,2,3,4\ldots}$$

At this point:

$$X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y$$

At y=0, $w(x,0,t) = 0 \Rightarrow$

$$A_1 \sin\alpha x \sin 0 + A_2 \sin\alpha x \cos 0 = 0 \Rightarrow A_2 = 0$$

So $X(x)Y(y) = A_1 \sin\alpha x \sin\gamma y$

At y=1, $w(x,1,t) = 0 \Rightarrow$

$A_1 \sin\alpha x \sin\gamma 1 = 0 \Rightarrow \sin\gamma = 0$

which gives $\underline{\gamma_m = m\pi,\ m = 1, 2, 3, 4...}$

Frequencies and Mode Shapes

$$\sigma_n = n\pi, \gamma_m = m\pi \Rightarrow \omega_{nm} = \sqrt{\gamma_m^2 + \sigma_n^2} \Rightarrow$$

frequencies are

$$\underline{\omega_{nm} = \pi\sqrt{n^2 + m^2}}$$

mode shapes are

$$\underline{\{\sin n\pi x \sin m\pi y\}_{n,m=1}^{\infty}}$$

Solution

$$w(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\sin n\pi x\sin m\pi y)\{A_{nm}\sin\omega_{nm}ct+B_{nm}\sin\omega_{nm}ct\}$$

The $\sin n\pi x\sin m\pi y$ are orthogonal:

$$\int_0^1\int_0^1 w(x,y,t)(\sin n\pi x\sin m\pi y)dxdy=\frac{1}{4}(A_{nm}\sin\omega_{nm}ct+B_{nm}\sin\omega_{nm}ct)$$

Apply the initial conditions at t=0 to get:

$$A_{nm}=4\int_0^1\int_0^1 w(x,y,0)(\sin n\pi x\sin m\pi y)dxdy$$

$$B_{nm}=\frac{4}{\omega_{nm}c}\int_0^1\int_0^1 w_t(x,y,0)(\sin n\pi x\sin m\pi y)dxdy$$

Suppose that $w_t(x, y, 0) = 0 \Rightarrow B_{nm} = 0,$

Suppose that $w(x, y, 0) = \sin \pi x \sin \pi y \Rightarrow A_{nm} = 0,$

except for A_{11} and $\omega_{11} = \pi\sqrt{2} \Rightarrow$

$w(x, y, t) = A_{11} \sin \pi x \sin \pi y \sin \omega_{11} ct$ first mode vibration

What does the second mode look like? There are two:

$$w_{12}(x, y, t) = A_{12} \sin \pi x \sin 2\pi y \sin \omega_{12} ct$$

$$w_{21}(x, y, t) = A_{21} \sin 2\pi x \sin \pi y \sin \omega_{21} ct$$

$\omega_{12} = \omega_{21} = \pi\sqrt{5}$, but $w_{12}(x, y, t) \neq w_{21}(x, y, t)$

w_{12} has node line at $y = 0.5$ while

w_{21} has a node line at $x = 0.5$.

Plate vibration:

$$D_E \nabla^4 w(x,y,t) = \rho w_{tt}(x,y,t), \quad x,y \in \Omega$$

$$D_E = \frac{Eh^3}{12(1-\nu^2)}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

- Think of it as adding bending stiffness to the membrane
- Assume small deflections with respect to the thickness *h*
- Plane through middle is called neutral plane and does not deform in bending
- No thickness stretch

Boundary Conditions:

- This is a 2D analog of the EB beam theory and is called *thin plate* theory
- Must be enough boundary conditions for the 4th order derivatives:

$$w(x,y,t) = 0 \quad \text{and} \quad \frac{\partial w(x,y,t)}{\partial n} = 0 \ \text{ for } x, y \in \partial\Omega$$

For a simply supported (pinned) rectangular plate

$$w(x,y,t) = 0, \ \text{ along } x = 0, y = 0, x = \ell_1, y = \ell_2$$

$$\frac{\partial^2 w(x,y,t)}{\partial x^2} = 0, \ \text{ along } x = 0, x = \ell_1$$

$$\frac{\partial^2 w(x,y,t)}{\partial y^2} = 0, \ \text{ along } y = 0, y = \ell_2$$

6.7 Models of Damping

Equations of motion for damped systems have the form:

$$w_{tt}(\mathbf{x},t) + L_1 w_t(\mathbf{x},t) + L_2 w(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \Omega$$

$$Bw(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \partial\Omega$$

Separation of variables will only work for certain forms of the operators L_1 and L_2

There are many models of damping and many ways to approach the inclusion of damping.

- Two ways to introduce damping
- Use the concept of modal damping
- Examine some physical possibilities

$$\ddot{T}_n(t) + \omega_n^2 T(t) = 0, \quad n = 1, 2, 3\ldots$$

Add modal damping via $2\zeta_n \omega_n \dot{T}_n(t)$

The string equation

$$\ddot{T}_n(t) + 2\zeta_n \omega_n \dot{T}_n(t) + \omega_n^2 T(t) = 0, \quad n = 1, 2, 3\ldots$$

$$\Rightarrow T_n(t) = A_n e^{-\zeta_n \omega_n} \sin(\omega_{dn} t + \phi_n), \quad \omega_{dn} = \omega_n \sqrt{1 - \zeta^2}$$

Example: **compute the response of a cantilevered bar with modal damping ratio 0.01 and IC's $w(x,0)=(x/L)$ and $w_t(x,0)=0$.**

Undamped modal solution

$$\Rightarrow X_n(x) = \sin\left[\frac{(2n-1)\pi x}{2L}\right], \quad \omega_n = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}}$$

Add modal damping

$$\omega_{dn} = \omega_n\sqrt{1-\zeta_n^2} = 0.9999\frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}}$$

$$T_n(t) = A_n e^{-0.01\omega_n t}\sin(\omega_{dn}t + \phi_n)$$

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-0.01\omega_n t}\sin(\omega_{dn}t + \phi_n)\sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$0.01\frac{x}{L} = \sum_{n=1}^{\infty} A_n \sin\phi_n \sin\sigma_n L$$

Multiplying the *m*th mode and integrating yields

$$\frac{0.01}{L}\int_0^L x \sin\sigma_m x dx = \frac{0.01}{L\sigma_m^2}(-1)^{m+1}$$

$$= \sum_{n=1}^{\infty} A_n \sin\phi_n \int_0^L \sin\sigma_m x \sin\sigma_n x dx$$

$$= A_m \sin\phi_m (\frac{L}{2})$$

$$\Rightarrow \underline{\frac{0.01}{L\sigma_m^2}(-1)^{m+1} = A_m \sin\phi_m (\frac{L}{2})}$$

$$w_t(x,0)=0 \Rightarrow$$

$$0=\sum_{n=1}^{\infty} A_n[-0.01\omega_n \sin(\phi_n)+\omega_{dn}\cos(\phi_n)]\sin\sigma_n x$$

$$\Rightarrow 0=A_m[-0.01\omega_m \sin(\phi_m)+\omega_{dm}\cos(\phi_m)]\frac{L}{2}$$

$$\Rightarrow \tan\phi_m=\frac{\sqrt{1-(0.01)^2}}{0.01}=99.9949, m=1,2,3\ldots \Rightarrow \phi_m \approx \frac{\pi}{2} \qquad \forall m$$

$$\Rightarrow A_m=\frac{0.02}{L^2\sigma_m^2}(-1)^{m+1}, m=1,2,3\ldots\ldots$$

$$\Rightarrow w(x,t)=\sum_{n=1}^{\infty}\left(\frac{0.02}{L^2\sigma_n^2}(-1)^{n+1}\right)e^{-0.01\omega_n t}\cos\omega_{dn}t\sin\sigma_n x$$

$$\sigma_n=(2n-1)\pi/2L, \omega_n=\sigma_n\sqrt{E/\rho}, \omega_{dn}=0.9999\omega_n$$

Physical models of damping

- Air damping
- Material damping
- Boundary damping
- Strain rate damping

Linear viscous approximation: $\gamma w_t(x,t)$

$$\rho w_{tt}(x,t) + \gamma w_t(x,t) - \tau w_{xx}(x,t) = 0$$

Clamped-clamped boundary conditions

$$w(0,t) = w(0,\ell) = 0$$

Example: **solve the damped fixed string using modal analysis**

$$w(x,t) = X(x)T(t) \Rightarrow$$

$$\frac{\rho\ddot{T}(t)+\gamma\dot{T}(t)}{\tau T(t)} = \frac{X''(x)}{X(x)} = -\sigma^2$$

$$X'' + \sigma^2 X = 0 \text{ plus bc } \Rightarrow X_n = \sin\frac{n\pi x}{\ell}, \sigma_n = \frac{n\pi}{\ell}$$

$$\Rightarrow \ddot{T}_n(t) + \underbrace{\frac{\gamma}{\rho}}_{2\zeta_n\omega_n} \dot{T}_n(t) + \frac{\tau}{\rho}\left(\frac{n\pi}{\ell}\right)^2 T_n(t) = 0$$

$$\Rightarrow \zeta_n = \frac{1}{2\omega_n}\frac{\gamma}{\rho} = \frac{\gamma\ell}{2n\pi\sqrt{\tau\rho}}$$

$$\Rightarrow T_n(t) = A_n e^{-\zeta_n \omega_n t} \sin(\omega_{dn} t + \phi_n)$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} A_n e^{-\zeta_n \omega_n t} \sin(\omega_{dn} t + \phi_n) \sin \frac{n\pi x}{\ell}$$

Where the remaining constants are determined by the initial conditions

Note that as ω_n increases with n, the series dies out faster and fewer modes need to be kept to fully represent the response

Viscous damping can be used to model a plate with energy dissipation.

$$\rho w_{tt}(x,y,t)+\gamma w_t(x,y,t)+D_E\nabla^4 w(x,y,t)=0,\quad x,y,\in\Omega$$

Damping for the longitudinal vibration of a bar

$$\rho w_{tt}(x,y,t)+2[\gamma-\beta\frac{\partial^2}{\partial x^2}w_t(x,t)]-\frac{EA}{\rho}w_{xx}(x,t)=0,\quad x,y,\in\Omega$$

A general model for damped layered systems would be of the form

$$L_0 w_{tt}(\mathbf{x},t)+L_1 w_t(\mathbf{x},t)+L_2 w(\mathbf{x},t)=0,\qquad \mathbf{x}\in\Omega$$

$$Bw(\mathbf{x},t)=0,\qquad \mathbf{x}\in\partial\Omega$$

Modal Analysis and the response

Also called “eigenfunction expansion” or “modal analysis”

Remove the constant multiplier of the eigenfunctions by normalizing such that:

$$\int_{\Omega} \phi_n(\mathbf{x})\phi_m(\mathbf{x})\,d\Omega = \delta_{nm}$$

Then

$$w(\mathbf{x},t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(\mathbf{x})$$

Substitute the expansion into the equation of motion, multiply by an eigenfuctution and integrate.

$$\int_{\Omega}\phi_n(\mathbf{x})L\phi_n(\mathbf{x})d\Omega=\lambda_n\int_{\Omega}\phi_n(\mathbf{x})\phi_n(\mathbf{x})d\Omega=\lambda_n$$

Then the equation of motion becomes the modal equation:

$$\ddot{a}_n(t)+\lambda_n a_n(t)=0, \qquad\qquad n=1,2,3\cdots\infty$$

An initial value problem for the temporal functions

Finding the time equation by mode eigenfunction expansion

$u_{tt}(x,t) + Lu(x,t) = 0,$ subject to boundary conditions.

Solve $L\phi_n = \lambda_n \phi_n,$ where $\int_0^\ell \phi_n \phi_m dx = \delta_{nm}$

$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$ and substitute into eom

$\sum_{n=1}^{\infty} \ddot{a}_n(t)\phi_n(x) + \sum_{n=1}^{\infty} a_n(t) L\phi_n(x) = 0.$ Next multiply by ϕ_m and integrate

$\sum_{n=1}^{\infty} \ddot{a}_n(t)\int_0^\ell \phi_n\phi_m dx + \sum_{n=1}^{\infty} a_n(t)\lambda_n \int_0^\ell \phi_n\phi_m dx = 0,$ where we used $L\phi_n = \lambda_n\phi_n$

$$\Rightarrow \ddot{a}_n(t) + \lambda_n a_n(t) = 0, \qquad n = 1, 2, 3 \cdots \infty$$

Modal Analysis in Damped Systems

$$w_{tt}(\mathbf{x},t) + L_1 w_t(\mathbf{x},t) + L_2 w(\mathbf{x},t) = 0, \quad \mathbf{x} \in \Omega$$

Let the operators L_1 and L_2 have the same eigenfunctions with eigenvalues $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ respectively, then

$$\sum_{n=1}^{\infty} [\ddot{a}_n(t)\phi_n(\mathbf{x}) + \lambda_n^{(1)}\dot{a}_n(t)\phi_n(\mathbf{x}) + \lambda_n^{(2)}a_n(t)\phi_n(\mathbf{x})] = 0$$

Multiply by $\phi_m(\mathbf{x})$ and integrate to get:

$$\ddot{a}_m(t) + \lambda_m^{(1)}\dot{a}_m(t) + \lambda_m^{(2)}a_m(t) = 0, \quad m = 1, 2, 3 \cdots \infty$$

Example A membrane in air

$$L_1 = 2\frac{\gamma}{\rho}, \quad L_2 = -\frac{T}{\rho}\nabla^2$$

If $l_1^{(2)} > g$ then:

$$a_n(t) = e^{-\gamma t}[A_n \sin(\sqrt{\lambda_n^{(2)} - \left(\frac{\gamma}{\rho}\right)^2})t + B_n \cos(\sqrt{\lambda_n^{(2)} - \left(\frac{\gamma}{\rho}\right)^2})t]$$

6.8 Modal analysis and the forced response

Consider a string with unit impulse applied and viscous damping present

$$\rho w_{tt}(x,t) + \gamma w_t(x,t) - \tau w_{xx}(x,t) = f(x,t)$$

$$f(x,t) = \delta(x - \frac{\ell}{4})\delta(t)$$

Boundary conditions are $w(0,t) = w(\ell,t) = 0$

$\Rightarrow X_n(x) = a_n \sin\dfrac{n\pi x}{\ell}$. Let $w_n(x,t) = a_n \sin\dfrac{n\pi x}{\ell} T_n(t)$

substitute into (1) $\Rightarrow$

$$\left\{\rho \ddot{T}_n + \gamma \dot{T}_n - \tau\left[-\left(\frac{n\pi}{\ell}\right)^2\right]T_n\right\}\sin\frac{n\pi x}{\ell} = \delta(x - \frac{\ell}{4})\delta(t)$$

Multiply by $X_m(x)$ and integrate to get

$$\left\{\rho\ddot{T}_n+\gamma\dot{T}_n-\tau\left[-\left(\frac{n\pi}{\ell}\right)^2\right]T_n\right\}\frac{\ell}{2}=\delta(t)\int_0^{\ell}\delta(x-\frac{\ell}{4})\sin\frac{n\pi x}{\ell}\,dx$$

$$=\delta(t)\sin\frac{n\pi}{4}$$

$$\ddot{T}_n(t)+\frac{\gamma}{\rho}\dot{T}_n(t)+\left(\frac{cn\pi}{\ell}\right)^2 T_n(t)=\left(\frac{2}{\ell\rho}\sin\frac{n\pi}{4}\right)\delta(t)$$

n=1,2,3,...

Using the impulse response function of equations (3.7) and (3.8) yields:

$$T_n(t) = \frac{4\sin(n\pi/4)}{\sqrt{(2\rho c n\pi)^2 - (\gamma\ell)^2}} e^{-\gamma t/2\rho} \sin\left[\frac{\sqrt{(2\rho c n\pi)^2 - (\gamma\ell)^2}}{2\rho\ell} t\right]$$

$$\Rightarrow$$

$$w(x,t) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/4)}{\sqrt{(2\rho c n\pi)^2 - (\gamma\ell)^2}} e^{-\gamma t/2\rho} \sin\left[\frac{\sqrt{(2\rho c n\pi)^2 - (\gamma\ell)^2}}{2\rho\ell} t\right] \sin\frac{n\pi x}{\ell}$$

General forced response (Sec. 3.2)

$$\ddot{T}_n(t) + 2\zeta_n \omega_n \dot{T}_n(t) + \omega_n^2 T_n(t) = f_n(t) \Rightarrow$$

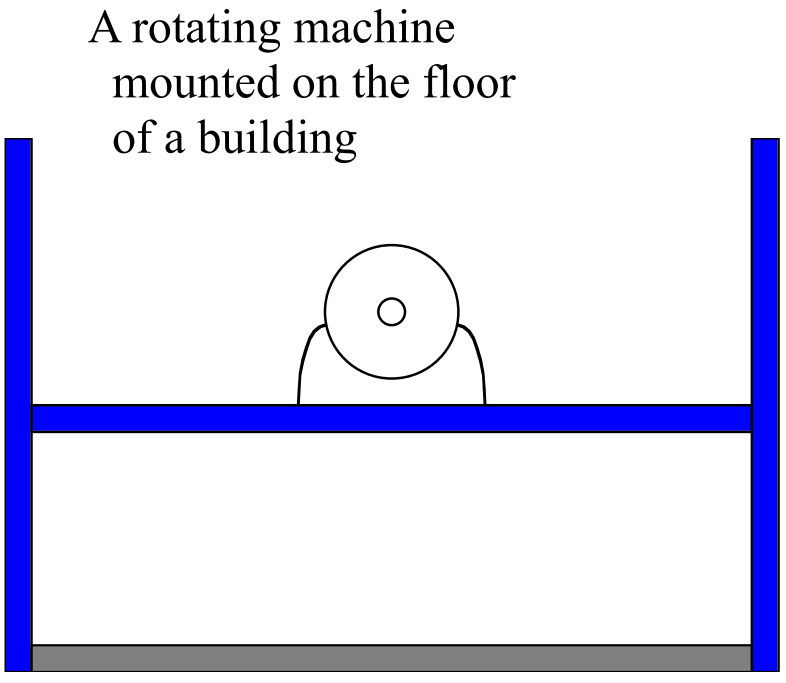
$$T_n(t) = A_n e^{-\zeta_n \omega_n t} \sin(\omega_{dn} t + \phi_n)$$

$$+ \frac{e^{-\zeta_n \omega_n t}}{\omega_{dn}} \int_0^t f_n(\tau) e^{\zeta_n \omega_n \tau} \sin \omega_{dn}(t - \tau) d\tau$$

Then combine with the spatial mode shapes to form the series solution (a summation of modes).

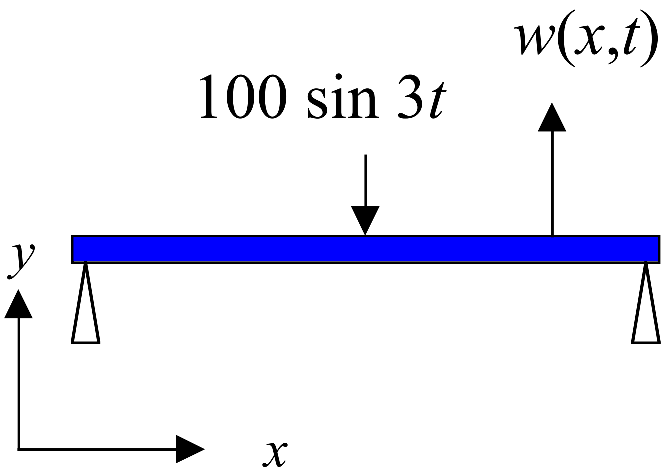
The idea of using modes is to take us back to sdof methods.

Example 6.8.2



- The machine exerts a harmonic load on the floor of 100 N amplitude at 3 rad/s
- Model the floor as a simply supported EB beam and compute the forced response

The mathematical model becomes



- $w_{tt}+c^2 w_{xxxx} =(100/\rho A)\sin 3t\,\delta(x-l/2)$
- $w(0,t)=w(l,t)=0,$ $w_{xx}(0,t)=w_{xx}(l,t)=0$
- $c^2 =EI/\rho A$
- From the spatial eigenvalue problem:
- $X_n(x)=A_n \sin(n\pi x/l)$

Recall from before that:

$$\omega_n = \sqrt{\frac{EI}{\rho A}}\left(\frac{n\pi}{\ell}\right)^2 \qquad n = 1, 2, 3\ldots$$

$$\text{Normalize } X_n(x) = \sin\frac{n\pi}{\ell}x \Rightarrow$$

$$\int_0^\ell X_n(x)X_n(x)dx = A_n^2 \underbrace{\int_0^\ell \sin\frac{n\pi}{\ell}x\sin\frac{n\pi}{\ell}xdx}_{\ell/2} = 1$$

$$\Rightarrow A_n = \sqrt{\frac{2}{\ell}}, \text{ and } X_n(x) = \sqrt{\frac{2}{\ell}}\sin\frac{n\pi}{\ell}x, n = 1, 2, 3\ldots.$$

$$\text{and } \int_o^\ell X_m(x)X_n(x)dx = 0, n \neq m$$

An orthonormal set

$$\ddot{T}_n(t)X_n(x) + c^2 T_n(t)X_n''''(x) = (100\sin 3t)\delta(x - \frac{\ell}{2})$$

But $X_n''''(x) = (\omega^2 / c^2)X_n(x) \Rightarrow$

$$[\ddot{T}_n(t) + \omega_n^2 T_n(t)]X_n(x) = (100\sin 3t)\delta(x - \frac{\ell}{2})$$

$$\ddot{T}_n(t) + \omega_n^2 T_n(t) = (100\sin 3t)\sqrt{\frac{2}{\ell}}\int_0^{\ell}\delta(x - \frac{\ell}{2})\sin\frac{n\pi x}{\ell}\,dx$$

$$\ddot{T}_n(t) + \omega_n^2 T_n(t) = \sqrt{\frac{2}{\ell}}(100\sin 3t)\sin\frac{n\pi}{2}, n = 1, 2, 3...$$

$\Rightarrow$

$$\ddot{T}_n(t) + \frac{EI}{\rho A}\left(\frac{n\pi}{\ell}\right)^4 T_n(t) = 0, \qquad n = 2,4,6,\ldots$$

$$\ddot{T}_n(t) + \frac{EI}{\rho A}\left(\frac{n\pi}{\ell}\right)^4 T_n(t) = 100\sqrt{\frac{2}{\ell}}\sin 3t, \quad n = 1,5,9,\ldots$$

$$\ddot{T}_n(t) + \frac{EI}{\rho A}\left(\frac{n\pi}{\ell}\right)^4 T_n(t) = -100\sqrt{\frac{2}{\ell}}\sin 3t, \quad n = 3,7,11,\ldots$$

$\Rightarrow$

$$T_n(t) = \frac{100\sqrt{2/\ell}}{(EI/\rho A)(n\pi/\ell)^4 - 9}\sin 3t, \quad n = 1,5,9,\ldots$$

$$T_n(t) = \frac{-100\sqrt{2/\ell}}{(EI/\rho A)(n\pi/\ell)^4 - 9}\sin 3t, \quad n = 3,7,11,\ldots$$

The total response is given by:

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) \Rightarrow$$

$$w(x,t) = \frac{200}{\ell}\left[\frac{\sin \pi x/\ell}{(\pi^4 EI/\ell^4 \rho A) - 9} - \frac{\sin 3\pi x/\ell}{(81\pi^4 EI/\ell^4 \rho A) - 9} + \frac{\sin 5\pi x/\ell}{(625\,\pi^4 EI/\ell^4 \rho A) - 9} \ldots\ldots\right]\sin 3t$$

A hint on approximation