Chap. 14. Complex Integration. <Cauchy integral theorem.> 14-9

14.1 Line Integral in the Complex Plane.

$$\int_C f(z)\,dz \qquad \oint_C f(z)\,dz$$

indefinite integral. closed path.

① 1st evaluation method: Indefinite integration and substitution of limits

Theorem 1

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

$f(z) = F'(z)$

indefinite integration of analytic functions.

② 2nd Evaluation method: Use of a representation of a path.

integration by the use of the path.

$$\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3$$

Theorem 2)

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,\dot{z}(t)\,dt$$

C: representation by $z = z(t)$, $a \le t \le b$.

$z = x + iy$. $\dot{z} = \dot{x} + i\dot{y}$

$$\int_a^b f(z(t))\,\dot{z}\,dt = \int_a^b \underbrace{(u+iv)}_{f(z)}(\dot{x}+i\dot{y})\,dt$$

$\dot{x}\,dt = dx \quad \dot{y}\,dt = dy$

$$= \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C [u\,dx - v\,dy] + i\int_C (u\,dy + v\,dx)$$

(A) represent the path C in the form $z(t)$ $a \le t \le b$

(B) Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$

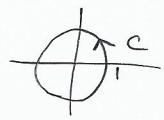
(C) Substitute $z(t)$ for every z in $f(z)$

(D) Integrate $f[z(t)]\,\dot{z}(t)$ over t

Ex5)

$$\oint_C \frac{dz}{z} \qquad C: \text{unit circle}$$

$|z| = 1$



$|z| = 1$

a) $z(t) = \cos t + i\sin t = e^{it} \qquad 0 \le t \le 2\pi$

b) $\dot{z}(t) = ie^{it}$

c) $f(z(t)) = 1/z(t) = e^{-it}$

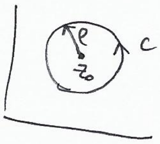
D)

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} \cdot ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

$$\therefore \oint_C \frac{dz}{z} = 2\pi i$$

very important

EX 6



$f(z) = (z - z_0)^m \qquad \oint_C f(z) = ?$

$z(t) = z_0 + \rho(\cos t + i \sin t) \qquad 0 \le t \le 2\pi$

$= z_0 + \rho e^{it}$

$\therefore (z - z_0)^m = \rho^m e^{imt}$

$dz = i\rho e^{it} \cdot dt$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} \cdot i\rho e^{it} dt = i\rho^{m+1} \underbrace{\int_0^{2\pi} e^{i(m+1)t} dt}$$

$$= i\rho^{m+1}\left[\int_0^{2\pi} \cos(m+1)t\,dt + i\int_0^{2\pi} \sin(m+1)t\,dt\right] \quad \text{Euler formula.}$$

if $m = -1$: $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$. $\longrightarrow 2\pi i$

if $m \ne -1$: $0 \quad (\because \int_0^{2\pi} \cos = \int_0^{2\pi} \sin = 0)$

$$\therefore \oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \ne -1 \text{ and integer}) \end{cases}$$

Dependence on path. : Generally path dependence of complex line integral.

Ex7) $f(z) = \mathrm{Re}\, z = x$ from 0 to $1+2i$

i) C^* : $z(t) = t + 2it \quad 0 \le t \le 1.$

$\dot{z}(t) = 1 + 2i$

$f(z(t)) = x(t) = t$

$$\int_{C^*} \mathrm{Re}\, z \cdot dz = \int_0^1 t(1+2i)\,dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i$$

ii) C_1 : $z(t) = t \qquad \dot{z}(t) = 1 \qquad f(z(t)) = x(t) = t. \qquad 0 \le t \le 1$

C_2 : $z(t) = 1 + it. \qquad \dot{z}(t) = i \qquad f(z) = x(t) = 1 \qquad 0 \le t \le 2$

$$\int_C \mathrm{Re}\, z\, dz = \int_{C_1 0}^{1} t\,dt + \int_{C_2 0}^{2} i\,dt = \quad = \frac{1}{2} + 2i$$

differ.

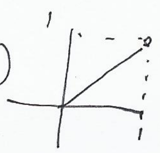
Bounds for integrals. ML-inequality.

$$\left|\int_C f(z)\,dz\right| \le ML.$$

L : length of C

M : a constant such that $|f(z)| \le M$.

Ex8)



$\int_C z^2 \cdot dz \qquad \sqrt{2} = L. \qquad |f(z)| = |z^2| \le 2$

14.2 Cauchy's Integral Theorem. (가장 중요한 part of this chapter)

시험 要

- Simple closed path = a closed path that does not intersect or touch itself as shown in Fig.

  (Simple vs not simple.)

- Simply connected domain D : vs multiply connected domain

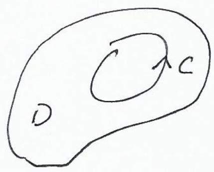
  vs. doubly, triply.

Theorem 1. Cauchy's Integral Theorem.

If $f(z)$ is analytic in a simply connected domain D,

for every simple closed path C in D → also called contour ∴ contour integral

$$\oint_C f(z)\,dz = 0.$$



proof) From (8) in sec 14.1 — assumption. " $f'(z)$ is continuous "

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$

↓ Green's Theorem ⇒ $\left(\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1 dx + F_2 dy) \right)$

∴ $F_1 = u$, $-v = F_2$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$(-v_x - u_y)$ 인 $(= 0$ ∵ Due to second Cauchy–Riemann equation)

$-v_x - u_y$ ($u_y = -v_x$).

in the same way

$$\oint_C (u\,dy + v\,dx) = \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) dx\,dy = 0$$

$(v_y - u_x)$ due to first C-R equ. $u_x = v_y$

Ex 1) $\oint_C e^z\,dz = 0$ $\quad \oint_C \cos z\,dz = 0$ $\quad \oint_C z^n\,dz = 0$ for any closed path.

since this functions are entire (or analytic for all z)

Ex 2) $\oint_C \sec z\,dz = 0$ $\quad \oint_C \frac{dz}{z^2+4} = 0.$ ⇒ C : unit circle"

Ex 3) $\oint_C \frac{1}{\cos z}$ ⇒ $\cos z = 0$ not analytic at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \cdots$, but outside C therefore okay

$\oint_C \frac{dz}{z^2+4}$ $\quad z = \pm 2i$ analytic, but outside C ∴ $\oint \frac{dz}{z^2+4} = 0$

Ex 4) Nonanalytic function

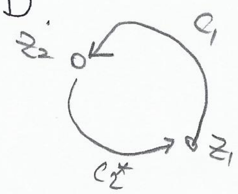
$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it}\, i e^{it}\,dt = 2\pi i$$, not applied Cauchy's integral theorem

Ex 5) $\oint_C \frac{dz}{z^2} = \int_0^{2\pi} e^{-2it}\, i e^{it}\,dt = i\int_0^{2\pi} e^{-it}\,dt = -e^{-it}\Big|_0^{2\pi} = 0$ $\quad$ (0 이라고 해서 analytic 이 아니다.)

(more more example in the back) However, $\frac{1}{z^2}$ is not analytic at $z=0$, ... f to be analytic in D is sufficient

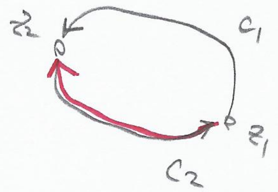
Independence of Path.

If $f(z)$ is analytic in a simply connected domain D,
then the integral of $f(z)$ is independent of path in D.



$$\oint_C f(z) = \int_{C_1} f(z)\,dz + \int_{C_2^*} f(\,dz = 0,$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Principle of Deformation of path.

(As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value.)

Existence of Indefinite integral.

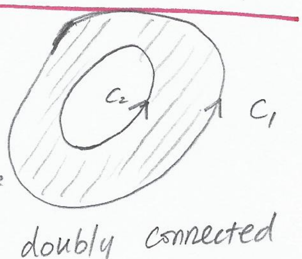
Theorem 3 If $f(z)$ is analytic in a simply connected domain

applied to multiply connected domains.

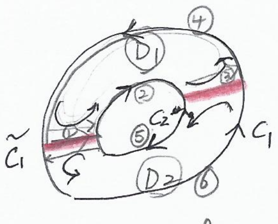
Cauchy's Integral Theorem for Multiply Connected domains.

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

both integral being taken counterclockwise



doubly connected domain.



proof $\int_{D_1} = 0$ $\quad \int_{D_2} = 0.$

$① + ② + ③ + ④ = 0$ A $\quad ⑥ - ③ + ⑤ - ① = 0$ B

(A)+(B) $\quad ② + ④ + ⑤ + ⑥ = 0$

$C_1 = ④ + ⑥$

$-C_2 = ② + ⑤$

$\therefore C_1 - C_2 = 0$

$C_1 = C_2$ //

Independence of Path

: < generally path dependent ! > $\int_{C_1} f(z)dz \neq \int_{C_2} f(z)dz$

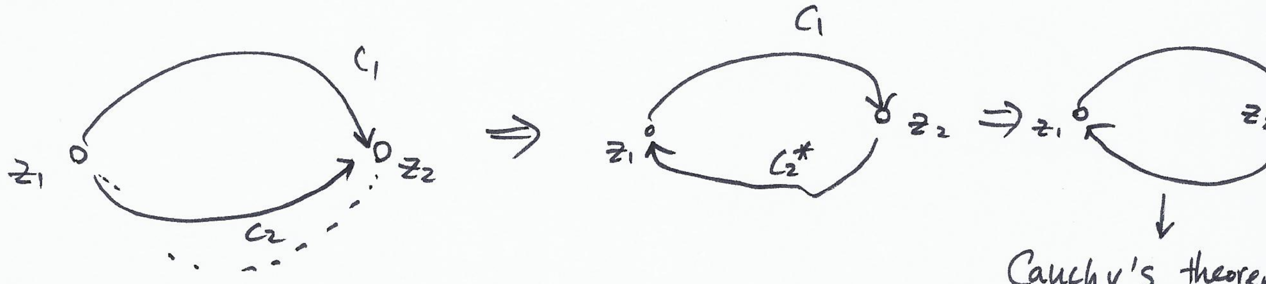
If $f(z)$ is analytic in a simply connected domain D,

then the integral of $f(z)$ is independent of path in D.

$$\oint_C f(z)\,dz = 0 \quad (\because \text{analytic})$$

Let z_1 and z_2 be any points in D.

consider two paths C_1 and C_2 from z_1 to z_2



Cauchy's theorem

$$\therefore \int_{C_1} f\,dz = (-)\int_{C_2^*} f\,dz \longleftarrow \int_{C_1} f\,dz + \int_{C_2^*} f\,dz = 0 \leftarrow \oint_C f(z)\,dz =$$

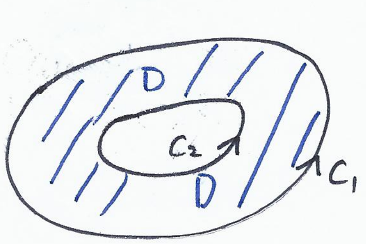
direction change

$$\int_{C_1} f\,dz = \int_{C_2} f\,dz$$

Principle of Deformation of Path

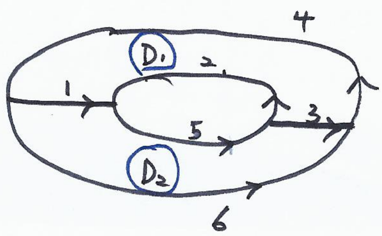
As long as deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value.

Cauchy's Integral Theorem for multply connected domains



$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

both integral being taken counterclockwise.



$$\int_{D_1} f(z)\,dz = 0 \qquad \int_{D_2} f(z)\,dz = 0.$$

$$\int_{1-2+3+4} f(z)\,dz = 0 \qquad \int_{6-3-5-1} f(z)\,dz = 0$$

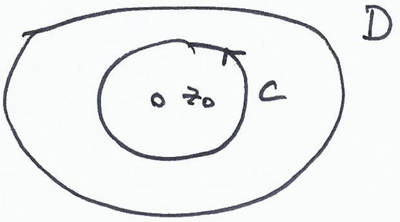
A + B

$$\int_{-2+4+6-5} f(z)\,dz = 0 \qquad \int_{4+6} f(z)\,dz = \int_{2+5} f(z)\,dz$$

⇒ Can be extended to triply

14.3 Cauchy's Integral formula. ⟵ Cauchy's integral theorem

Let $f(z)$ be analytic in a simply connected domain, for any point z_0 in D & any simple closed path C in D that encloses z_0



$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0),$$

or $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

proof) $$f(z) = f(z_0) + [f(z) - f(z_0)]$$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \underbrace{\oint_C \boxed{\frac{f(z)-f(z_0)}{z-z_0}}\,dz}_{}$$

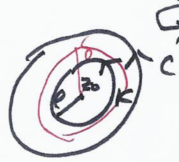
$= 2\pi i f(z_0)$ $\qquad$ $=0$ 임을 증명해야 함.

$$\oint_K \frac{f(z)-f(z_0)}{z-z_0}$$ $\parallel$

~~→ analytic 임을 증명~~

→ contour C 보다 작은 K 를 도입함.

①



with radius ρ and center z_0.

ML- inequality

$$\left|\int_C f(z)\,dz\right| \le ML$$

L : length of C

M : a constant such that $|f(z)| \le M$

Hence an $\varepsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for all z in the disk $|z - z_0| < \delta$. $\qquad (\rho < \delta)$

Choosing the radius ρ of K smaller than δ.

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho} \cdot 2\pi\rho$$

$|z - z_0| < \rho < \delta$

$$\left|\oint_K \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| = \frac{\varepsilon}{\rho}\cdot 2\pi\rho = 2\pi\varepsilon.$$

<좀더 쉬운 proof ?>

Since ε can be chosen arbitrarily small, $|\ | \le 0$. $\therefore = 0$

Ex 1

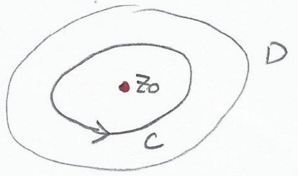
Ex 2

Ex 3.

14.3 Cauchy's Integral formula. ← Cauchy's integral theorem.

(Theorem 1) Let $f(z)$ be analytic, in a simply connected domain, for any point z_0 in D and any closed path C in D that encloses z_0

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)$$ Cauchy's integral formula.



or $$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz.$$

proof) $f(z) = f(z_0) + [f(z) - f(z_0)]$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \oint_C \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{\text{analytic except } z=z_0}\,dz$$

$= f(z_0)\cdot 2\pi i$ ↓ 앞에서.

$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$

$$\boxed{\oint_C \frac{dz}{z-0} = 2\pi i f(0) = 2\pi i}$$

그러므로. C 보다 작은 contour K of radius ρ and center z_0.

그러므로 an $\varepsilon>0$ being given, we can find a $\delta>0$ - such that $|f(z)-f(z_0)|<\varepsilon$ for all z in the disk $|z-z_0|<\delta$. ($\rho<\delta$)

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}$$

integration $\oint_K \left|\frac{f(z)-f(z_0)}{z-z_0}\right| dz < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon$

Ex 1. $$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i\, e^z\Big|_{z=2} = 2\pi i e^2$$ $f(z)=e^z$

(C encloses 2) $2\pi i f(z_0) = 2\pi i f(2)$

for $z_0 = 2$ is outside C. 0.

Ex 2 $$\oint_C \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}\,dz = 2\pi i\, f\left(\tfrac{1}{2}i\right) = 2\pi i\left(\tfrac{1}{2}\left(\tfrac{1}{2}i\right)^3-3\right)$$

↳ if c enclose $\frac{1}{2}i$ $f(z) = \frac{z^3}{2}-3$ $\left(-\frac{1}{2}\frac{1}{8}i\right)$

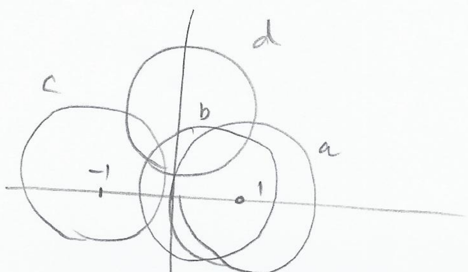
$= \frac{\pi}{8} - 6\pi i$

Ex 3. $$\oint_C \left[g(z) = \frac{z^2+1}{z^2-1}\right] dz$$ where 4 C exists (Fig 358)

[not analytic at $z=1, -1$]

(a) C : $|z-1|=1$ enclosed the point $z_0=1$, where $g(z)$ is not analytic

$$\oint_C \frac{(z^2+1)/z+1}{z-1}\,dz = 2\pi i f(1) = 2\pi i$$ $f(z)$



(b) same

(c) $$\oint_C \frac{(z^2+1)/z-1}{z+1}\,dz = 2\pi i f(-1) = -2\pi i$$

(d) 0

14.4 Derivatives of Analytic Functions.

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\cdot dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\, dz$$

$$\cdots$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\cdot dz.$$

C: any simple closed path in D that encloses z_0 and whose full interior belongs to D.

proof)

$$f'(z_0) = \lim_{\Delta z \to 0} \boxed{\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}}$$

Cauchy's integral formula

$$f(z_0+\Delta z) \qquad \boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz}$$

$$\to \frac{1}{2\pi i\,\Delta z}\left[\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\, dz - \oint \frac{f(z)}{z-z_0}\, dz\right]$$

$$\frac{1}{2\pi i\,\Delta z}\oint_C \frac{\cancel{zf(z)} - \cancel{z_0 f(z)} - \cancel{zf(z)} + \cancel{z_0 f(z)} + \Delta z f(z)}{(z-z_0-\Delta z)(z-z_0)}\, dz$$

$$\frac{1}{2\pi i\,\cancel{\Delta z}}\oint_C \frac{\cancel{\Delta z}\, f(z)}{(z-z_0-\Delta z)(z-z_0)}\, dz$$

as we take $\Delta z \to 0$.

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\, dz$$

. . . proof by induction

Ex1)

$$\oint_C \frac{\overset{f(z)}{\cos z}}{(z-\pi i)^2}\, dz = 2\pi i\, f'(\pi i) = -2\pi i \sin \pi i = -2\pi \sinh \pi$$

$-\sin \pi i$

any contour enclosing πi

Ex2)

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\, dz$$

$f(z) = 4z^3 - 6z$ $\quad f''(z) = 12z^2 - 6$

$$\frac{2\pi i f''(-i)}{2!} = \pi i(-12-6) = -18\pi i.$$

for any contour enclosing $-i$

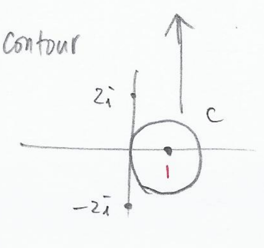
Ex3)

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\, dz = \oint_C \frac{f(z)}{(z-1)^2}\, dz$$

where $f(z) = \dfrac{e^z}{z^2+4}$

$\therefore\ 2\pi i f'(1)$

for any contour



Derivatives of analytic functions.

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

C : ~~any~~ simple closed path in D that encloses z_0

proof)

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

From Cauchy's integral formula.

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

$$\therefore \lim_{\Delta z \to 0} \frac{1}{\Delta z}\cdot\frac{1}{2\pi i}\Big[\overbrace{\oint_C \frac{f(z)}{z - z_0 - \Delta z}\,dz}^{f(z_0+\Delta z)} - \overbrace{\oint_C \frac{f(z)}{z-z_0}\,dz}^{f(z_0)}\Big]$$

$$= \lim \frac{1}{2\pi i \Delta z}\Big[\oint_C \frac{\cancel{zf(z)} - \cancel{z_0 f(z)} - \cancel{zf(z)} + \cancel{z_0 f(z)} + \Delta z f(z)}{(z - z_0 - \Delta z)(z-z_0)}\,dz$$

$$= \lim_{\Delta z \to 0} \frac{1}{2\pi i \cancel{\Delta z}}\oint_C \frac{\cancel{\Delta z} f(z)}{(z - z_0 - \Delta z)(z-z_0)}\,dz$$

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

proof by induction ......

Ex 1

Ex 2

Ex 3.

Cauchy's Inequality. Liouville's and Morera's Theorem.

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

C : circle of radius ~~and~~

$|f(z)| \leq M$

Liouville's Theorem.

If an entire function is bounded in an absolute value in the whole complex plane,

This function is constant.

Morera's Theorem.

If $f(z)$ is continuous in a simply connected domain D

if $\oint_C f(z)\,dz = 0$

for every closed path in D

then $f(z)$ is analytic in D.

14.1 26. 27

14.2 12. 22. 23. 25.

14.3 2. 13. 17

14.4

~~14.5~~ 3. 4 10. 13