Chap 13. Complex Numbers and Functions. Complex Differentiation

Complex analysis 13-1
↳ real calculus to complex calculus

13.1. Complex Numbers and Their Geometric Representation. (Nothing New. Just Reminder)

Complex number $z = (x, y) = x + iy$. $\text{Re}\, z = x$, $\text{Im}\, z = y$.

x: real part, iy: imaginary part

$\oint_C \frac{dz}{z}$ $\int_0^\infty \frac{dx}{1+x^3}$

✓ addition, Multiplication.

$$z_1 = (x_1, y_1) \qquad z_2 = (x_2, y_2)$$

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$

✓ $$z_1 z_2 = (x_1, y_1)\cdot(x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1)$$

when $x = 0$, pure imaginary $z = iy$

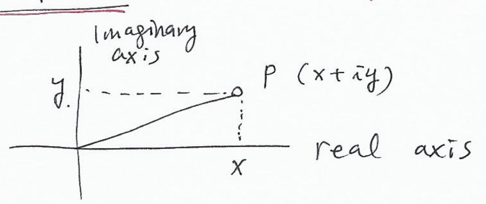
✓ subtraction,

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

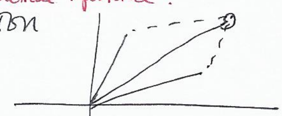
✓ Division $\frac{z_1}{z_2} = x + iy$ $x = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}$ $y = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$

Consider the geometric representations of complex numbers, which is of great practical importance.

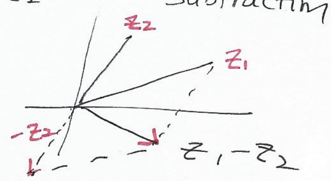
Complex plane



addition

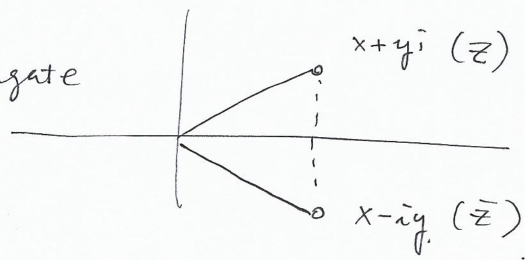


Subtraction



Conjugate Number

$\bar{z} = x - iy$. Complex Conjugate



$$\text{Re}(z) = x = \frac{1}{2}(z + \bar{z})$$

$$\text{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$$

if z is real $z = \bar{z}$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

13.2 Polar Form of Complex Numbers. Powers and Roots

(Another reminder)

$x = r\cos\theta \qquad y = r\sin\theta$

$z = x + yi = r(\cos\theta + i\sin\theta)$

$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$

$|z|$: distance of the point z from the origin

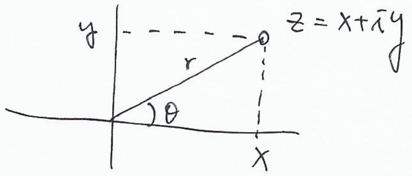
$|z_1 - z_2|$: distance between z_1 and z_2.

θ geometric interpretation $\quad \theta$ is called argument of z

$\arg z$

$\therefore \theta = \arg z$, $\quad \tan\theta = \frac{y}{x}$

$\arg z = \operatorname{Arg} z \pm 2n\pi \quad (n = \pm 1, \pm 2, \ldots)$



$r = |z|$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$-\pi < \operatorname{Arg} z \le \pi$

principal value.

Ex 1)

$z = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

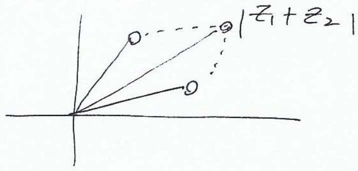
$|z| = \sqrt{2}$

$\arg z = \frac{\pi}{4} \pm 2n\pi$

$\operatorname{Arg} z = \frac{\pi}{4}$

Triangle inequality.

$|z_1 + z_2| \le |z_1| + |z_2|$



generalized triangle inequality

$|z_1 + z_2 + \cdots z_n| \le |z_1| + |z_2| + \cdots |z_n|$.

Multiplication and Division in Polar Form

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\right]$

$= r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$

$|z_1 z_2| = |z_1||z_2|$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Division $\quad z_1 = \left(\frac{z_1}{z_2}\right)z_2 \qquad |z_1| = \left|\left(\frac{z_1}{z_2}\right)z_2\right| = \left|\frac{z_1}{z_2}\right||z_2| \qquad \therefore \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$

By analogy,

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \qquad \arg z_1 = \arg\left[\left(\frac{z_1}{z_2}\right)z_2\right] = \arg\left(\frac{z_1}{z_2}\right) + \arg z_2$

therefore

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

De Moivre's Formula. $(z_1 = z_2 = z)$, $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$ $\leftarrow$ $(n = 0, 1, 2 \dots)$: by induction for $n = 1, 2, \dots$)

For $|z| = r = 1$

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

for $n=2$ $(\cos\theta + i\sin\theta)^2 = \underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta} + i\,\underbrace{2\sin\theta\cos\theta}_{\sin 2\theta}$ $= \cos 2\theta + i\sin 2\theta$

$z^3 = 1$ $(z^3 - 1) = 0$ $z^2 + z + 1 = 0$

$z = 1$, ω, ω^2

Roots. if $z = w^n$ $w = \sqrt[n]{z}$

we write z and w in the polar form,

$z = r(\cos\theta + i\sin\theta)$ and $w = R(\cos\phi + i\sin\phi)$

By De Moivre's Formula

$w^n = R^n(\cos n\phi + i\sin n\phi)$,

$R^n = r$ $R = \sqrt[n]{r}$

$n\phi = \theta + 2k\pi$, thus $\phi = \dfrac{\theta}{n} + \dfrac{2k\pi}{n}$ where k is integer.

For $k = 0, 1, \dots, n-1$, we get n distinct values of w.

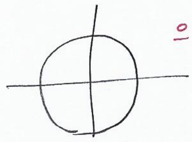
$\therefore$ $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values $(k = 0, 1, 2, \dots n-1)$

$$w = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

for example) Taking $z = 1$. $|z| = r = 1$ $\operatorname{Arg} z = 0$.

$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\frac{2k\pi}{n} \qquad k = 0, 1, \dots n-1$$

n values : nth roots of unity. 으로 설명 (one by one.)

Fig. 

$\sqrt[3]{1}$ $\sqrt[4]{1}$ $\sqrt[5]{1}$

$z^3 - 1 = 0$ $z^4 - 1 = 0$ $z^5 - 1 = 0$ ...

If w denotes the value corresponding to $k = 1$, then the n values of $\sqrt[n]{1}$

$1, w, w^2 \dots w^{n-1}$ (n values of $\sqrt[n]{1}$)

more generally,

if w_1 is any nth root of an arbitrary complex number z,

$w_1, w_1 w, w_1 w^2 \dots w_1 w^{n-1}$

because multiplying w by w^k correspond to increasing the argument of w_1 by $2k\pi/n$.

$(z-1)(z^2 + z + 1) = 0$

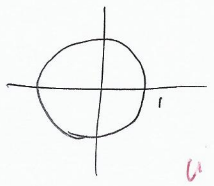
$\hookrightarrow$ if ω is solution, then ω^2 as well

$\hookrightarrow$ $\omega^4 + \omega^2 + 1 = \omega + \omega^2 + 1 = 0$.

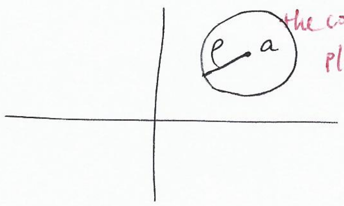
13.3 Derivative. Analytic Function.

Since the functions live in the complex plane, the concepts ('domain, derivative') are slightly different.

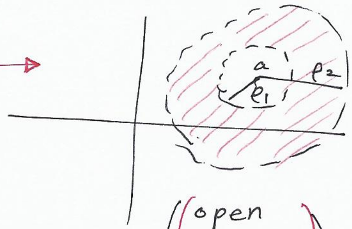
$|z| = 1$ unit circle



$|z-a| = \rho$ circle in the complex plane



$\rho_1 < |z-a| < \rho_2$



((open annulus))

neighborhood of a : $|z-a| < \rho$

if $\rho_1 \le |z-a| \le \rho_2$ closed annulus.

$$w = f(z)$$

Complex function $\quad w = f(z) = u(x,y) + i\, v(x,y)$ (Complex function $f(z)$ = equivalent to a pair of real functions $u(x,y)$ and $v(x,y)$)

Let $w = f(z) = z^2 + 3z$ Find u, v at $z = 1+3i$

$z = (x+yi)$

$x^2 - y^2 + 2xyi + 3(x+yi)$

$\cancel{(x^2-y^2+3x)}$

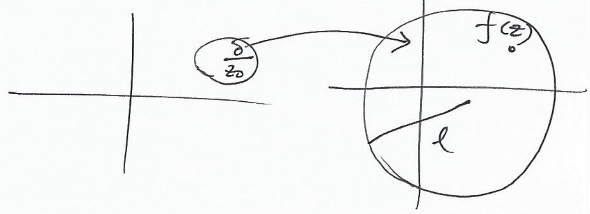
$(x^2-y^2+3x) + (3y+2xy)i$

$\therefore \begin{matrix} x=1 \\ y=3 \end{matrix} \Rightarrow (1-9+3) + (9+6)i$

$(-5+15i)$

limit $\quad \lim_{z \to z_0} f(z) = \ell$. such that $|f(z) - \ell| < \varepsilon$ for $|z - z_0| < \delta$

continuous $\quad \lim_{z \to z_0} f(z) = f(z_0)$



$(cf)' = cf'$ $\quad (fg)' = f'g + fg'$ ...

$(f+g)' = f' + g'$

Derivative

Same as in real calculus.

$f(z) = z^2 \quad f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

(or)

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

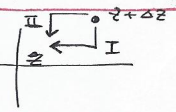
approach from any direction.

Ex. 4 $\quad f(z) = \bar{z} = x - iy$.

$\Delta z = \Delta x + i\Delta y$.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

if $\begin{cases} \Delta y = 0, & 1 \quad (\mathrm{I}) \\ \Delta x = 0 & -1 \quad (\mathrm{II}) \end{cases}$



$\boxed{\text{not differentiable}}$

Def) Analytic Function (functions that are differentiable in some domain)

if $f(z)$ is defined and differentiable at any points in D.

Said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

analyticity of $f(z)$ at z_0

$\longrightarrow$ $f(z)$ has a derivative at every point in some neighborhood of z_0

13.4 Cauchy - Rieman Equations. Laplace's Equation. → most important equation in this chapter!

provide a criterion for the analyticity of a complex function

$$w = f(z) = u(x,y) + i\,v(x,y)$$

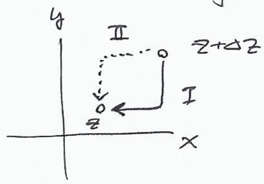
Theorem 1

f is analytic in a domain D. if and only if

$$u_x = v_y, \quad u_y = -v_x \quad : \text{Cauchy - Rieman Eq'n}.$$

proof)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



two paths I and II

$\Delta z = \Delta x + i\Delta y \qquad z+\Delta z = x+\Delta x + i(y+\Delta y)$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)] - [u(x,y) + i\,v(x,y)]}{\Delta x + i\Delta y}$$

① path I, $\Delta y \to 0$, first then $\Delta x \to 0$.

after $\Delta y = 0$, $\Delta z = \Delta x$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= u_x + i v_x$$

Two must be the same.

② Similarly, path II, $\Delta x \to 0$, → $\Delta y \to 0$.

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$f'(z) = -i u_y + v_y$$

from previous example.

Ex 1)

$f(z) = z^2 = (x^2 - y^2) + 2xy\,i \qquad u_x = 2x,\ -u_y = 2y \qquad v_y = 2x,\ v_x = 2y \qquad \left(\begin{matrix} u_x = v_y \\ -u_y = v_x \end{matrix}\right)$ okay analytic.

$f(z) = \bar{z} = x - iy \qquad u_x = 1,\ u_y = 0 \qquad v_y = -1,\ v_x = 0. \qquad u_x \neq -v_y \Rightarrow$ not analytic

Theorem 2. Cauchy - Rieman Eq'n. proof is optional

If $u(x,y)$, $v(x,y)$ have continuous first partial derivatives and satisfy the Cauchy-Rieman Eq'n in some domain D. Then complex function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D.

Ex 2) $f(z) = e^x(\cos y + i \sin y)$ analytic ?

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y = v_y = e^x \cos y$

$u_y = -e^x \sin y \quad -u_y = v_x \quad v_x = e^x \sin y$

Ex3) if $f(z)$ is analytic in a domain D

$f(z)=u+iv$ $\quad |f(z)| = k = \text{const}$ in D.

then $f(z) = \text{const}$ in D.

proof) $|f|^2 = |u+iv|^2 = u^2+v^2 = k^2$

w.r.t.

differentiation (x) $uu_x + vv_x = 0$

(y) $uu_y + vv_y = 0$.

$v_x = -u_y$, $\quad v_y = u_x$

$$
\begin{array}{lr}
(uu_x - vu_y = 0) & \times u \\
+\ (uu_y + vu_x = 0) & \times v \\
\hline
(u^2+v^2)u_x = 0 & \\
(u^2+v^2)u_y = 0 &
\end{array}
$$

To get rid of u_y

u_x 제거

마찬가지로 $\begin{pmatrix} \times \leftrightarrow v \\ \times \ \ u \end{pmatrix}$

$u_x = u_y = 0$, $\Rightarrow$ $u_x = v_y = 0$

$u_y = -v_x = 0$.

$u = \text{const}$ and $v = \text{const}$. hence $f = \text{const}$

(13.4) #1

① Cauchy - Rieman Eq'n in polar form

$$
\left.
\begin{aligned}
u_r &= \frac{1}{r} v_\theta \\
v_r &= -\frac{1}{r} u_\theta
\end{aligned}
\right)
$$

homework

Laplace Eq'n Harmonic functions.

Theorem 3) if $f(z) = u(x,y) + iv(x,y)$ is analytic.

Then both u. v satisfy Laplace's equation. ✳

gravitation

$$\nabla^2 u = u_{xx} + u_{yy} = 0.$$

$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

$\Rightarrow$ v is said to be a harmonic conjugate function of u in D.

proof) $u_x = v_y$ $\qquad$ $u_y = -v_x$

x D'ff. $\downarrow$ $\qquad$ $\downarrow$ y Diff'

$u_{xx} = v_{yx}$ $\qquad$ $u_{yy} = -v_{xy}$

$$\left[\begin{aligned} \frac{\partial u}{\partial t} &= c^2 \nabla^2 u \\ \nabla^2 u &= 0 \end{aligned} \right]$$

Solution = Harmonic function

$\downarrow$

$\nabla^2 u = 0$

since. $v_{yx} = v_{xy}$ $\qquad$ $\therefore$ $u_{xx} + u_{yy} = 0$.

like wise $\qquad$ $\therefore$ $v_{xx} + v_{yy} = 0$. $\quad \nabla^2 v = 0$.

Ex4) How to find a harmonic conjugate function by the Cauchy - Riemann Eq'n.

$u = x^2 - y^2 - y$, $\qquad$ ① harmonic $\quad$ v

② harmonic conjugate function : v of u.

$\nabla^2 u = 0$. $\qquad \begin{pmatrix} u_x = 2x \\ u_y = -2y-1 \end{pmatrix}$

$\therefore$ $v_y = u_x = 2x \longrightarrow v = 2xy + h(x)$

$v_x = -u_y = 2y+1 \longrightarrow$ $\qquad \downarrow$

$v_x = 2y + h'(x)$

$\dfrac{dh}{dx} = 1$ $\quad h = x + c$

$\therefore f(z) = (x^2-y^2-y) + i(2xy+x+c)$, $\qquad \therefore v = 2xy + x + c$.

$= z^2 + iz + ic$

13.5 Exponential Function

$$e^z = e^x(\cos y + i \sin y)$$

$\to e^x e^{iy}$ Euler Formula

$re^{i\theta}$

$|e^z| = e^x$

$\arg e^z = y \pm 2n\pi \quad (n = 0, 1, 2 \ldots)$ (z: infinitely many solutions)

* 비슷하나, 유일한 차이.

$-\pi < y \le \pi$: horizontal strip of width 2π

fundamental region of e^z

$e^{z+2\pi i} = e^z$ for all z ($2\pi i n$)

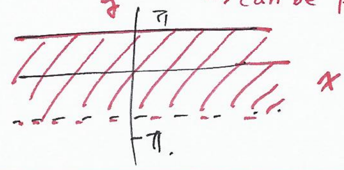
<periodicity of e^x with period $2\pi i$>

e^z extends the real exponential function e^x of calculus in a natural fashion

i) $e^z = e^x$ for real $z = x$, $\cos y = 1$

ii) e^z is analytic. for all z

iii) $(e^z)' = e^z$, $e^{z_1+z_2} = e^{z_1}e^{z_2}$ ↳ can be proved



13.6 Trigonometric and Hyperbolic Function. Euler Formula

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

suggests the following definitions

$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

for $z = x + yi$ ~~by definition~~

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$\tan z = \frac{\sin z}{\cos z}$ $\quad (\cos z)' = -\sin z$

Euler's formula is valid in complex.

$[e^{iz} = \cos z + i \sin z]$

Ex1) show that

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\cos z = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \frac{1}{2}e^{-y}(\underbrace{\cos x + i\sin x}_{e^{ix}}) + \frac{1}{2}e^{y}(\underbrace{\cos x - i\sin x}_{e^{-ix}})$

$= \underbrace{\frac{1}{2}(e^{y} + e^{-y})}_{\cosh y}\cos x - \underbrace{\frac{1}{2}i(e^{y} - e^{-y})}_{\sinh y}\sin x$

$= \cosh y \cos x - i \sin x \sinh y$

Ex2) $\cos z = 5$ (?) → no real solution! but complex ~~Number~~ Solution is possible.

$\frac{1}{2}(e^{iz} + e^{-iz}) = 5 \Rightarrow e^{2iz} - 10e^{iz} + 1 = 0$

$(e^{iz}) = (e^{-y+xi}) = 5 \pm \sqrt{25-1} = 9.899$ & 0.101

$\therefore e^{-y} = 9.899$ or 0.101

↳ $y = \pm 2.292$

$e^{ix} = 1$ only real part

↳ $x = \pm 2n\pi$

$\therefore z = \pm 2n\pi \pm 2.292 i \quad (n = 0, 1, 2 \ldots)$

General formula. hold for complex values

$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

$\cos^2 z + \sin^2 z = 1$

$\cos z = 0$

$\cos z = \underset{0}{\cos x}\,\underset{\neq 0}{\cosh y} - i \sin x \underset{0}{\sinh y}$

$\frac{\pi}{2}, \frac{3}{2}\pi \ldots$

$\cos x = 0 \quad \sinh y = 0 \quad y = 0$

↳ $\pm\frac{1}{2}(2n+1)\pi$

Hyperbolic functions

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$\left\{ \begin{array}{l} (\cosh z)' = \sinh z \\ (\sinh z)' = \cosh z \end{array} \right\}$$

$$\boxed{\cosh z = \frac{1}{2}(e^z + e^{-z})} \Rightarrow \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z \qquad \underline{\cosh iz} = \underline{\cos z}$$

< related >

$$\left\{ \begin{array}{ll} \cosh iz = \cos z & \sinh iz = i \sin z \\ \cos iz = \cosh z & \sin iz = i \sinh z \end{array} \right\} ?$$

more complicated than real Logarithm

13.7 Logarithm. General Power. Principal Value.

$w = \ln z$ for $z = x + yi$ is defined for $z \neq 0$

$e^w = z$ → $w = u + iv$, $z = re^{i\theta}$

$$e^w = e^{u+iv} = re^{i\theta} = z \quad \left(\begin{array}{l} e^u = r \Rightarrow u = \ln r \\ v = \theta \end{array} \right)$$

$\ln z = u + iv$

$$\left(\begin{array}{l} \therefore w = u + iv \\ \ln z = \ln r + i\theta \end{array} \right)$$

$$\ln z = \ln r + i\theta. \quad \left(\begin{array}{l} \theta = \arg z \\ r = |z| > 0 \end{array} \right)$$

↳ $\theta = \arg z$ is infinitely many

since the argument of z is determined only up to integer multiples of 2π, the complex natural logarithm $\ln z$ is infinitely many-valued.

principal value of $\ln z$

$$\boxed{\mathrm{Ln}\, z = \ln|z| + i\, \mathrm{Arg}\, z \quad (z \neq 0)}$$

vs. $\ln z = \ln|z| + i \arg z$

since the other values of $\arg z$ differ by integer multiples of 2π

$$\langle \ln z = \mathrm{Ln}\, z \pm 2n\pi i \rangle \quad (n = 1, 2 \ldots)$$

(real part are same, imaginary part differ by integer multiples of 2π)

For example, ① z is positive real, $\mathrm{Arg}\, z = 0$

② If z is negative real then $\mathrm{Arg}\, z = \pi$

$$\mathrm{Ln}\, z = \ln|z| + \pi i \quad (\text{from } \mathrm{Ln}\, z = \ln|z| + i\, \mathrm{Arg}\, z)$$

$$\ln e^z = z \pm 2n\pi i \qquad (n = 0, 1, \ldots)$$

$\ln(z_1 z_2) = \ln z_1 + \ln z_2$, $\quad \ln(z_1/z_2) = \ln z_1 - \ln z_2$

✓ $\ln 1 = 0, \pm 2\pi i, \pm 4\pi i \ldots$ $\qquad \mathrm{Ln}\, 1 = 0$

✓ $\ln 4 = 1.386294 \pm 2n\pi i$ $\qquad \mathrm{Ln}\, 4 = 1.386294$

✓ $\ln(-1) = \pm\pi i, \pm 3\pi i$ $\qquad \mathrm{Ln}(-1) = \pi i$

$\ln i = \frac{\pi}{2} i, -\frac{3\pi}{2} i$ $\qquad \mathrm{Ln}(i) = \pi i/2$

Ex 2) Let $z_1 = z_2 = e^{\pi i} = -1$

$3-4i$ $\quad \tan\theta = -\frac{4}{3}$

$\ln(3-4i)$ − 0.9

$= \ln 5 + i \arg(3-4i) = 1.609438 - 0.927295i$

$\pm 2n\pi i$

Theorem 1. Analyticity of the Logarithm.

For every $n = 0, \pm 1, \pm 2 \ldots$, $(\ln z = \mathrm{Ln}\, z \pm 2n\pi i)$, which is analytic, $\boxed{\text{except at } 0, \text{ and on the negative real axis,}}$ has the derivative

$$(\ln z)' = \frac{1}{z}$$

analyticity check (Cauchy-Rieman equation satified?)

$u + iv$

proof) $\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln(x^2+y^2) + i\left(\arctan\frac{y}{x} + c\right)$

$$u_x = \frac{x}{x^2+y^2} = v_y = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2+y^2} = -v_x = -\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)$$

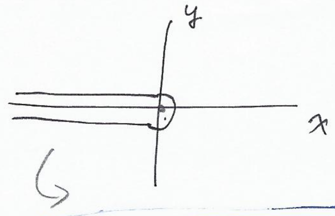
→ Cauchy-Riemann eq'n holds.

$$\therefore (\ln z)' = u_x + i v_x = \frac{x}{x^2+y^2} + i\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2+y^2} = \frac{1}{z}$$

branch

branch cut : negative real axis

principal branch : The branch for $n=0$.



$\boxed{z: \text{not } 0 \text{ or negative real}}$

General Powers.

$z = x+iy$.

$$\boxed{z^c = e^{c \ln z}}$$

$$z^c = e^{c \operatorname{Ln} z} \text{ : principal value of } z^c.$$

If $c = n = 1, 2, \ldots$

If $c = -1, -2, \ldots$ Similar.

If $c = 1/n$. $z^c = \sqrt[n]{z} = e^{(1/n)\ln z}$

Ex 3) General Power

$$\ln|z| + i\arg z = i\frac{\pi}{2} + \pm 2n\pi i \quad i\frac{\pi}{2} = (\operatorname{Ln} z) \quad \ln z$$

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left(i\left(\frac{\pi}{2}i \pm 2n\pi i\right)\right)$$

$$= e^{-(\pi/2) \mp 2n\pi}$$

all real : principal value $(n=0)$

$$e^{-\pi/2}$$

Similarly,

$$(1+i)^{2-i} = \exp((2-i)\ln(1+i)) = \exp\left[(2-i)\left[\ln|z| + i\arg z \pm 2n\pi i\right]\right]$$

$\ln|z| = \ln\sqrt{2}$, $\arg z = \frac{\pi}{4}$

$$\therefore \exp\left[2\ln\sqrt{2} + \frac{\pi}{4} \mp 2n\pi + i\left(-\ln\sqrt{2} + \frac{\pi}{2} \pm 4n\pi\right)\right]$$

$e^{\ln 2^{-\frac{1}{2}}}$

$\left((2)^{-\frac{1}{2}} + \frac{\pi}{2}\right)i$

$$2 \cdot e^{(\pi/4 \pm 2n\pi)}\left[\cos\left(\frac{\pi}{2} + \ln 2^{-\frac{1}{2}}\right)\right.$$

$\rightarrow -\sin\left(-\frac{1}{2}\ln 2\right)$

$+\cos$

$\sin\left(\frac{1}{2}\ln 2\right)$

$$\left. + i\sin\left(\frac{\pi}{2} + \ln 2^{-\frac{1}{2}}\right)\right]$$

$\cos\left(\frac{1}{2}\ln 2\right)$

$+ i\cos\left(\frac{1}{2}\ln 2\right)$

h.w	by 11/29
13.1	13, 14, 18
13.2	15, 22, 29
13.3	21
13.4	1, 15, 22
13.5	2, 13, 16
13.6	9, 11
13.7	8, 19, 25

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\theta}$

$\frac{1}{2}\ln\left(\frac{1}{2}\right)^{\frac{1}{2}}$

$$\left[\cos\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right) + i\sin\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right)\right]$$

$$= \sin\frac{1}{2}\ln 2 + i\cos\frac{1}{2}\ln 2$$

$2^{-\frac{1}{2}}$

$\cos\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right)$

$-\sin\left(-\frac{1}{2}\ln 2\right) = \sin(=$

$$\cos\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right) + i\sin\left(\frac{\pi}{2} - \frac{1}{2}\ln 2\right)$$

$$\sin\left(\frac{1}{2}\ln 2\right) + \cos\left(\frac{1}{2}\ln 2\right)$$