Chap 12. PDEs

12.1 Basic Concepts of PDEs

· PDE $\longleftrightarrow$ ODE

· equation that contains one or more partial derivatives of an unknown function that depends on at least two variables.

for example,

- wave equation — $u(t,x)$: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\xrightarrow{\text{2-D}}$ if $u(t.x.y)$: $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
- heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ | $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
- Laplace equation for electrostatic potential: $\nabla^2 u = 0 \xrightarrow{2D} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\xrightarrow{3D}$

Especially, our interest.

· order : The order of highest derivative.

· linear : if PDE is of the first degree in

· homogeneous

nonhomogeneous : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x.y)$

Solution is in the very different group.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$u = x^2 - y^2$

$u = e^x \cos y$

$u = \ln(x^2+y^2)$

Unique solution for a PDE corresponding to a given physical problem.

$\longrightarrow$ obtained by the additional conditions

- $\hookrightarrow$ Boundary Condition (B.C.)
- $\hookrightarrow$ Initial condition (I.C.) — t is variable. at $t=0$

Like ODEs, "Fundamental Theorem on Superposition" is valid.
only if homogeneous & linear,

If u_1 and u_2 are solutions of homogeneous linear PDE in some region R,

Then $u = c_1 u_1 + c_2 u_2$ (c_1, c_2 any constants)

are also a solution of the PDE.

$\longrightarrow$ Can expand to infinite u_i ($i = 1 \dots \infty$) for example.

$$u = \sum_{i=1}^{\infty} c_i u_i$$

also solution

Ex 1. $u_{xx} - u = 0$ where $u(x, y)$

if ODE, $u(x)$ $\lambda^2 - 1 = 0$ $\lambda = \pm 1$ $u = c_1 e^{x} + c_2 e^{-x}$

if PDE $u(x, y)$ c_1 and c_2 are the function of y.

$\therefore$ $u = c_1(y) e^{x} + c_2(y) e^{-x}$

Ex $u_{xy} = -u_x$

Solve it like ODE

$u_x = p$ $p_y = -p$ $p = c_1 e^{-y}$

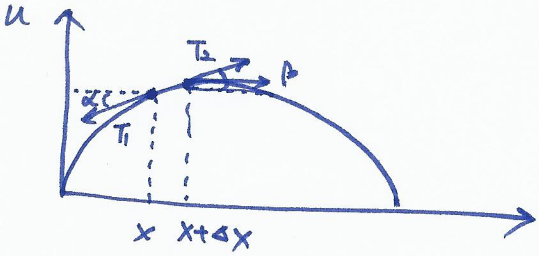
actually $c(x)$

$u_x = c(x) e^{-y}$

$\longrightarrow$ $u(x, y) = \int c_1(x) e^{-y}\, dx + c_2(y)$

$= e^{-y} \int c_1(x)\, dx + c_2(y)$

12.2 Modelling : 1-D wave equation



assumption : 1. homogeneous string

2. no gravitational force

3. only vertical motion.

at fixed time t

consider the forces acting on a small portion of the string.

horizonal component : $T_1 \cos\alpha = T_2 \cos\beta = T = \text{const}$

vertical " : $T_2 \sin\beta - T_1 \sin\alpha = \underbrace{\rho \Delta x}_{m} \underbrace{\frac{\partial^2 u}{\partial t^2}}_{a}$ (Newton's 2nd law)

small section mass

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

u : deflection

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}.$$

$\Longrightarrow$ slope of the string at x and $x+\Delta x$

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Big|_x \qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x}$$

$$\therefore \frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x\right)\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

let $\Delta x \to 0$

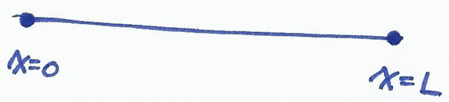
$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2} \qquad \text{where } c^2 = \frac{T}{\rho}.$$

$\longrightarrow$ one-D wave equation.

$\longrightarrow$ vibration of elastic string (guitar)

12.3 Solution by Separating Variables. Use of Fourier series

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$



B.C. $u(0,t)=0$, $u(L,t)=0$ for all $t \geq 0$

fastened at $x=0, L$.
$\therefore$ $u=0$

I.C. $u(x,0) = f(x)$ initial deflection
$(0 \leq x \leq L)$
$u_t(x,0) = g(x)$ initial velocity.

$u_t \to \frac{\partial u}{\partial t}$.

① method of separating variables or product method.

$$u(x,t) = F(x)\,G(t)$$

② ~~Two~~ ODEs with B.C.

③ Use of Fourier series from I.C.

$\longrightarrow$ solution.

$\to$ very general method

① $u(x,t) = F(x)\,G(t)$

$$F\ddot{G} = c^2 G F'' \quad \text{where} \quad \ddot{G} = \frac{d^2G}{dt^2} \quad F'' = \frac{d^2F}{dx^2}$$

$$\underbrace{\frac{\ddot{G}}{c^2 G}}_{f(t)} = \underbrace{\frac{F''}{F}}_{g(x)}$$

(now separated

Changing x or t does not alter the other side.

Thus.

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k$$

must be constant.

$\therefore$ two ODEs $\ddot{G} - c^2 k G = 0$, $F'' - kF = 0$

separation constant k is arbitrary

② Solving ODE with B.C.

$$u(0,t) = F(0)G(t) = 0$$
$$u(L,t) = F(L)G(t) = 0 \qquad \text{for all } t \geq 0.$$

$G(t) \not\equiv 0$; otherwise $u \equiv 0$

$\therefore F(0)=0, \quad F(L)=0$

$$F'' - kF = 0.$$

k arbitrary.

i) $k=0$. $F''=0$ $\quad F(x) = ax+b \qquad F(0)=b=0.$

$F(L) = aL = 0 \quad \therefore a=0$

$F(x) = 0 \longrightarrow$ trivial solution

ii) $k = \mu^2 > 0$

$F'' - \mu^2 F = 0 \qquad F(x) = Ae^{\mu x} + Be^{-\mu x}$

$F(0) = A+B = 0 \qquad A = -B$

$F(x) = A(e^{\mu x} - e^{-\mu x})$

$F(L) = A(e^{\mu L} - e^{-\mu L}) = 0$

$A = 0$

$\therefore F(x) = 0 \qquad$ trivial solution

iii) $k = -p^2 < 0$

$$F'' + p^2 F = 0$$

$F(x) = A\cos px + B\sin px$

$F(0) = A = 0. \quad , \quad F(L) = B\sin pL = 0$

$pL = n\pi \qquad \therefore \underline{p = \frac{n\pi}{L}} \quad (n=1,2,3\ldots)$

setting $B=1$.

we obtain infinitely many solutions

$$F(x) = F_n(x) = \sin\frac{n\pi x}{L} \quad (n=1,2,3\ldots)$$

$$k = -\left(\frac{n\pi}{L}\right)^2$$

$\ddot{G} - c^2 k G = 0, \qquad k = -\left(\frac{n\pi}{L}\right)^2$

$\ddot{G} + c^2\left(\frac{n\pi}{L}\right)^2 G = 0. \qquad \ddot{G} + \lambda_n^2 G = 0$

where $\lambda_n = \frac{cn\pi}{L}$ $(n = 1, \dots)$

$G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$ → unknown.

$\therefore u_n(x,t) = G_n(t) F_n(x) = \underline{(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x}$

→ Eigenfunction

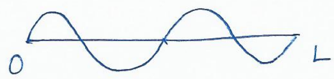
λ_n : Eigenvalues.

harmonic motion having the frequency $\frac{\lambda_n}{2\pi} = \frac{cn}{2L}$ per time.

nodes : points of the string that do not move.

$n=1$.

$n=4$



$(n-1)$ nodes for n.

with t is changing.

Step 3. Solution of the Entire problem. Fourier series.

$u_n(x,t) = G_n(t) F_n(x) = (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x$

I.C. $u(x,0) = G_n(0) F_n(x) = (\underline{B_n} \cos\lambda_n t + \cancel{B_n^* \sin\lambda_n t})$

$= B_n \sin\frac{n\pi}{L}x \neq f(x)$ (since n is changing.)

can't satisfy IC

Since the wave equation is linear and homogeneous,

it follows from the Fundamental Theorem of Superposition.

$\sum_{n=1}^{\infty} u_n$ is also a solution.

$$u = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x$$

$$u(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x} \qquad 0 \le x \le L$$

Fourier sine series $\qquad B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(-B_n \lambda_n \sin\lambda_n t + B_n^* \lambda_n \cos\lambda_n t\right) \sin\frac{n\pi x}{L} \Bigg|_{t=0}$$

$$\sum_{n=1}^{\infty} B_n^* \lambda_n \sin\frac{n\pi}{L}x = g(x)$$

$$B_n^* \lambda_n = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi}{L}x\, dx$$

$$\lambda_n = \frac{cn\pi}{L}$$

$$\therefore \quad B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\, dx .$$

$\therefore$ For simplicity $\quad g(x) = 0 \longrightarrow B_n^* = 0.$

$$\lambda_n = \frac{cn\pi}{L}$$

$$\therefore\ u = \sum_{n=1}^{\infty} \underbrace{(B_n \cos\lambda_n t)\sin\frac{n\pi}{L}x}_{\frac{1}{2}\left[\sin\left(\frac{n\pi}{L}(x-ct)\right) + \sin\left(\frac{n\pi}{L}(x+ct)\right)\right]} \quad \text{where } B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx .$$

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

$$u(x,t) = \frac{1}{2}\left\{f^*(x-ct) + f^*(x+ct)\right\}$$

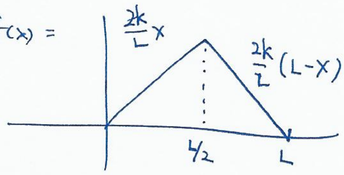
$$\because\ f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$$

average (or superposition of $\underline{f(x-ct)} + \underline{f(x+ct)}$

at $t=$

Ex.

$f(x) =$

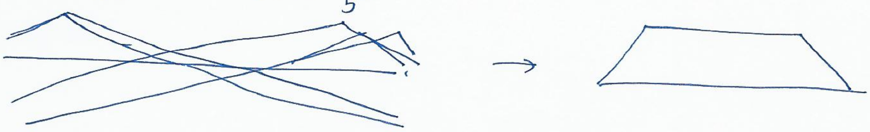


$g(x) = 0. \longrightarrow B_n^* = 0. \longrightarrow u(x,t) =$

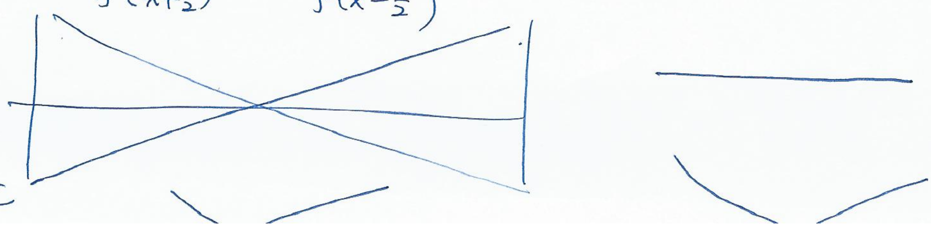
$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx} = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + \cdots\right.$$

at $t=0$. $\quad u(x,0) = f(x)$

at $t = L/5c \quad \frac{1}{2}f(x+\frac{L}{5}) \qquad \frac{1}{2}f(x-\frac{L}{5})$



at $t = L/2c \quad f(x+\frac{L}{2}) \qquad f(x-\frac{L}{2})$



at $x = L/c$

12.4 D'Alembert's solution of the Wave Equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Big) \Rightarrow \quad u_{tt} = c^2 u_{xx}$$

$v = x + ct \qquad u(x,t) \Rightarrow u(v,w)$

$w = x - ct \qquad \begin{pmatrix} v_x = 1 & v_t = c \\ w_x = 1 & w_t = -c \end{pmatrix}$

$u_x = u_v v_x + u_w w_x = u_v + u_w$

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x$$
$$= u_{vv} + 2u_{vw} + u_{ww}$$

$u_t = u_v v_t + u_w w_t = c(u_v - u_w)$

$u_{tt} = c(u_v - u_w)_t = c^2 (u_{vv} - 2u_{vw} + u_{ww})$

$\therefore c^2(u_{vv} - 2u_{vw} + u_{ww}) = c^2(u_{vv} + 2u_{vw} + u_{ww})$

$\therefore u_{vw} = \frac{\partial^2 u}{\partial v \partial w} = 0.$

integration w.r.t. w

$\frac{\partial u}{\partial v} = h(v)$

$$u = \underbrace{\int h(v)\,dv}_{\phi(v)} + \psi(w) = \phi(v) + \psi(w)$$

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

$\therefore$ D'Alembert's solution.

I. C. $\quad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$

$u_t = c\phi'(x+ct) - c\psi'(x-ct)$

$\phi(x) + \psi(x) = f(x)$

$c\phi'(x) - c\psi'(x) = g(x) \Rightarrow \int dx \quad \therefore \phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + k(x_0)$

where $k(x_0) = \phi(x_0) - \psi(x_0)$

$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}k(x_0)$

$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{1}{2}k(x_0)$

$\phi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \frac{1}{2}k(x_0)$

$+)\ \psi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds - \frac{1}{2}k(x_0)$

$\frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$

if $g(s) = 0$.

Same as 12.2

Heat flow from a Body in space. Heat Equation.

assumption

1. ρ, C_p, k is constant

2. $\underline{\underline{v}} = -K \nabla u$ heat flow 의 velocity는 온도 gradient

T : region in the body bounded by a surface S with outer unit normal vector $\underline{\underline{n}}$

$|\underline{\underline{v}} \cdot \underline{\underline{n}}| \Delta A$ heat leaving T.

Total amount of heat ~~that~~ flows leaving across S

$$\iint_S \underline{\underline{v}} \cdot \underline{\underline{n}} \, dA$$

$\underline{\underline{v}} = -K \nabla u$

$$= -k \iint_S (\nabla u) \cdot \underline{n} \, dA = -K \iiint_T \operatorname{div}(\nabla u) \, dx\,dy\,dz$$

Gauss Divergence Theorem

$$= -k \iiint_T \nabla^2 u \, dx\,dy\,dz \quad \text{—— ①}$$

Total amount of heat in T.

$$H = \iiint_T C_p \rho u \, dx\,dy\,dz$$

rate of H decrease

$$\frac{\partial H}{\partial t} = -\iiint_T C_p \rho \frac{\partial u}{\partial t} \, dx\,dy\,dz \quad \text{—— ②}$$

no heat generation.

$$-k \iiint_T \nabla^2 u \, dx\,dy\,dz = -\iiint C_p \rho \frac{\partial u}{\partial t} \, dx\,dy\,dz$$

$$\iiint_T \left(\frac{\partial u}{\partial t} - c^2 \nabla^2 u \right) dx\,dy\,dz = 0$$

$$\therefore \frac{\partial u}{\partial t} = c^2 \nabla^2 u.$$

12.6. One-D heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2 BC $u(0,t)=0$. $u(L,t)=0$ for all $t \geq 0$

1 IC $u(x,0) = f(x)$

Separation of variables

$$u(x,t) = F(x)G(t)$$

$$F(x)\dot{G}(t) = c^2 F''(x) G(t)$$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2 \quad (k \quad =0$$

$$F'' + p^2 F = 0 \qquad \dot{G} + c^2p^2G = 0.$$

$F(x) = A\cos px + B\sin px$

$F(0)=0 \quad F(L)=0$

$A=0.\ \sin pL = 0$

$pL = n\pi$

$p = \dfrac{n\pi}{L} \quad (n=1,2\ldots)$

$\therefore F_n(x) = \sin\dfrac{n\pi x}{L}.$

$\dot{G} + c^2p^2G = 0$

since $p = \dfrac{n\pi}{L} \quad (n=1,2,\ldots)$

$\dot{G}_n + \left(\dfrac{cn\pi}{L}\right)^2 G_n = 0.$

$G_n(t) = B_n e^{-\lambda_n^2 t}$ where $\lambda_n = \dfrac{cn\pi}{L}$

$$\therefore u_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

From I.C. $u(x,0) = f(x) \Rightarrow u_n(x,0) = B_n \sin\dfrac{n\pi x}{L} \neq f(x)$

So series must be considered.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$\therefore B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Fourier sine series

Difference Between the wave eqn

as $t \to \infty$, $u(x,t) \longrightarrow 0.$

12.7 Heat Eq'n :

<300ft> wire . bar.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

no B.Cs

only ICs. $f(x) = u(x,0)$ $\quad -\infty < x < \infty$

initial temperature.

$u(x,t) = F(x)G(t)$

$$F'' + p^2 F = 0 \qquad \dot{G} + c^2p^2 G = 0.$$

$$F(x) = A\cos px + B\sin px \qquad G(t) = e^{-c^2p^2t}$$

$$u(x,t;p) = F(x)G(t) = (A\cos px + B\sin px)e^{-c^2p^2t}$$

Since $f(x)$ is not periodic, better to use Fourier integral instead of Fourier series.

A, B are arbitrary, so regard them as functions of p.

$\therefore$ $A(p)$, $B(p)$.

$$u(x,t) = \int_0^\infty u(x,t;p)\,dp = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, e^{-c^2p^2t}\,dp$$

$A(p)$, $B(p)$?

$$u(x,0) = f(x) = \int_0^\infty [A(p)\cos px + B(p)\sin px)\,dp$$

where.

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv, \quad B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv$$

By entering $A(p)$ & $B(p)$

$$u(x,0) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px - pv)\,dv\right] dp$$

Similarly, $u(x,t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(px-pv)\, e^{-c^2p^2t}\,dv\right] dp.$

reverse the order of integration

$$\boxed{u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\boxed{\int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp}\right]dv} \quad ①$$

need to solve the integral

From $\int_0^\infty e^{-s^2}\cos 2bs\,ds = \frac{\sqrt{\pi}}{2}e^{-b^2}$ $\qquad p = \frac{s}{c\sqrt{t}}$ $\qquad$ ⑥ $b = \frac{x-v}{2c\sqrt{t}}$

① $s^2 = c^2p^2t$ $\qquad$ ② $b = \frac{p(x-v)}{2s} = \frac{(x-v)}{2c\sqrt{t}}$

$s = cp\sqrt{t}$

③ $ds = c\sqrt{t}\,dp$, ④ $2bs = (x-v)p$, ⑤ $b = \frac{p(x-v)}{2s}$

$dp = \frac{ds}{c\sqrt{t}}$

$$\therefore \int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp = \frac{\sqrt{\pi}}{2}\cdot\frac{1}{c\sqrt{t}}\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}$$

$$* \quad \boxed{u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left\{\frac{-(x-v)^2}{4c^2t}\right\}dv}$$

② taking $z = \frac{v-x}{2c\sqrt{t}}$ $\qquad$ $v = x + 2c z\sqrt{t}$ $\qquad$ $dz = \frac{1}{2c\sqrt{t}}dv$

$\therefore u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\,e^{-z^2}\,dz$

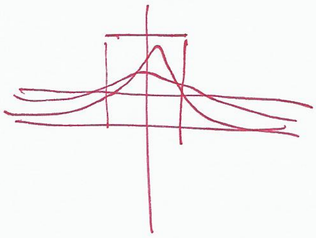
Ex1. Temperature in an infinite bar.

if $f(x) = \begin{cases} u_0 = \text{const.} & \text{if } |x|<1 \\ 0 & |x|>1 \end{cases}$

① Fourier integral method

$$\therefore u(x,t) = \frac{u_0}{2c\sqrt{\pi t}} \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv$$

$-1<v<1 \;\downarrow\; \dfrac{-1-x}{2c\sqrt{t}} < z < \dfrac{1-x}{2c\sqrt{t}}$



$$\text{or} = \frac{u_0}{\sqrt{\pi}} \int_{\frac{-1-x}{2c\sqrt{t}}}^{\frac{(1-x)}{2c\sqrt{t}}} e^{-z^2}\, dz$$

error function.

Fig 299.

② Use of Fourier transform. Since Fourier transform is closely related to the Fourier integral

Take Fourier transform w.r.t. x.

$$u_t = c^2 u_{xx}$$

$$\mathcal{F}(u_t) = c^2 \mathcal{F}(u_{xx}) = c^2(-\omega^2)\mathcal{F}(u) = -c^2\omega^2 \mathcal{F}(u) = -c^2\omega^2 \hat{u}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \underbrace{\int_{-\infty}^{\infty} u e^{-i\omega x} dx}_{\hat{u}} = \frac{\partial \hat{u}}{\partial t}$$

$$\therefore \frac{\partial \hat{u}}{\partial t} = -c^2\omega^2 \hat{u} \longrightarrow \text{1st ODE with } t.$$

$$\hat{u}(\omega,t) = C(\omega)\, e^{-c^2\omega^2 t} \quad \text{not constant.}$$

I.C. $\hat{u}(\omega,0) = C(\omega) = \hat{f}(\omega) = \mathcal{F}(f)$ $\qquad \hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-c^2\omega^2 t}$

inversion formula $\quad u = \mathcal{F}^{-1}(\hat{u}) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-c^2\omega^2 t} e^{i\omega x} d\omega$

since $\hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(v)\, e^{-i\omega v} dv$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v) \int_{-\infty}^{\infty} e^{-c^2\omega^2 t}\, e^{i(\omega x - \omega v)} d\omega\, dv$$

$$\downarrow$$

$\cos(\omega x + \omega v) + i\sin(\omega x - \omega v)$

0 due to odd

$$\therefore \boxed{\frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \int_{0}^{\infty} e^{-c^2\omega^2 t} \cos(\omega x - \omega v)\, d\omega\, dv} \quad (1)$$

③ Method of Convolution. inverse F.T.!

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\mathcal{F}(g) \Longrightarrow f*g = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) e^{i\omega x} d\omega$$

$$u(x,t) = (f*g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega) e^{i\omega x} d\omega$$

$\boxed{\text{question is what is } g(x)}$ $\qquad$ where $\hat{g}(\omega) = \dfrac{1}{\sqrt{2\pi}} e^{-c^2\omega^2 t}$

From Formula. $\mathcal{F}(e^{-ax^2}) = \dfrac{1}{\sqrt{2a}} e^{-\omega^2/4a}$

$$\therefore \mathcal{F}\left(e^{-\frac{x^2}{4c^2t}}\right) = \sqrt{2c^2t}\, e^{-c^2\omega^2 t}$$

$c^2 t = \dfrac{1}{4a} \qquad a = \dfrac{1}{4c^2 t}$

$$= \sqrt{2c^2t}\,\sqrt{2\pi}\, \hat{g}(\omega)$$

$$g(x) = \mathcal{F}^{-1}(\hat{g}(\omega)) = \frac{1}{\sqrt{2c^2t}\sqrt{2\pi}} e^{-\frac{x^2}{4c^2t}}$$

$\left(\hat{g}(\omega) = \dfrac{1}{\sqrt{2c^2t}\sqrt{2\pi}} e^{-\frac{x^2}{4c^2t}} \right)$

by the definition of convolution

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp$$

②

$$\therefore\ u(x,t) = (f*g)(x) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \exp\left\{-\frac{(x-p)^2}{4c^2 t}\right\} dp$$

Semi-infinite bar

skip 해도 무방.!

$0 \longrightarrow \infty$

$u(x,0) = f(x)$

$u(0,t) = 0. \qquad f(0) = u(0,0) = 0.$

application of Fourier Sine transform.

$$u_t = c^2 u_{xx}$$

$$\mathcal{F}_s(u_t) = \frac{\partial \hat{u}_s}{\partial t} = c^2 \mathcal{F}_s(u_{xx}) = -c^2 w^2 \mathcal{F}_s(u) = -c^2 w^2 \hat{u}_s(w,x)$$

$$\hat{u}_s = C(w)\, e^{-c^2 w^2 t}$$

$$= \hat{f}_s(w)\, e^{-c^2 w^2 t}$$

inverse sine transform.

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(p) \sin wp\, dp.$$

$$\therefore\ u(x,t) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(p) \sin wp\, e^{-c^2 w^2 t} \sin wx\, dp\, dw.$$

12.1 : 4. 7. 22

12.3 : 5. 9

12.6 : 7. 12.

12.7 : 5. 6

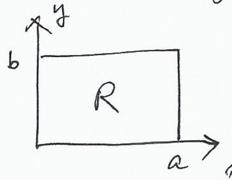
~~by [redacted]. 10/22~~

(by 2015 11/2)

12.9

Rectangular membrane.

2-D wave equation



< rectangular region >

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \longrightarrow \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(4-sides)

$$\begin{cases} u = 0 \text{ on } \cancel{\text{the}} \text{ boundary} \\ u(x,y,0) = f(x,y) & \text{initial deflection} \\ u_t(x,y,0) = g(x,y) & \text{initial velocity.} \end{cases}$$

Step. 1. → 3 ODEs

$u(x,y,t) = F(x,y)G(t)$, then $F(x,y) = H(x)Q(y)$

$$F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$$

$$\frac{\ddot{G}}{c^2 G} = \frac{F_{xx} + F_{yy}}{F} = -\nu^2$$

Time Function: ① $\ddot{G} + \lambda^2 G = 0$ where $\lambda = c\nu$

Amplitude Function: $F_{xx} + F_{yy} + \nu^2 F = 0$ 2-D Helmholtz eq'n.

$$F(x,y) = H(x)Q(y)$$

$$H''Q + HQ'' + \nu^2 HQ = 0.$$

$$\frac{H''}{H} = -\frac{1}{Q}(Q'' + \nu^2 Q) = -k^2$$

(p, k, ν, λ) all coupled.

② $\therefore H'' + k^2 H = 0$ ③ $Q'' + p^2 Q = 0$ where $p^2 = \nu^2 - k^2$

Step 2. Satisfying the BCs

$H(x) = A\cos kx + B\sin kx$ ✓ $Q = C\cos py + D\sin py$ ✓

BC. $u = 0$ at the boundary.

$H(0) = H(a) = 0$ $Q(0) = Q(b) = 0$.

$A = 0$, $ka = m\pi$ $C = 0$, $pb = n\pi$.

$k = \frac{m\pi}{a}$ (m: integer) $p = \frac{n\pi}{b}$ (n: integer)

Solution

$H_m(x) = \sin\frac{m\pi}{a}x$ ✓ $Q_n(y) = \sin\frac{n\pi}{b}y$.

$$\therefore F_{mn}(x,y) = H_m(x)Q_n(y) = \sin\frac{m\pi}{a}x \,\sin\frac{n\pi}{b}y.$$

since $p^2 = \nu^2 - k^2$ and $\lambda = c\nu$ $\nu = \sqrt{p^2 + k^2} = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

$$\ddot{G} + \lambda^2 G = 0 \qquad \ddot{G}_{mn} + \lambda_{mn}^2 G_{mn} = 0$$

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\sin\lambda_{mn}t$$

$$\boxed{\text{Eigenfunction.} \quad u_{mn}(x,y,t) = (B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}$$

λ_{mn} : eigenvalues. $\begin{pmatrix} m = 1.2.3\ldots \\ n = 1.2.3\ldots \end{pmatrix}$

frequency $\left(\frac{\lambda_{mn}}{2\pi}\right)$

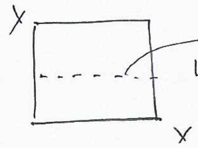
- node : eigenfunction $= 0$ → 1-D 경우
- nodal line : where eigenfunction is 0. (2-D 경우)
  ↓
  or physically does not move.

Ex. Skip

for square membrane $a = b = 1$. $\lambda_{mn} = c\pi\sqrt{m^2+n^2}$, $\lambda_{mn} = \lambda_{nm}$ symmetric.

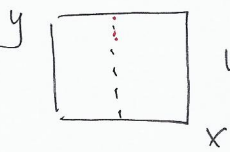
$$u_{12} = (B_{12}\cos c\pi\sqrt{5}t + B_{12}^{*}\sin c\pi\sqrt{5}t)F_{12}$$

$2\pi\left(\frac{1}{2}\right)$



u_{12} → nodal line: at $y = 1/2$

$$u_{21} = (B_{21}\cos c\pi\sqrt{5}t + B_{12}^{*}\sin c\pi\sqrt{5}t)F_{21}$$



u_{21} nodal line at $x = 1/2$.

Step 3. Solution of

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\sin\cos\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

I.C. $u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\underbrace{\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}_{k_m(y)}$?

$$f(x,y) = \sum_{m=1}^{\infty} K_m(y)\sin\frac{m\pi x}{a}.$$

$$\therefore K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}dx$$

$$\sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b} = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}dx$$

$$\therefore B_{mn} = \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}dx \cdot \sin\frac{n\pi y}{b}dy$$

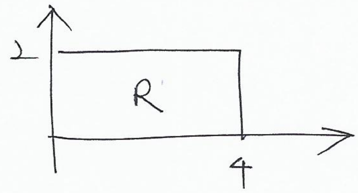
$$\therefore = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dx\,dy$$

To determine B_{mn}^{*}

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^{*}\lambda_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = g(x,y)$$

$$\therefore\ B_{mn}^{*} = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

Ex.



Tension $= 12.5$ lb/ft.

density $= 2.5$ slug/ft^2

$g(x,y)=0 \quad B_{mn}^{*}=0$

$f(x,y) = 0.1\,(4x-x^2)(2y-y^2)$

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x-x^2)(2y-y^2) \sin\frac{m\pi x}{4} \sin\frac{n\pi x}{2}\, dx\, dy$$

$$= \frac{1}{20}\underbrace{\int_0^4 (4x-x^2)\sin\frac{m\pi x}{4}\, dx}_{\frac{256}{m^3\pi^3}\ (m:\ \text{odd})} \cdot \underbrace{\int_0^2 (2y-y^2)\sin\frac{n\pi x}{2}\, dy}_{\frac{32}{n^3\pi^3}\ (n:\ \text{odd})}$$

$$\therefore\ B_{mn} = \frac{256\cdot 32}{20\cdot m^3 n^3 \pi^6} \simeq \frac{0.426050}{m^3 n^3} \quad (\text{for odd } m, n)$$

$$u(x,y,t) = 0.426050 \sum_{m,n\ \text{odd}}\sum \frac{\cos\sqrt{\frac{5\pi}{4}}\sqrt{(m^2+4n^2)}\, t}{m^3 n^3} \sin\frac{m\pi x}{4} \sin\frac{n\pi y}{2}$$

how about (3, 3)

↳ two nodal line

(1, 1)

no nodal line

(3, 1)

$\sin\frac{3\pi x}{4} \sin\frac{n y}{2}$

$x = \frac{4}{3}, \frac{8}{3}$

$0 \quad \frac{4}{3} \quad \frac{8}{3} \quad \frac{12}{3}$

$0,\ \pi,\ 2\pi,\ 3\pi,$

(1, 3)

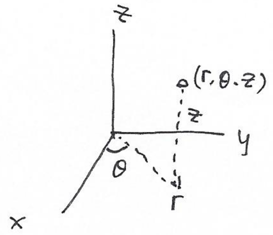
$$\boxed{\sin\frac{\pi x}{4} \sin\frac{3\pi y}{2}}$$

$y = \frac{2}{3} \quad \frac{4}{3} \quad \frac{6}{3}$

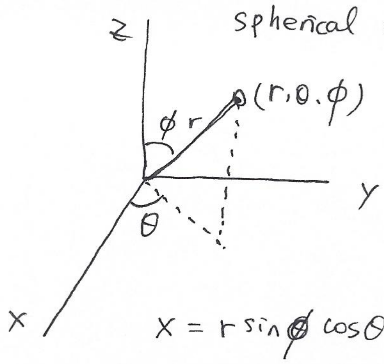
two nodal line

Cylindrical coordinate

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$



Spherical coordinate



$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

or

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right]$$

Dirichlet problem. $\frac{\partial^2 u}{\partial \theta^2} = 0$

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)\right] = 0$$

$u(R, \phi) = f(\phi)$ on the surface

$\lim_{r\to\infty} u(r,\phi) = 0$ potential at infinity will be zero.

① Separating variables.

$u(r, \phi) = G(r)H(\phi)$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial (GH)}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\, G\frac{dH}{d\phi}\right)\right] = 0.$$

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{H\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) = k.$$

i) $\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = k$ $\quad r^2\frac{d^2G}{dr^2} + 2r\frac{dG}{dr} = kG.$

ii) $\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0$

i)

take k as $n(n+1)$

$$r^2G'' + 2rG' - n(n+1)G = 0.$$

Euler Cauchy Equation. $G = r^a$ $\quad a(a-1) + 2a - n(n+1)$

$a = n$ & $-n-1$

$\therefore G_n(r) = r^n \qquad G_n^*(r) = \frac{1}{r^{n+1}}$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0$$

$$\cos\phi = w \qquad \sin^2\phi = 1-\omega^2$$

$$\frac{d}{d\phi} = \frac{d}{dw}\frac{dw}{d\phi} = -\sin\phi\frac{d}{dw} \longrightarrow \left(\frac{-d}{dw}\right) = \frac{1}{\sin\phi}\frac{d}{d\phi}.$$

$$\therefore \quad -\frac{d}{dw}\left(-\sin^2\phi\frac{dH}{dw}\right) + kH = 0. \qquad \frac{d}{dw}\left[(1-\omega^2)\frac{dH}{dw}\right] + n(n+1)W = 0.$$

$$(1-\omega^2)H'' - 2wH' + n(n+1)H = 0 \qquad \text{For integer } n = 0, 1, \ldots$$

$$H(w) = P_n(w) = P_n(\cos\phi) \qquad \text{solutions.}$$

Two sets of solution $\quad u_n(r,\phi) = A_n r^n P_n(\cos\phi) \qquad u_n^*(r,\phi) = \frac{B_n}{r^{n+1}} P_n(\cos\phi)$

i) interior problem : potential within the sphere S.

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi)$$

at $r=R$. $\quad u(R,\phi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\phi) = f(\phi).$

Fourier – LeGendre series.

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) \qquad a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\,dx$$

$$\therefore \quad A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} \tilde{f}(w) P_n(w)\,dw$$

$$w = \cos\phi \qquad dw = -\sin\phi\,d\phi$$

$$A_n = \frac{2n+1}{2R^n}\int_{-1}^{0} f(\phi) P_n(\cos\phi)(-\sin\phi\,d\phi)$$

$$= \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi) P_n(\cos\phi)\sin\phi\,d\phi$$

ii) Exterior problem.

we can't use u_n ~~for~~ outside the sphere.

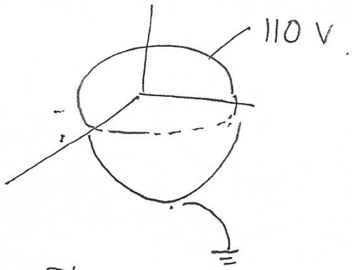
$$u(r,\phi) = \sum u_n^*(r,\phi) = \sum \frac{B_n}{r^{n+1}} P_n(\cos\phi)$$

in the similar way.

$$B_n = \frac{2n+1}{2}R^{n+1}\int_0^{\pi} f(\phi) P_n(\cos\phi)\sin\phi\,d\phi.$$

Ex1. Spherical capacitor.

(two metallic capacitor)



$$f(\phi) = \begin{cases} 110 & 0 \le \phi < \pi/2 \\ 0 & \pi/2 < \phi < \pi \end{cases}$$

$$A_n = \frac{2n+1}{2} \cdot 110 \int_0^{\pi/2} P_n(\cos\phi) \sin\phi \, d\phi$$

$$= \frac{2n+1}{2} \cdot 110 \int_0^1 P_w(w)\, dw \qquad \text{where } w = \cos\phi .$$

$$A_n = 55(2n+1) \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} \boxed{\int_0^1 w^{n-2m}\, dw}$$

$$\frac{1}{n-2m+1}$$

$M = \cancel{0}\ n/2$ for even n

$M = (n-1)/2$ for odd n.

$$A_n = \frac{55(2n+1)}{2^n} \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m+1)!}$$

$n=0$. $A_0 = 55$

$n=1$ $A_1 = 165/2$.

$A=2$ $A_2 = 0$

$A=3$ $\frac{-385}{8}$

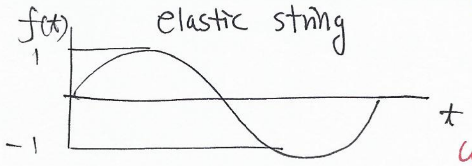
$$\therefore\ u(r.\phi) = 55 + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi) + \cdots$$

Since $R=1$. $\longrightarrow$ $A_n = B_n$.

12.12. Solution of PDEs by Laplace Transforms

① Take Laplace Transform with respect to ~~the~~ one of two variables, usually t.

Ex 1. Semi-infinite string, elastic string



② (motion of the left end of the string) B.C.

$$w(0,t) = f(t) = \begin{cases} \sin t & \text{if } 0 \le t \le 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

wave equation. $\lim_{x\to\infty} w(x,t) = 0$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

I.C. $\begin{pmatrix} w(x,0)=0 \\ w_t(x,0)=0 \end{pmatrix}$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\left(\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)\right) \Rightarrow \mathcal{L}\left(\frac{\partial^2 w}{\partial t^2}\right) = s^2\mathcal{L}(w) - s\underbrace{w(x,0)}_{0} - \underbrace{w_t(x,0)}_{0} = c^2\mathcal{L}\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$\mathcal{L}\left(\frac{\partial^2 w}{\partial x^2}\right) = \int_0^\infty e^{-st}\frac{\partial^2 w}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st} w\,dt = \frac{\partial^2}{\partial x^2}\mathcal{L}(w(x,t))$$

$\therefore$ ~~$s^2\mathcal{L}(w)$~~ $W(x,s) = \mathcal{L}(w(x,t))$

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2} \quad \text{thus} \quad \frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2}W = 0$$

2nd ODE. ~~w.r.t.~~ as a function of x

$$\therefore W(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c}$$

$s, c > 0$

$$W(0,s) = \mathcal{L}(w(0,t)) = \mathcal{L}(f(t)) = F(s)$$

$\left(\lim_{x\to\infty} w(x,t) = 0\right)$

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st} w(x,t)\,dt = \int_0^\infty e^{-st}\lim_{x\to\infty} w(x,t)\,dt = 0$$

Therefore $A(s) = 0$ since $c > 0$.

$$W(0,s) = F(s) = B(s)$$

$$\therefore W(x,s) = F(s)e^{-sx/c}$$

s-shifting

inverse Transform.

$$w(x,t) = f\left(t - \frac{x}{c}\right)u\left(t - \frac{x}{c}\right),$$

unit function $u(t) = 0$

since $f(x) = \sin x$

that is $W(x,t) = \sin\left(t - \frac{x}{c}\right)$ if $\frac{1}{c}x < t < \frac{x}{c} + 2\pi$

or $ct > x > (t-2\pi)c$

$0 < t - \frac{x}{c} < 2\pi$

Traveling wave Fig 317

(Very similar to Fourier Transform method)