Part C. Fourier Analysis. PDEs. → Chap. 12.

↳ allows us to model periodic phenomena. which appear frequently in enginee

The idea is to represent complicated functions in terms of simple periodic functio

what is that Cosine, sine

Chap 11. Fourier Analysis.

① 1st part deals with $\boxed{\text{Fourier series}}$ → 다 like Taylor series. $\sum_{n=0}^{\infty}(a_n \sin nx + b_n \cos nx)$ ...

infinite series designed to represent general periodic functions like sine, Cosine

Characteristic is "orthogonality."

② 2nd part. orthogonality b/w sine cosine function can be extended to general orthogonal function (like P_n, J_n)

what is that?

③ 3rd part applying Fourier series to non-periodic ~~function~~ phenomena, leading to obtain (Fourier integral & Fourier Transform.)

which is useful to solve PDEs

11.1. Fourier Series.

$$f(x+p) = f(x) \quad \text{for all } x.$$

$\sin x$	$\cos x$	→	2π
$\sin 2x$	$\cos 2x$	→	π. 2π
...			
$\sin nx$	$\cos nx$	→	$\frac{2\pi}{n}$, ... 2π

Then p is called period.

$f(x)$ is a periodic function.

The smallest positive period → fundamental period.

If $f(x)$ has period of P,

$$f(x+np) = f(x)$$

it also has the period $2P$

because $f(x+2p) = f(x+p) = f(x)$

Suppose that $f(x)$ is a given function of period 2π. (1)(sinx cosx) (sin2x cos2x) ...

Then $f(x)$ can be represented by a trigonometric series.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

→ Fourier series.

What is unknowns in the series. a_0, a_n, b_n.

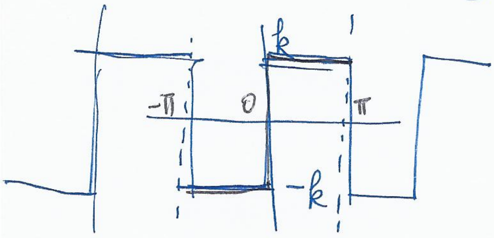
Euler Formula gives,

Before derivation,

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx & n=1,2\ldots \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx & n=1,2\ldots \end{cases}$$

For example,

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases} \qquad f(x+2\pi) = f(x)$$



→ odd function! (key information)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_0^{\pi} k\,dx\right] \to 0$$

why? (Think about area)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underline{f(x)}\,\underline{\cos nx}\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} k\cos nx\,dx$$

(odd · even) = (odd) = 0

$$= \frac{1}{\pi}\left[-k\frac{1}{n}\sin nx\Big|_{-\pi}^{0} + \frac{1}{\pi}\,k\frac{1}{n}\sin nx\Big|_0^{\pi}\right] = 0$$

or

$\because \sin n\pi$ (n = integer) $= 0$.

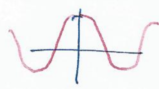
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underline{f(x)}\,\underline{\sin nx}\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -k\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} k\sin nx\,dx$$

(odd · odd)

even.

$$= \frac{1}{\pi}\left[\frac{-k}{n}[-\cos nx]\Big|_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{k}{n}\cdot -\cos nx\Big|_0^{\pi}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n} - \frac{k}{n}\cos n\pi + \left(-\frac{k}{n}\cos n\pi + \frac{k}{n}\right)\right]$$



$$= \frac{2k}{n\pi}(1-\cos n\pi)$$

$$\cos n\pi = \begin{cases} -1 & \text{odd} \\ 1 & \text{even} \end{cases}$$

$$1-\cos n\pi : \begin{cases} 0 & \text{even} \\ \boxed{2 \quad \text{odd}} \end{cases}$$

$\therefore$ = only odd term survives.

$b_1 = \frac{4k}{\pi}$ $\quad b_2 = 0$ $\quad b_3 = \frac{4k}{3\pi}$ $\quad b_4 = 0$ $\quad b_5 = \frac{4k}{5\pi}$ $\ldots$

$$\therefore f(x) = \underbrace{\frac{4k}{\pi}\left(\sin x\right.}_{S_1} + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \cdots \left.\right)$$

Fig 261.

S_1

S_2

S_3 . .

at $x = \frac{\pi}{2}$

$$k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

about

Converges to the original f(x) keeping ...

Derivation of Euler Formula, Before doing it,

$\boxed{\text{Trigonometric system is orthogonal}}$ on the interval $-\pi \le x \le \pi$

in other words,

① $\int_{-\pi}^{\pi} \cos Mx \, dx = 0$

② $\int_{-\pi}^{\pi} \sin Nx \, dx = 0$

for M, N, integer

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \ne m)$$

Evaluation test.

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \ne m)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad \left(\begin{matrix} n \ne m \\ n = m \end{matrix}\right)$$

if $m = n$ $\quad \int (\cos mx)^2 dx$ $\quad \frac{1 + \cos 2mx}{2}$

proof.)

$$\cos nx \cos mx = \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x$$

$$\sin nx \sin mx = \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x$$

$$\sin nx \cos mx = \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x$$

$\boxed{\text{Except } m = n}$

Anyhow,

$$\int \cos(Mx)dx \quad \text{or} \quad \int \sin Nx \, dx = 0.$$

application of Theorem 1 to the Fourier series

① integration on both sides from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

Termwise integration.

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-\pi}^{\pi} \cos nx \, dx}_{0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin nx \, dx}_{0}\right]$$

$$= 2\pi a_0$$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

multiplying both sides by $\cos mx$ → constant m, fixed integer m.

$\cos 2x = \cos^2 x - \sin^2 x$

$2\cos^2 x - 1$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx\right]$$

0

0 for any m, n.

if $n \neq m$, $\int dx = 0$

only when $n = m$.

$a_m \pi$,

$$\therefore\ a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

Likewise,

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = a_0 \int_{-\pi}^{\pi} \sin mx\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx \sin mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx\,dx\right]$$

$b_m \pi$.

$$\therefore\ b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

Next class

① period 2π to ~~Arbitrary~~ period $2L$

expansion ② $\boxed{\text{simplification}}$ for odd $f(x)$ and even $f(x)$ function

③ Half-range expansions.

11.2 Arbitrary Period. Even & Odd functions. Half-range expansions

$f(x+2\pi) = f(x)$

Reminder : $f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

① Expansion from period 2π to $2L$ any period

Let $f(x)$ have period $p = 2L$,

introduction of new variable v, such that $f(x)$, as a function v, has period 2π.

$-L < x < L \longrightarrow -\pi < v < \pi$

(a) $x = \frac{p}{2\pi} v$, so that $v = \frac{2\pi}{p} x = \frac{\pi}{L} x$

$v = \pm\pi \longrightarrow x = \pm L$

$\because x = \frac{L}{\pi} v$

$f(x)$

$f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$

$\downarrow$

$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) dv \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos nv\, dv$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \sin nv\, dv$

$v = \frac{\pi}{L} x \qquad dv = \frac{\pi}{L} dx \qquad -\pi < dv < \pi \Rightarrow -L < dx < L$

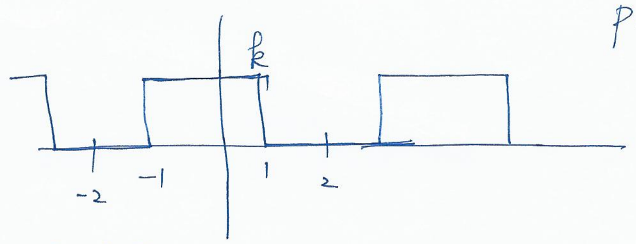
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx$

Ex 1. $f(x)$



$p = 4 \Rightarrow L = 2.$

Find the Fourier series.

$$a_0 = \frac{1}{2\cdot 2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2\cdot 2}\int_{-1}^{1} k\,dx = \frac{k}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2k}{n\pi}\cdot \sin\frac{n\pi x}{2}\Big|_0^1 = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$n =$ even $\quad a_n = 0$

$n =$ odd. $\quad 1, 5 \cdots (1) \quad \frac{2k}{n\pi}$

$\quad 3, 7 \cdots (-1) \longrightarrow -\frac{2k}{n\pi}$

$$\therefore f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right)$$

$b_n = 0$ since $f(x)\sin\frac{n\pi}{2}x$ is an odd function

$\rightarrow$ Fourier Cosine series ($\because$ even $f(x)$)

② simplification

a) $f(x)$ even function $\quad f(-x) = f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$

$b_n = 0 \quad \underline{\int f(x)\sin\frac{n\pi}{L}x\,dx}$

$\quad \hookrightarrow$ odd function!

go back, & check,

"Directly we can see that it is Fourier Cosine series since $f(x)$ is an even function."

b) $f(x)$ odd function $\quad f(-x) = -f(x)$

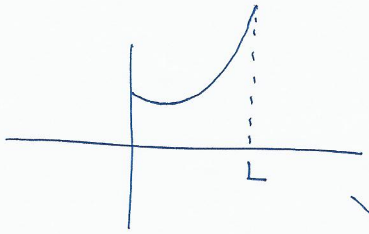
$a_0,\ a_n = 0 \quad \int_{-L}^{L}$ odd function $\cdot\,dx = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

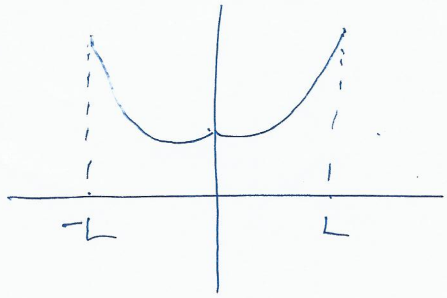
odd · odd = even.

③ Half-range expansion.

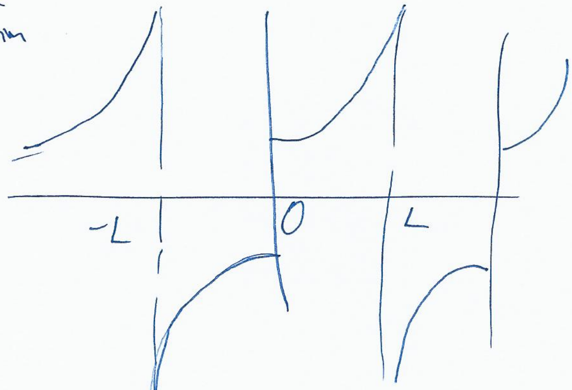
given func. f(x)



even periodic function

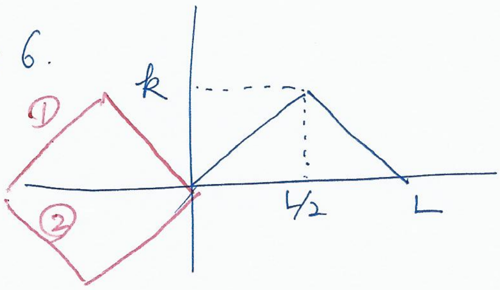


odd periodic function



both extensions have period 2L

Ex 6.



$$f(x) = \begin{cases} \frac{2k}{L}x & 0<x<\frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2}<x<L. \end{cases}$$

① even periodic function

Think about area.

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\cdot\frac{1}{2}\cdot k\cdot L = \frac{k}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = \frac{2}{L}\left[\underbrace{\int_0^{L/2}\frac{2k}{L}x\cos\frac{n\pi}{L}x\,dx}_{①} + \int_{L/2}^{L}\frac{2k}{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right]$$

i)

$$\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx = \frac{2k}{L}\left[x\frac{L}{n\pi}\sin\frac{n\pi}{L}x\Big|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi}{L}x\,dx\right]$$

$$\frac{L}{n\pi}\left(-\cos\frac{n\pi}{L}x\right)\Big|_0^{L/2}$$

$$= \frac{2k}{L}\left[\cancel{\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\right]$$

②

$$\frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx = \frac{2k}{L}\left[\frac{L}{n\pi}\sin\frac{n\pi}{L}x\cdot(L-x)\Big|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{2k}{L}\left[-\frac{L}{n\pi}\sin\frac{n\pi}{2}\cdot\frac{L}{2}\right] + \frac{L}{n\pi}\left(\frac{L}{n\pi}\right)\left(-\cos\frac{n\pi}{L}x\right)\Big|_{L/2}^{L}$$

$$= \frac{2k}{L}\left[-\cancel{\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] \quad 2kL,\ -\cos n\pi \quad + \frac{L^2}{n^2\pi^2}\left(-\cos n\pi + \cos\frac{n\pi}{2}\right)$$

$$\therefore\ a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

a_1 $\quad (0 - (-1) - 1) = 0$

$a_2 :$ $\quad 2(-1) - 1 - 1 = -4$

$a_3 :$ $\quad (0 - (-1) - 1) = 0$

$a_4 :$ $\quad 2 - 1 - 1 = 0$

$\vdots$

only $a_2, a_6, a_{10} \ldots$ survives.

otherwise $a_n = 0$

$$\therefore\ f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{\cancel{L}}x + \frac{1}{6^2}\cos\frac{6\pi}{\cancel{L}}x + \cdots\right)$$

even periodic function

or Fourier cosine series.

b) odd periodic ~~func~~ expansion.

$a_0, a_n = 0$

only $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x \, dx$

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L} - \frac{1}{3^2}\sin\frac{3\pi}{L} + \frac{1}{5^2}\sin\frac{5\pi}{L} - + \cdots\right)$$

odd periodic function

Theorem 1. Sum & scalar multiple.

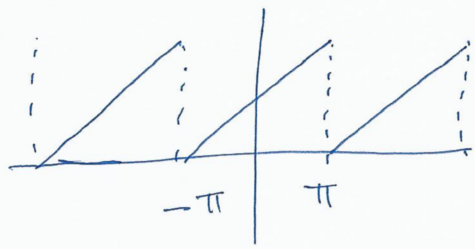
Fourier coefficient of a sum f_1 and f_2

= Sum of the corresponding Fourier coefficient f_1 and f_2.

Fourier coefficien of cf_1 = $c \times$ Corresponding Fourier coefficient.

Too easy proof!

Example $f(x) = x + \pi \quad -\pi < x < \pi \quad f(x+2\pi) = f(x)$



$f_1 = x \qquad f_2 = \pi \rightarrow \pi$

odd function Fourier sine series, or b_n only survives

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n}\Big|_0^{\pi} + \frac{1}{n}\cancel{\int_0^{\pi} \cos nx \, dx} \right] \quad 0$$

$$= 2\left(-\frac{1}{n}\cos n\pi\right)$$

	$\cos n\pi$
$n=1$	-1
$n=2$	1
$n=3$	-1

$-\frac{(-1)^n \cdot 2}{n}$ b_n. $\quad \sum b_n \sin nx$

$\therefore f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right)$

11.3 Forced oscillations. (말로 설명하고 skip 가능)

$$my'' + cy' + ky = r(t)$$

damping constant.
spring constant

what ~~is~~ do m, c, k stand for ?

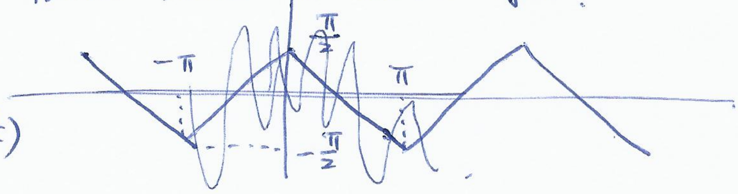
How about $r(t)$?

external force.

$r(t)$: sine or cosine function, or e^t, polyonomial in Eng. Math 1.

If $r(t)$ is not a pure sine or cosine function, but periodic function.
we can use Fourier series to solve $y(t)$.

Ex)



$$y'' + 0.05y' + 25y = r(t)$$

$$\begin{cases} t + \frac{\pi}{2} & -\pi < t < 0 \\ -t + \frac{\pi}{2} & 0 < t < \pi \end{cases}$$

$y = y_h + y_p$ (as $t \to \infty$ steadystate solution y_p)

$y_h \Rightarrow$ $\lambda^2 + 0.05\lambda + 25 = 0$. $\lambda = \dfrac{-0.05 \pm \sqrt{0.05^2 - 100}}{2} \simeq \dfrac{-0.05 \pm 10i}{2} = -0.025 \pm 5i$

y_p ? $r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$

[Steadystate solution.] → Fourier series (especially cosine series)

$\therefore$ $y'' + 0.05y' + 25y = \dfrac{4}{n^2\pi}\cos nt$ ($n = 1, 3, 5 \ldots$)
for any n.
$\because \cos nt$

y_p ?? y_p : $y_n = A_n \cos nt + B_n \sin nt$ ⟸ undetermined coefficient meth

enter y_n into original equation.

Then $A_n = \dfrac{4(25 - n^2)}{n^2\pi D_n}$ $B_n = \dfrac{0.2}{n\pi D_n}$ $D_n = (25 - n^2)^2 + (0.0$

linear. $\therefore$ y_p : $y = y_1 + y_3 + y_5 + \cdots$

steadystate solution

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}}$$

what is dominant term ?

when C_n is maximized ? D_n is minimized. esp. $n = 5$.

$C_1 = 0.0531$ $C_3 = 0.0088$ $C_5 = 0.2037$ $C_7 = 0.0011$

$n = 5$. is most important. ⟹ output is harmonic oscillation of 5 times

11.4. Approximation by Trigonometric Polynomials.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

f(x) is a periodic function on the interval $-\pi \le x \le \pi$.

How about the Nth partial sum. of the Fourier series.

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \quad (2N+1)$$

: Can be regarded as "approximation" of f(x).

Whether (1) is the "best" approximation of f by a trigonometric polynomial of the same degree N, "error" approximation is as small as possible.

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \quad (2N+1)$$

$|f-F|$

$$E = \int_{-\pi}^{\pi} (f-F)^2 dx \rightarrow \text{square error}$$

need to find coefficient A_0, A_n, B_n. such that E is min.

$$E = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx \quad ①$$

$$+ \int_{-\pi}^{\pi} \Big[A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \Big]^2 dx$$

②

only $(m=n)$ survive.

$\rightarrow \pi (2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots B_N b_N)$

again

$\int_{-\pi}^{\pi} (\cos nx \cdot \sin nx) = 0.$

$(\cos nx \cdot \cos mx)$

$\int_{-\pi}^{\pi} (\cos^2 nx)\, dx = \pi.$

$\rightarrow \pi [2A_0^2 + A_1^2 + \cdots A_N^2 + B_1^2 + B_2^2 + \cdots + B_N^2]$

$$\therefore E = \int_{-\pi}^{\pi} f^2 dx - 2\pi \Big[2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n)\Big] + \pi \Big[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2)\Big]$$

We take $A_n = a_n$ and $B_n = b_n$ (coefficient of Fourier series)

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \Big[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)\Big]$$

Cannot be negative

$$E - E^* = \pi \Big\{ 2(A_0 - a_0)^2 + \sum_{n=1}^{N} \big[(A_n - a_n)^2 + (B_n - b_n)^2\big] \Big\} \ge 0.$$

$E = E^* \iff A_0 = a_0 \cdots \quad B_N = b_n.$

Simply speaking, Partial sum of Fourier series (with Fourier coefficient)
minimizes the square error.
or best "approximation" of ~~the~~ $f(x)$

and as N increase, better approximation to f,
E^* equation을 보면..

$\boxed{\text{Bessel's inequality}}$

since $E^* \geq 0$ ($\because$ square error)

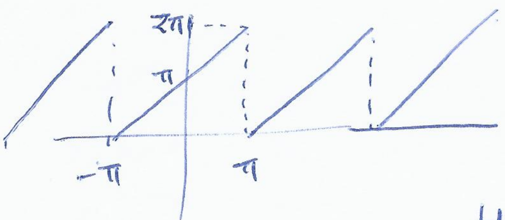
becomes $\underset{\text{equal}}{=}$ as $n \to \infty$

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx.$$

$=$: parseral's identity.

as N increases.

Ex 1) E^* for sawtooth wave.



$f(x) = x + \pi \quad -\pi \leq x \leq \pi$

already solve solve

How to obtain the Fourier series?

divide into

$f_1 + f_2$

$$F(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx\right)$$

$\therefore \quad a_N = 0. \qquad b_N = \dfrac{2(-1)^{N+1}}{N}$

$a_0 = \pi$

$$E^* = \int_{-\pi}^{\pi}(x+\pi)^2\,dx - \pi\left[2\pi^2 + 4\sum_{n=1}^{N}\frac{1}{n^2}\right]$$

$N = 10 \longrightarrow 1.1959$

$100 \qquad 0.1250$

11.5 Sturm - Liouville Problems. Orthogonal Functions.

Fourier series : represent general periodic function in terms of cosine & sine

orthogonal ⟵ trigonometric system

Generalization.

replace the trigonometric system by other orthogonal system.

{ i.e. Fourier-Legendre series ? Fourier-Bessel series. }

To prepare for generalization, introduce the S-L. problem.

consider a 2nd-order ODE of the form.

(1) $[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \qquad a \leq x \leq b$

S-L equation

$$\begin{cases} k_1 y + k_2 y' = 0 & \text{at } x = a \\ l_1 y + l_2 y' = 0 & \text{at } x = b. \end{cases}$$

S-L problem.

where λ : a parameter

$k_1\ k_2\ l_1\ l_2$: real constants.

⟹ goal is to solve these types of problems.

i) $y \equiv 0$. solution, but trivial solution. not our interest.

we ~~code~~ want to find eigenfunction $y(x)$ with corresponding λ ⟶ eigen value.

Easy Example. $y'' + \lambda y = 0 \qquad y(0) = 0, \quad y(\pi) = 0.$

$[p(x)y']' + [\lambda r(x)]y = 0$

$$\begin{cases} y(0) + 0 \cdot y'(0) = 0 \\ y(\pi) + 0 \cdot y'(\pi) = 0. \end{cases}$$

where $p(x) = 1$, $r(x) = 1$, $q(x) = 0$.

i) $\lambda = -\nu^2 \qquad y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \qquad c_1 + c_2 = 0. \quad c_1 e^{\nu\pi} + c_2 (e^{-\nu\pi}) = 0$

trivial solution $c_1 = c_2 = 0$.

ii) $\lambda = 0$

iii) $\lambda = \nu^2 \qquad y(x) = A\cos\nu x + B\sin\nu x \qquad A = 0. \quad y(\pi) = B\sin\nu\pi = 0$

$\nu = \cancel{0}, \pm 1, \pm 2 \ldots$

Orthogonal function

all eigenvalues of the S-L. problem are real.

A 77 (appendix 4)

Def.)

$$(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0 \quad (m \neq n)$$

orthogonal with respect to the weight function $r(x)$

norm $\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)\, y_m^2\, dx}$

if $r(x) = 1$. $\Rightarrow$ just orthogonal !!

orthonormal

$$(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Krönecker symbol.

Ex2) $y_m(x) = \sin mx$ $\quad m = 1, 2, \ldots$ $\Rightarrow$ find the orthonormal set

$$(y_m, y_n) = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx\, dx = 0 \quad (m \neq n)$$

$$\|y_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi. \qquad \text{norm} = \sqrt{\pi}$$

$\therefore$ orthonormal set. : $\dfrac{\sin x}{\sqrt{\pi}}, \dfrac{\sin 2x}{\sqrt{\pi}}, \dfrac{\sin 3x}{\sqrt{\pi}} \ldots$.

Theorem) orthogonality of eigenfunctions of S-L. problem.

Let (y_m, y_n) eigenfunctions with corresponding eigenvalues. λ_m, λ_n

Then $(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0$

$\hookrightarrow$ orthogonal

proof in the following.

Orthogonality of Eigenfunctions of S-L problem

$$(py_m')' + (q+\lambda_m r)y_m = 0 \quad \times\ y_n$$

$$+)\ (py_n')' + (q+\lambda_n r)y_n = 0 \quad \times(-y_m)$$

<q 소거>

$$(\lambda_m - \lambda_n) r y_m y_n = y_m (py_n')' - y_n (py_m')'$$

$$= +(y_m' \cdot py_n') + py_m' y_n' - py_m' y_n'$$

$$= (y_m \cdot py_n' - y_n py_m')'$$

integration from a to b.

$$(\lambda_m - \lambda_n)\int_a^b r y_m y_n \, dx = \int_a^b \left[p(y_m y_n' - y_n y_m')\right]' dx$$

need to prove that integration is zero.

$$= \left[p y_m y_n' - p y_n y_m'\right]_a^b$$

$$= p(b)\left[y_n'(b) y_m(b) - y_m'(b) y_n(b)\right] - p(a)\left[y_m(a) y_n'(a) - y_n(a) y_m'(a)\right]$$

① $p(a) = p(b) = 0$. right term 0. → no BC is needed.

② $p(a) \neq 0$, $p(b) = 0$. $-p(a)\left[y_m(a)y_n'(a) - y_n(a)y_m'(a)\right]$

at x=a

$$k_1 y_n(a) + k_2 y_n'(a) = 0 \quad \times\ y_m(a)$$

$$-)\ k_1 y_m(a) + k_2 y_m'(a) = 0 \quad \times\ y_n(a)$$

k_1 소거

$$k_2 (y_m(a) y_n'(a) - y_n(a) y_m'(a)) = 0$$

since $k_2 \neq 0$, $[\quad] = 0$.

또는 if $k_2 = 0$

$k_1 y_n(a) = 0$

$k_1 y_m(a) = 0$

$k_1 \neq 0$

$y_n(a) = 0$

$y_m(a) = 0$

→ okay, $= 0$

③ $p(a) = 0$, $p(b) \neq 0$ same

$$k_2 (y_m(b) y_n'(b) - y_n(b) y_m'(b)) = 0$$

Y_n J_n

④ $p(a) \neq 0$, $p(b) \neq 0$, BC ②, ③ 을 동시에 고려.

⑤ $p(a) = p(b)$. $p(b)\left[y_n'(b)y_m(b) - y_m'(b)y_n(b) - y_m(a)y_n'(a) + y_n(a)y_m'(a)\right]$

~~or~~ $y(a) = y(b)$, $y'(a) = y'(b)$ okay

periodic B.C. 만족하면

같은 BC 또는 or.

Ex 1. Fourier – LeGendre Series $\int_{-1}^{1} P_n P_m \, dx = 0$

We can use because of its orthogonality $\left(\begin{array}{l} -1 \le x \le 1 \\ r(x) = 1 \end{array} \right.$

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) \cdots$$

$$a_m \ ? \quad \frac{1}{\|P_m\|^2} \int_{-1}^{1} \cancel{r(x)} f(x) P_m(x) \, dx$$

"tricky" $\|P_m\|^2 = \sqrt{\int_{-1}^{1} P_m(x)^2 dx} = \sqrt{\frac{2}{2m+1}}$ $m = 0, 1, \ldots$

$$\therefore \quad a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx$$

let $f(x) = \sin \pi x$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} \sin \pi x \, P_m(x) \, dx$$

$a_1 \Rightarrow P_1(x) = x$

$P_2(x) = \left(\frac{3}{2}x^2 - \frac{1}{2} \right)$

$\Downarrow$

$a_2 = 0$ ($\because$ odd funct

only odd term survives

$$a_1 = \frac{3}{2} \int_{-1}^{1} x \sin \pi x \, dx = \frac{3}{\pi}$$

$$\sin \pi x = \sum_{n=1}^{\infty} a_n P_n(x)$$

$$= 0.95493 P_1(x) - 1.15824 P_3(x) + 0.21929 P_5(x) - + \ldots$$

Fourier – Bessel series

① is S-L problem ?

② orthogonal ?

③ Fourier – Bessel series. Coefficients.

For example. LeGendre Polynomials

$$(1-x^2)y'' - 2xy' + \overset{n(n+1)}{\lambda} y = 0. \qquad (-1 \le x \le 1)$$

$$((1-x^2)y')' + \lambda y = 0$$

$$\longrightarrow \quad \begin{aligned} p(x) &= 1 - x^2 \\ q(x) &= 0 \\ r(x) &= 1 \end{aligned}$$

Since $p(-1) = p(1) = 0$. We need no B.C.

P_n is a solution.

$$\Longrightarrow \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0. \qquad \text{orthogonal } P_n.$$

11.6 Orthogonal series. Generalized Fourier series.

Let y_0, y_1, y_2 be orthogonal with respect to $r(x)$ on $a \le x \le b$.

$f(x)$ can be represended by convergent series.

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$$

$\longrightarrow$ orthogonal series.
generalized Fourier Series.

Given $f(x)$, problem is a_m? use orthogonality

$\longrightarrow \quad \times\ r(x) y_n(x)$ (n= fixed). then $\int_a^b dx$.

$$(f, y_n) = \int_a^b r f y_n\, dx = \int_a^b r\left(\sum_{m=0}^{\infty} a_m y_m\right) y_n\, dx = \sum_{m=0}^{\infty} a_m \underbrace{\int_a^b r y_m y_n\, dx}_{(y_m,\, y_n)}$$

(m is changing)

orthogonality $\quad 0$ except $m = n$.

$$a_n (y_n, y_n) = a_n \|y_n\|^2 \qquad \therefore a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r f(x) y_m(x) dx$$

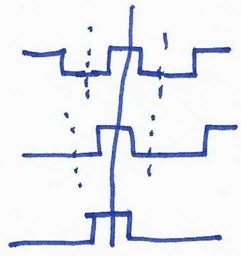
11.7 Fourier Integral

extension to non-periodic function.

Ex 1. Rectangular wave $f_L(x)$ with period $2L > 2$

$$f_L(x) = \begin{cases} 0 & -L < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < L \end{cases}$$

$2L = 4$
$= 8$
$= 16$
$\vdots$
∞



$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What happens to the Fourier coefficient of f_L as $L \uparrow$?

1st thing is to know even or odd.

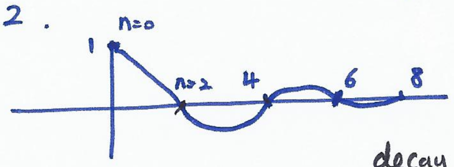
$\to$ so $b_n = 0$

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L} \qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi}{L}x \, dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi}{L}x \cdot dx = \frac{2}{L}\cdot\frac{\sin n\pi/L}{n\pi/L}$$

$$= \left(\frac{2}{L}\,\frac{\sin w_n}{w_n}\right)$$

How a_n behaves with $L = 2, 8$

$L = 2$.



$w_n = \frac{n\pi}{2}$

decay to 0.

$\therefore f(x) =$

$\frac{1}{L}$ a_0

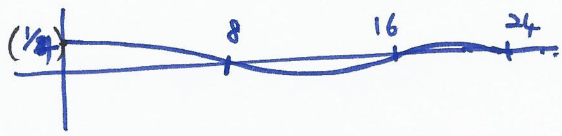
$+$

a_n.

$$\therefore f_L(x) = \frac{2}{L}\sum_{n=1}^{\infty} \frac{\sin w_n}{w_n} \cdot \cos w_n x$$

where $w_n = \frac{n\pi}{L}$.

$L = 8$



From Fourier series to Fourier integral.

Consider any periodic function $f_L(x)$ of period 2L.

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

What happens to $f_L(x)$ if we let $L \to \infty$

a_0, a_n, b_n from Euler formula.

$$f_L(x) = \frac{1}{2L}\int_{-1}^{1} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v \, dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\, dv\right]$$

$$\Delta W = W_{n+1} - W_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \qquad \frac{1}{L} = \frac{\Delta W}{\pi}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\boxed{\sum_{n=1}^{\infty}\Big[(\cos w_n x)\Delta W}\int_{-L}^{L} f_L(v)\cos w_n v\,dv$$

$$+ \boxed{(\sin w_n x)\Delta W}\int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big]$$

Now let $L \to \infty$, $\quad f(x) = \lim_{L\to\infty} f_L(x)$

$$\frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv = 0.$$

$\Sigma(\ )\Delta W$ becomes integral from 0 to ∞

$\underline{L \to \infty}$, $\underline{W_n \to W}$ $\quad \Delta W = dw$

$$\therefore\ f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos wx \underbrace{\int_{-\infty}^{\infty} f(v)\cos wv\,dv}_{A(w)} + \sin wx \underbrace{\int_{-\infty}^{\infty} f(v)\sin wv\,dv}_{B(w)}\Big]dw$$

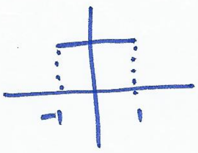
$$\therefore\ f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw$$

where $A(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\cos wv\,dv$

$B(w) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v)\sin wv\,dv$

Fourier integral !!

Application of Fourier integral



$$f(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

Find the Fourier integral representation

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv = \frac{1}{\pi}\int_{-1}^{1}\cos wv\,dv = \frac{\sin wv}{\pi w}\Big|_{-1}^{1} = \frac{2\sin w}{\pi w}$$

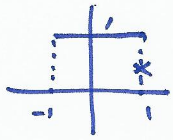
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv = 0.$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin w}{w}\cos wx\,dw$$

//

Further information

① $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin w \cos wx}{w}\, dw$



$$= \begin{cases} 1 & 0 \le x < 1 \\ \frac{1+0}{2} = \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases} \quad (\text{by Theorem})$$

$$\therefore \int_0^\infty \frac{\cos wx \sin w}{w}\, dw = \begin{cases} \pi/2 & 0 \le x < 1 \\ \pi/4 & x = 1 \\ 0 & x > 1 \end{cases}$$

Dirichlet's Discontinuous factor.

Especially, ② when $x = 0$. $\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}$.

$$Si(u) = \int_0^u \frac{\sin w}{w}\, dw : \text{sine integral}$$

as $u \to \infty$ $Si(u) = \frac{\pi}{2}$.

Fourier Cosine Integral and Fourier Sine Integral.

If f is even and has a Fourier integral representation.

Then $B(w) = 0$.

$\therefore f(x) = \int_0^\infty A(w) \cos wx\, dw$ (Fourier Cosine Integral)

where $A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv\, dv$

If f is odd

Then $A(w) = 0$.

(Fourier Sine Integral)

$\therefore f(x) = \int_0^\infty B(w) \sin wx\, dw$

where $B(w) = \frac{2}{\pi}\int_0^\infty f(v) \sin wv\, dv$

$\Rightarrow$ help in evaluating integrals.

For $f(x) = e^{-kx}$ $(k>0,\ x>0)$.

① Derive the Fourier cosine integral

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv \rightarrow \underbrace{\frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\,dv}$$

integral by parts.

$$-\frac{1}{k}e^{-kv}\cos wv\Big|_0^\infty + \underline{\frac{w}{k}\int_0^\infty e^{-kv}(-\sin wv)\,dv}$$

one more integration!

$$+\frac{w}{k}\left(-\frac{1}{k}e^{-kw}(-\sin wv)\Big|_0^\infty - \frac{w}{k}\int_0^\infty e^{-kv}\cos wv\,dv\right)$$

$\rightarrow 0$

$$= \left[+\frac{1}{k} - \frac{w^2}{k^2}\int_0^\infty e^{-kw}\cos wv\,dv\right]$$

$$A\left(1+\frac{w^2}{k^2}\right) = \frac{1}{k} \qquad A = \frac{k}{w^2+k^2}$$

$$\therefore\ A(w) = \frac{2k/\pi}{w^2+k^2}$$

$$\therefore\ f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{w^2+k^2}\,dw \Rightarrow \int_0^\infty \frac{\cos wx}{w^2+k^2}\,dw = \frac{\pi}{2k}e^{-kx}$$

② Fourier sine integral.

$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\,dv$$

integral by parts twice!

$$B(w) = \frac{2w/\pi}{w^2+k^2}$$

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}\,dw$$

$$\therefore\ \int_0^\infty \frac{w\sin wx}{k^2+w^2}\,dw = \frac{\pi}{2}e^{-kx}$$

so-called

Laplace integral!

11.8 Fourier Cosine & Sine Transforms

Laplace Transform.

$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$

$f(t) = \mathcal{L}^{-1}(F)$

Fourier Cosine Transform

$$f(x) = \int_0^\infty A(w) \cos wx \, dw$$

where $A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv$ → symmetric distribution of $\frac{2}{\pi}$, $\sqrt{\frac{2}{\pi}}$

we set $A(w) = \sqrt{\frac{2}{\pi}} \hat{f}_c(w)$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx \quad \text{or} \quad \mathcal{F}_c(f)$$

$\mathcal{F}_c^{-1} = f$

and $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

Fourier Sine Transform.

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \quad \text{where } B(w) = \frac{2}{\pi}\int_0^\infty f(v) \sin wv \, dv$$

$$B(w) = \sqrt{\frac{2}{\pi}} \hat{f}_s(w)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx \quad \text{or} \quad \mathcal{F}_s(f)$$

and $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw$$

Ex. 1 Fourier $\binom{\text{Cosine}}{\text{Sine}}$ Transform

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a. \end{cases}$$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos wx \, dx = \sqrt{\frac{2}{\pi}} \cdot k \left.\frac{\sin wx}{w}\right|_0^a = \sqrt{\frac{2}{\pi}}\, k \frac{\sin wa}{w}$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin wx \, dx = \sqrt{\frac{2}{\pi}}\, k \cdot \left.\frac{-\cos wx}{w}\right|_0^a = \sqrt{\frac{2}{\pi}}\, k \frac{1-\cos wa}{w}$$

integration by parts

Ex 2. $\mathcal{F}_c(e^{-x})$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx = \sqrt{\frac{2}{\pi}} \left.\frac{e^{-x}}{1+w^2}(-\cos wx + w \sin wx)\right|_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+w^2}\right)$$

Linearity
$$\left(\begin{aligned} \mathcal{F}_c(af+bg) &= a\mathcal{F}_c(f) + b\mathcal{F}_c(g) \\ \mathcal{F}_s(af+bg) &= a\mathcal{F}_s(f) + b\mathcal{F}_s(g) \end{aligned} \right).$$

by definition

linear operator!

linearity

$$\mathcal{F}_c(af+bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(f)$$
$$\mathcal{F}_s(af+bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(f)$$

linear operations

Theorem 1. Cosine and sine Transforms of derivatives let $(f(x) \to 0 \text{ as } x \to \infty)$

$$\mathcal{F}_c(f'(x)) = w\,\mathcal{F}_s(f(x)) - \sqrt{\tfrac{2}{\pi}}\,f(0)$$
$$\mathcal{F}_s(f'(x)) = -w\,\mathcal{F}_c(f(x))$$

proof)

$$F_c(f'(x)) = \sqrt{\tfrac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx = \sqrt{\tfrac{2}{\pi}}\left[\left[f(x)\cos wx\right]_0^\infty + w\int_0^\infty f(x)\sin wx\,dx\right]$$
$$= -\sqrt{\tfrac{2}{\pi}}f(0) + w\,\mathcal{F}_s(f(x))$$

$$F_s(f'(x)) = \sqrt{\tfrac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx = \sqrt{\tfrac{2}{\pi}}\left[f(x)\sin wx\right]_0^\infty - w\int_0^\infty f(x)\cos wx\cdot dx$$
$$= 0 - w\,\mathcal{F}_c(f(x))$$

$$\mathcal{F}_c\{f''(x)\} = w\,\underbrace{F_s(f'(x))}_{-wF_c(f(x))} - \sqrt{\tfrac{2}{\pi}}\,f'(0) = \cancel{-w^2 F_c}$$

$$\mathcal{F}_c(f''(x)) = -w^2\,\mathcal{F}_c(f(x)) - \sqrt{\tfrac{2}{\pi}}f'(0)$$
$$\mathcal{F}_s(f''(x)) = -w^2\,\mathcal{F}_s(f(x)) + \sqrt{\tfrac{2}{\pi}}\,wf(0)$$
$$\searrow -wF_c(f'(x)) = -w^2\mathcal{F}_s(f(x)) + \sqrt{\tfrac{2}{\pi}}\,wf(0)$$

after 2nd derivative.
($\mathcal{F}_c$ returns to $\mathcal{F}_c$
F_s 〃 F_s)

Ex3. ?? $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a>0$

$$(e^{-ax})'' = a^2 e^{-ax} \qquad f'(x) = -a e^{-ax}$$
$$a^2 f(x) = f''(x)$$

Cosine Transform

$$a^2\,\mathcal{F}_c(f) = \mathcal{F}_c(f'') \qquad f'(0) = -a.$$
$$= -w^2 F_c(f) - \sqrt{\tfrac{2}{\pi}}f'(0)$$
$$= -w^2 F_c(f) + a\sqrt{\tfrac{2}{\pi}}$$
$$\therefore F_c(a^2+w^2) = a\sqrt{2/\pi}$$
$$\therefore \mathcal{F}_c(e^{-ax}) = \sqrt{\tfrac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right)$$

11.9 Fourier Transform $\xrightarrow{\text{obtained from}}$ complex form of the Fourier integral

Fourier Sine/cosine Transform $\longrightarrow$ real transform

From Fourier integral

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\, dw$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$ and $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\,[\cos wv \cos wx + \sin wv \sin wx]\, dv\, dw$$

$f(v)$ $\downarrow$ not function of w

$\cos(wx - wv) = \cos w(x-v)$ $\downarrow$ only function of w,

even function of w $\{0 < w < \infty\}$

change to $\frac{1}{2}\{-\infty < w < \infty\}$ whole range.

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv\, dw \quad \text{—①}$$

Let's consider

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\, dv\, dw = 0 \quad \text{—②}$$

Since $\sin(wx - wv)$ is an odd function of w

Euler formula $e^{ix} = \cos x + i\sin x$

$$f(v)\, e^{i(wx - wv)} = f(v)\,[\cos(wx - wv) + i\sin(wx - wv)]$$

① + ② i

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{iw(x-v)}\, dv\, dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv\right]}_{\hat{f}(w)} e^{iwx}\, dw$$

$f(v)$

$$\therefore \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

Theorem 1.

If $f(x)$ is absolutely integrable, F.T. exists.

Ex 1) $f(x) = 1$ if $|x|<1$

0 otherwise

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-iw}\left[e^{-iw} - e^{iw}\right]$$

end ?

$e^{iw} = \cos w + i \sin w$

$e^{-iw} = \cos w - i \sin w$

$-2i \sin w$

$$= \frac{1}{-iw\sqrt{2\pi}}(-2i \sin w) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w},$$

Linearity $\mathcal{F}(af+bg) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af+bg) e^{-iwx} d\cancel{x}$

$$= a\mathcal{F}(f) + b\mathcal{F}(g)$$

easy !

since $f(\infty), f(-\infty) = 0$

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}}\left[f(x) e^{-iwx}\Big|_{-\infty}^{\infty} + iw\int_{-\infty} f(x) e^{-iwx} dx\right.$$

$$= iw\,\mathcal{F}(f)$$

Likewise $\mathcal{F}(f''(x)) = iw\,\mathcal{F}(f'(x)) = -w^2 \mathcal{F}(f)$

Ex3) $\mathcal{F}(xe^{-x^2})$ $\quad f(x) = xe^{-x^2} = -\frac{1}{2}(e^{-x^2})'$

$$\mathcal{F}(f) = -\frac{1}{2}\mathcal{F}((e^{-x^2})') = -\frac{1}{2}iw\,\mathcal{F}(e^{-x^2})$$

$$= -\frac{1}{2}iw\left(\frac{1}{\sqrt{2}} e^{-w^2/4}\right)$$

From Table.

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/4a}$$

$a=1$

$\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$?? No.

Convolution : similar to multiplication or product.

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp = \int_{-\infty}^{\infty} f(x-p)\, g(p)\, dp.$$

Convolution Theorem of Fourier Transform.

$$\left(\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g) \right)$$

proof) by definition

$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp \cdot e^{-iwx}\, dx$$

exchange.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x-p)\, e^{-iwx}\, dx\, dp.$$

$(x-p = q \quad x = p+q \quad dx = dq)$

same range
$-\infty < x < \infty$
$-\infty < q < \infty$

$$\therefore = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) g(q)\, e^{-iw(p)}\, e^{-iwq}\, dq\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\boxed{\int_{-\infty}^{\infty} f(p)\, e^{-iwp}\, dp}}_{\sqrt{2\pi}\hat{f}(w)} \underbrace{\boxed{\int_{-\infty}^{\infty} g(q)\, e^{-iwq}\, dq}}_{\sqrt{2\pi}\hat{g}(w)}$$

$$= \frac{1}{\sqrt{2\pi}} [\sqrt{2\pi}\mathcal{F}(f)\ \sqrt{2\pi}\mathcal{F}(g)) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g))$$

Taking inverse Transform.

$$f*g(x) = \mathcal{F}^{-1}\{\sqrt{2\pi}\, \hat{f}(w)\, \hat{g}(w)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\, \hat{f}(w)\, \hat{g}(w)\, e^{iwx}\, dw$$

$$= \int_{-\infty}^{\infty} \hat{f}(w)\, \hat{g}(w)\, e^{iwx}\, dw$$

useful to ~~obtain~~ solve the PDE in Chap 12.

Summary of Chap 11.

"set Ground work for solving PDE in Chap 12."

$\boxed{\text{Fourier series}}$: periodic function $f(x)$ of period $p = 2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

If $f(x)$ is even → fourier cosine series
odd → fourier sine series

· $\boxed{\text{Orthogonality}}$ of trigonometric system.

· Replacement the trigonometric system by orthogonal.

→ Sturm-Liouville problems.

→ Solution of S-L problems : eigenfunctions.

Generalized fourier series : Fourier-Legendre, Fourier-Bessel series

$\boxed{\text{Fourier integral}}$: extension to nonperiodic functions $f(x)$

$$f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\,dw.$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$, $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$.

in complex form

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\,e^{iwx}\,dw$$

where $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx$.

$\boxed{\text{Fourier transform.}}$

Fourier cosine transform

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx$$

Fourier sine transform

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin wx\,dx$$

h.w. (1#1)

11.1 : 9. 13. 20
11.2 : 10. 29
11.3 : 6
11.4 : 4. 11
11.5 : 10. 12 by 10/23 (화)

11.6 : 2. 8 (11-2)

11.7 : 1. 7. 17 by 10/13
11.8 : 1, 12 by 10/22 (화)
11.9 : 7. 13 by 11/6 (화)