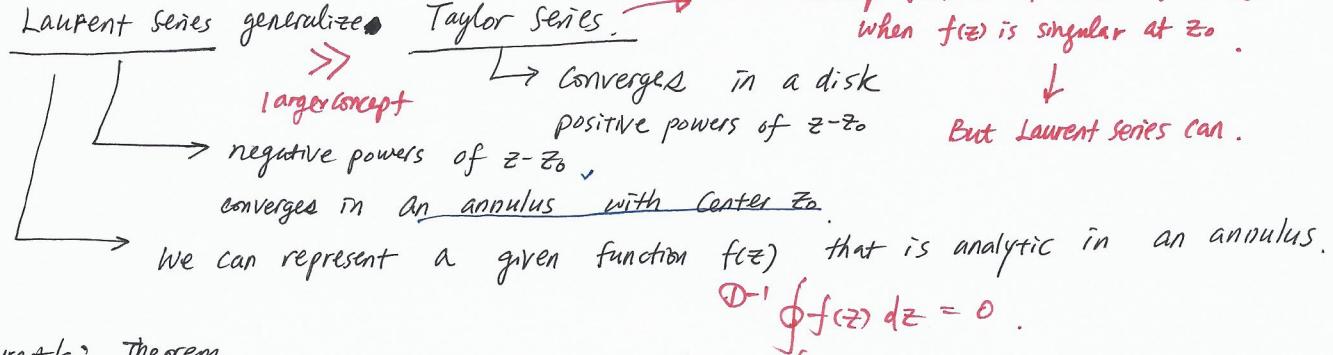
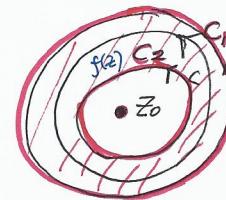


- i) Cauchy's integral formula  $\textcircled{1} f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z-z_0} dz^*$   
 ii) Another method for evaluating complex integrals and certain real integral : residue integration.



### Laurent's Theorem.

$f(z)$  be analytic in a domain containing two concentric circles  $C_1$  and  $C_2$  with center  $z_0$  and the annulus between them



(for path  $C$  between  $C_1$  and  $C_2$ )

Then 
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where  $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$  and

$$b_n = \frac{1}{2\pi i} \oint_C (z^*-z_0)^{n-1} f(z^*) dz^* \quad \text{(n=1 or even Residue integration)}$$

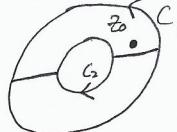
alternatively 
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where  $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$

[principal part of  $f(z)$  at  $z_0$ : series (or finite term) of the negative powers] where ( $n=0, \pm 1, \pm 2, \pm 3 \dots$ )

Proof.) (C) nonnegative powers : Taylor series.

from (3) in 14.3



$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z) dz}{z-z_0} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z) dz}{z-z_0}$$

Cauchy's integral formula

(point: skip)

$$\textcircled{1} f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$$

Cauchy's integral theorem

$$\therefore f(z) = g(z) + h(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^*-z} dz^* - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z^*)}{z^*-z} dz^*$$



$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

integrate counterclockwise over both  $C_1$  and  $C_2$ .

$z$  any point in the given annulus.

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^*-z} dz^* = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Taylor series. with coefficients  $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$

$C_1$  can be replaced by  $C$ .

$z_0$ , the point where the integrand ~~is~~ is not analytic

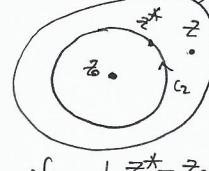
(b) negative powers.

16-2

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{where } b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*) dz^*$$

$$-\frac{1}{2\pi i} \oint_{C_2} \frac{f(z^*)}{z^* - z} dz^* \quad \text{where } z^* \text{ is in the annulus outside of the path } C_2.$$

$$\left| \frac{z^* - z_0}{z - z_0} \right| < 1$$



$\therefore \frac{1}{z^* - z} \rightarrow \text{develop in the powers of } \left| \frac{z^* - z_0}{z - z_0} \right|$

$$\therefore \frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \frac{z^* - z_0}{z - z_0}\right)} = \frac{-1}{z - z_0} \left(1 + \frac{z^* - z_0}{z - z_0} + \left(\frac{z^* - z_0}{z - z_0}\right)^2 + \dots + \left(\frac{z^* - z_0}{z - z_0}\right)^n\right)$$

$$- \frac{1}{z - z^*} \left(\frac{z^* - z_0}{z - z_0}\right)^{n+1}$$

multiplication  $- f(z^*)/2\pi i$  integration over  $C_2$

$$h(z) = - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z^*)}{z^* - z} dz^*$$

$$= \frac{1}{2\pi i} \left\{ \frac{1}{z - z_0} \oint_{C_2} f(z^*) dz^* + \frac{1}{(z - z_0)^2} \oint_{C_2} (z^* - z_0) f(z^*) dz^* + \dots + \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} (z^* - z_0)^n f(z^*) dz^* \right\} + R_n^*(z)$$

skip!

Ex 1)

Lauren series of  $z^{-5} \sin z$  with center  $0$  (not analytic)

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\therefore z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \underbrace{\frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040} z^2}_{\text{principal part}} \dots |z| > 0.$$

annulus of convergence

Ex 2)

$z^2 e^{1/z}$  with center  $0$ .

(not analytic)

$$e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z^n}\right)$$

$$\therefore z^2 e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2-n} = z^2 + z + \frac{1}{2} + \underbrace{\frac{1}{3!} \frac{1}{z} + \dots}_{\text{principal part}} |z| > 0.$$

is infinite series

Ex 3)

$\frac{1}{1-z}$  ① nonnegative power

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z| < 1 \quad \checkmark$$

② negative power

$$\frac{1}{1-z} = \frac{-1}{z(1-\frac{1}{z})} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = -\frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \dots \quad |z| > 1$$

$$\hookrightarrow |z| < 1$$

$$\text{Ex 4. } \frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)} \quad \text{from Ex 3.}$$

*skip*

$$\text{if } |z| < 1 \quad \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots$$

$$\text{if } |z| > 1 \quad -\frac{1}{z^4} - \frac{1}{z^5} - \dots$$

Ex 5. Taylor and Laurent series  $f(z) = \frac{-2z+3}{z^2-3z+2}$  with center 0

$$f(z) = \left(-\frac{1}{z-1} - \frac{1}{z-2}\right)$$

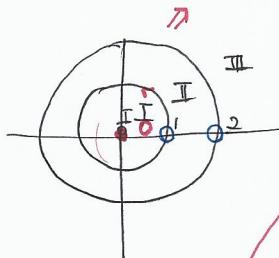
from Ex 3.

$$\sum_{n=0}^{\infty} z^n \quad |z| < 1 \quad \checkmark$$

$$\sum_{n=0}^{\infty} -\frac{1}{z^{n+1}} \quad |z| > 1 \quad \checkmark$$

$$-\frac{1}{z-2} = -\frac{1}{z(1-\frac{1}{2}z)} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad |z| < 2 \quad \checkmark$$

$$-\frac{1}{z-1} = -\frac{1}{z(1-\frac{1}{z})} = -\sum_{n=0}^{\infty} \frac{z^n}{z^{n+1}} \quad |z| > 2 \quad \checkmark$$



$$\text{I : } |z| < 1$$

$$\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n = \sum \left(1 + \frac{1}{2^{n+1}}\right) z^n \quad //$$

$$\text{II : } 1 < |z| < 2$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{z^{n+1}} + \frac{1}{2^{n+1}} z^n\right) = \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 \dots - \frac{1}{2} - \frac{1}{z^2} \dots$$

$$\text{III : } |z| > 2$$

$$\sum \left(-\frac{1}{z^{n+1}} - \frac{z^n}{z^{n+1}}\right) = -\sum \left(1 + z^n\right) \frac{1}{z^{n+1}}$$

all  $b_n$

$$\oint_C f(z) dz = 0$$

*analytic.*

If  $f(z)$  in Laurent theorem is analytic inside  $C_2$ ,

the coefficient  $b_n = 0$ . by Cauchy's integral theorem.

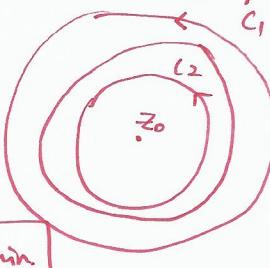
So that Laurent series becomes Taylor series.

For example.

$$\begin{cases} \text{Ex 5. I,} \\ \text{Ex 3. (a).} \end{cases}$$

$$b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*) dz^*$$

*analytic*  $\downarrow$   $\uparrow$  *analytic*



## 16.2 Singularity and Zeros. Infinity.

16-4

- i)  $f(z)$  is singular or has a singularity at a point  $z=z_0$  if  $f(z)$  is not analytic at  $z=z_0$ .
  - ii) isolated singularity if  $z=z_0$  has a neighborhood without further singularities of  $f(z)$ .
- $\tan z =$  isolated singularities at  $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$
- $\tan \frac{1}{z}$  : nonisolated singularities at  $z=0$ .

Isolated singularities of  $f(z)$  at  $z=z_0$  can be classified by the Laurent series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

principal part

valid in the intermediate

neighborhood of the singular point  $z=z_0$  except at  $z_0$  itself

$$0 < |z-z_0| < R$$

analytic at  $z=z_0$

i) If it has only finitely many terms.

$$\frac{b_1}{z-z_0} + \dots + \frac{b_m}{(z-z_0)^m} \quad (b_m \neq 0)$$

singularity of  $f(z)$  at  $z=z_0$  called

= a pole with the order  $m$ .

✓ 1st order pole : simple poles.

ii) infinitely many terms : isolated essential singularity

Ex1) i)  $f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2} \Rightarrow$  Simple pole at  $z=0$   
 5th order pole at  $z=2$ .

ii)  $e^{1/z} = \sum \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \dots$  isolated essential singularity at  $z=0$

$$\sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{2n+1}} = \frac{1}{z} - \frac{1}{3! z^3} + \frac{1}{5! z^5} - \dots$$

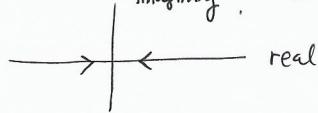
iii)  $z^{-5} \sin z = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040} z^2 + \dots$  : 4th order pole at  $z=0$ .

Ex2) Behavior near a pole

$$f(z) = \frac{1}{z^2} \quad \text{pole at } z=0. \quad \text{as } z \rightarrow 0, |f(z)| \rightarrow \infty$$

Theorem 4) Poles

If  $f(z)$  is analytic and has a pole at  $z=z_0$ ,  
 $|f(z)| \rightarrow \infty$  as  $z \rightarrow z_0$  in any manner.

Ex 3)  $e^{1/z}$ essential singularity at  $z=0$ .

$$e^{1/z} = \sum \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \dots$$

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as  $z \rightarrow 0^+$  :  $\infty$        $e^\infty$   
 $z \rightarrow 0^-$  :  $0$        $e^{-\infty}$

It takes on any given value  $c = c_0 e^{i\alpha} \neq 0$  in an arbitrarily small  $\epsilon$ -neighborhood of  $z=0$ .

**Skip.**

Set  $z = r e^{i\theta}$

$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha}$$

$$e^{\cos\theta/r} = c_0$$

$$\cos\theta = r \ln c_0$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$-\frac{\sin\theta}{r} = \alpha$$

$$-\sin\theta = \alpha r$$

$$r^2 \ln^2 c_0 + \alpha^2 r^2 = 1$$

$$\tan\theta = -\frac{\alpha}{\ln c_0}$$

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2}$$

Hence  $r$  can be made arbitrarily small by adding multiples of  $2\pi$  to  $\alpha$  leaving  $c$  unaltered.

→ Picard's Theorem.

If  $f(z)$  is analytic and has an isolated essential singularity at a point  $z_0$ , it takes on every value, (with at most one ~~exceptional~~ value,) in an arbitrarily small  $\epsilon$ -neighborhood of  $z_0$ .

**Removable singularities.**if  $f(z)$  is not analytic at  $z=z_0$ .but can be made analytic there by assigning a suitable value  $f(z_0)$ Ex.  $f(z) = \frac{\sin z}{z} \Rightarrow$  becomes analytic at  $z=0$  if we define  $f(0)=1$ .**Zeros of analytic function  $f(z)$  in a domain**: a  $z=z_0$  in  $D$  such that  $f(z_0)=0$ .

A zero has order  $n$   $\longleftrightarrow$   $f, f', f'' \dots f^{(n-1)}$  all 0 at  $z=z_0$  but  $f^{(n)}(z_0) \neq 0$ .

Ex) 2nd order zero,  $f(z_0) = f'(z_0) = 0, f''(z_0) \neq 0$ .

Ex 4. zeros.

 $1+z^2$  : simple zeros at  $\pm i$  $(1-z^4)^2$  : 2nd order zeros at  $\pm 1, \pm i$  ✓ $\sin z$  : simple zeros at  $0, \pm \pi, \pm 2\pi$ . ✓ $\sin^2 z$  : 2nd order zeros at  $0, \pm 2\pi, \pm 4\pi$ 

$$2\sin z \cos z = \sin 2z$$

Taylor Series at a zero.

at an nth-order zero  $z=z_0$  of  $f(z) \rightarrow f'(z_0) \dots f^{(n-1)}(z_0) = 0$ .

$$\rightarrow a_0 \dots a_{n-1} = 0$$

Theorem 3. zeros.

The zeros of an analytic function  $f(z) (\neq 0)$  are isolated  
→ each of them has a neighborhood that contains no further zeros of  $f(z)$ 

Theorem 4. poles and zeros.

Let  $f(z)$  be analytic at  $z=z_0$  and have a zero of nth order at  $z=z_0$ .Then  $1/f(z)$  has a pole of nth order at  $z=z_0$ .and so does  $h(z)/f(z)$ , provided that  $h(z)$  is analytic at  $z=z_0$  and  $h(z_0) \neq 0$ ,

✓ Riemann Sphere. Point at infinity.

✓ Analytic or singular at infinity.

Investigate a function  $f(z)$  for large  $|z|$ , set  $z=1/w$  $f(z) = f(1/w) \equiv g(w)$  in a neighborhood of  $w=0$ .We define  $f(z)$  to be analytic or singular at infinityif  $g(w)$  is analytic or singular, respectively at  $w=0$ .

$$g(0) = \lim_{w \rightarrow 0} g(w)$$

Ex 5) i)  $f(z) = \frac{1}{z^2}$  analytic at infinity (since  $g(w) = f(1/w) = w^2$  is analytic at  $w=0$ )→ 2nd order zero at  $\infty$ ii)  $f(z) = z^3$  singular at  $\infty$  and 3rd order pole since  $g(w) = f(1/w) = 1/w^3$  has a pole at  $w=0$ .iii)  $f(z) = e^z$  essential singularity at  $\infty$  since  $e^{1/w}$  has an essential singularity at  $w=0$ .

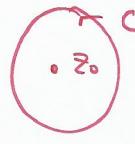
### 16.3 Residue integration method.

16-7

Purpose of Cauchy's residue integration method

is the evaluation of integrals

$$\oint_C f(z) dz$$



If  $f(z)$  has a singularity at a point  $z = z_0$  inside  $C$ , but is otherwise analytic on  $C$  and inside  $C$ .

Then  $f(z)$  has a Laurent series.

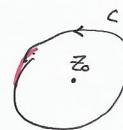
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

converges for all points near  $z = z_0$  (except  $z = z_0$  itself)  
 $0 < |z - z_0| < R$ .

$b_1$  = 1st negative power of  $\frac{1}{z - z_0}$  of the Laurent series.) from Laurent's Theorem

$$-b_1 = \frac{1}{2\pi i} \oint_C f(z) dz \quad \text{or } \left( \underset{z=z_0}{\operatorname{Res}} f(z) \right)$$

$$\oint_C f(z) dz = 2\pi i b_1$$



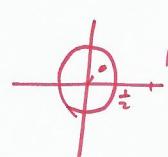
Ex1)  $f(z) = z^{-4} \sin z$  Counterclockwise around the unit circle  $C$ .

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!z} - \dots \quad |z| > 0$$

$f(z)$  has a pole of third order at  $z=0$ .

$$\text{residue } b_1 = -\frac{1}{3!} \quad \oint_C \frac{\sin z}{z^4} dz = 2\pi i b_1 = -\frac{\pi i}{3}$$

Ex2)  $f(z) = \frac{1}{z^3 - z^4}$  integrate clockwise around the circle  $|z| = \frac{1}{2}$   
 Singular at  $(z=0)$  and  $(z=1)$  outside  $C$   
inside  $C$ .



∴ need the residue at  $z=0$ .

from Laurent series that converges ~~at~~ for  $|z| < 1$

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \left(\frac{1}{z}\right) + \dots \quad (0 < |z| < 1)$$

$$\therefore -2\pi i \left[ \underset{z=0}{\operatorname{Res}} f(z) \right] = -2\pi i$$

1.

### Formulas for residues

Simple poles at  $z_0$ :

$$\underset{z=z_0}{\operatorname{Res}} f(z) = b_1 = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

or

$$\underset{z=z_0}{\operatorname{Res}} f(z) = \underset{z=z_0}{\operatorname{Res}} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

PROOF

① For a simple pole at  $z = z_0 \rightarrow f(z) = \frac{P(z)}{q(z)}$   $f(z) \rightarrow$  simple zero at  $z = z_0$

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \quad (0 < |z-z_0| < R)$$

$$\lim_{z \rightarrow z_0} (z-z_0)f(z) = b_1 + \lim_{z \rightarrow z_0} (z-z_0) [a_0 + a_1(z-z_0) \dots] = b_1$$

$$② f(z) = \frac{f(z_0)=0}{(z-z_0)} f'(z_0) + \frac{(z-z_0)^2}{z!} f''(z_0) + \dots \text{ and set } f = \frac{P}{q}$$

$$\text{Res } f(z) = \lim_{z \rightarrow z_0} (z-z_0) \frac{P(z)}{q(z)} = \lim_{z \rightarrow z_0} \frac{(z-z_0) P(z)}{(z-z_0) [f(z_0) + (z-z_0) f''(z_0)/z + \dots]} = \frac{P(z_0)}{f'(z_0)}$$

Ex 3) Residue at a simple pole

$$f(z) = (9z+i) / (z^3 + i) \quad i$$

$$① \text{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \lim_{z \rightarrow i} (z-i) \frac{9z+i}{z(z+i)(z-i)} = \frac{10i}{-2} = -5i \quad \checkmark$$

$$② \text{or} \text{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \lim_{z \rightarrow i} \frac{9z+i}{3z^2+1} = \frac{10i}{-2} = -5i \quad \checkmark$$

$$f(z) = \frac{b_m}{(z-z_0)^m} + \dots + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + b_1(z-z_0)^m$$

$$b_1 = \frac{1}{(m-1)!} g^{(m)}(z_0)$$

$$\text{coefficient of the power } (z-z_0)^{m-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

Poles of any order at  $z_0$  / at  $m$ th-order pole at  $z_0$

$$\text{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}$$

For 2nd order pole ( $m=2$ )

$$\text{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} \left\{ (z-z_0)^2 f(z) \right\}'$$

$$\text{ex) } f(z) = \frac{50z}{z^3 + 2z^2 - 7z + 4} = \frac{50z}{(z-1)^2(z+4)}$$

$$\text{Res}_{z=1} f(z) = \lim_{z \rightarrow 1} \frac{d}{dz} \left[ (z-1)^2 f(z) \right] = \lim_{z \rightarrow 1} \frac{d}{dz} \left( \frac{50z}{z+4} \right) = \frac{200}{5} = 8$$

$$\frac{50(2+4) - 50z}{(z+4)^2} = \frac{200}{(z+4)^2}$$

2nd order pole at  $z=1$  \checkmark

Several singularities inside the contour. Residue theorem.

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f(z)$$

↳ singular point inside  $C$ .

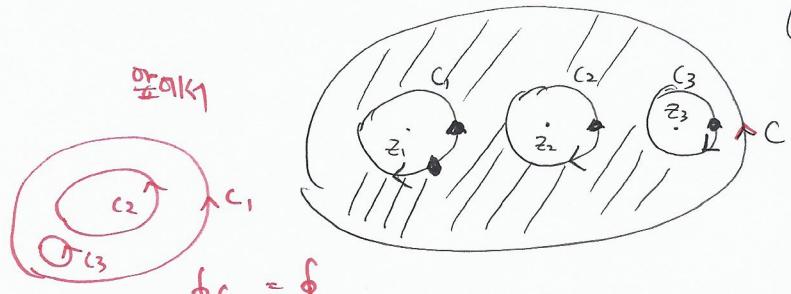
Cauchy's integral theorem.

$$\oint_C f(z) dz + \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \dots + \oint_{C_k} f(z) dz = 0$$

↓

Counterclockwise

$C_k$  : clockwise.



$$\oint_{C_1} = \oint_{C_2}$$

$$\therefore \oint_C f(z) dz = 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f(z)$$

$$\oint_C f(z) dz = \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \dots + \oint_{C_k} f(z) dz$$

Counterclockwise =  $\text{clockwise} = 2\pi i \text{Res } f(z)$

Ex 5.

$$\oint_C \frac{4-3z}{z^2-z} dz$$

4 cases

- i) C encloses 0, 1
- ii) 0
- iii) 1
- iv) 0 and 1 are outside.

$$2\pi i(-4+1) = -6\pi i$$

$$2\pi i(-4) = -8\pi i$$

$$2\pi i(1) = 2\pi i$$

16-9

$$\text{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left. \frac{4-3z}{z-1} \right|_{z=0} = -4$$

$$\text{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left. \frac{4-3z}{z} \right|_{z=1} = 1.$$

Ex 6.

$$\oint_C \frac{\tan z}{z^2-1} dz$$

C: counterclockwise  $|z| = \frac{3}{2}$

① Check singular points ✓ two possibilities  
 $\tan z: \pm \frac{\pi}{2}, \pm \frac{3\pi}{2} \dots$  not analytic., but outside C.

② check inside or outside C

③ Count the singular points inside C.

$$\oint_C \frac{\tan z}{z^2-1} dz = 2\pi i \left( \text{Res}_{z=1} \frac{\tan z}{z^2-1} + \text{Res}_{z=-1} \frac{\tan z}{z^2-1} \right) = 2\pi i \left( \left. \frac{\tan z}{2z} \right|_{z=1} + \left. \frac{\tan z}{2z} \right|_{z=-1} \right)$$

$$= 2\pi i \tan \frac{\pi}{2}$$

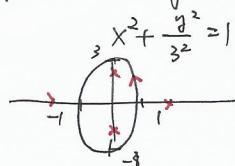
Ex 7.

$$\oint_C \left( \frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z} \right) dz$$

1st term

$$z^4-16=0 \quad \stackrel{\pm 2\bar{z}}{\uparrow}, \quad \text{inside } C \quad \rightarrow \text{no interest.}$$

C: ellipse  $4x^2+y^2=9$  counterclockwise.



$$\text{Res}_{z=2i} \left( \frac{ze^{\pi z}}{z^4-16} \right) = \left. \frac{ze^{\pi z}}{4z^3} \right|_{z=2i} = -\frac{1}{16}$$

$$\text{Res}_{z=-2i} \left( \frac{ze^{\pi z}}{z^4-16} \right) = \left. \frac{ze^{\pi z}}{4z^3} \right|_{z=-2i} = -\frac{1}{16}$$

2nd term

$$ze^{\pi/z} = z \left( 1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \dots \right)$$

$$= z + \pi + \left( \frac{\pi^2}{2} \right) \frac{1}{z} + \frac{\pi^3}{6} \frac{1}{z^2} + \dots$$

$$2\pi i \left( -\frac{1}{16} - \frac{1}{16} + \frac{\pi^2}{2} \right)$$

# 16.4 Residue Integration of Real Integrals.

16-10

$$J = \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$$

*substitution.*

$$\begin{aligned} z &= e^{i\theta} \\ \cos \theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z}) \\ \sin \theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z}) \end{aligned}$$

$$\frac{dz}{d\theta} = ie^{i\theta} = iz \quad (d\theta = \frac{dz}{iz})$$

$$\rightarrow J = \oint_C f(z) \frac{dz}{iz} \quad \underline{C: \text{unit circle.}}$$

$$\text{Ex 1)} \quad \int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos \theta} = 2\pi$$

$$\cos \theta = \frac{1}{2}(z + \frac{1}{z}) \quad d\theta = \frac{dz}{iz}$$

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}(z + \frac{1}{z})} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)} = -\frac{2}{i} \oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}$$

$$z = \sqrt{2} + 1, \quad \sqrt{2} - 1$$

↑  
no interest

(∴ outside C)

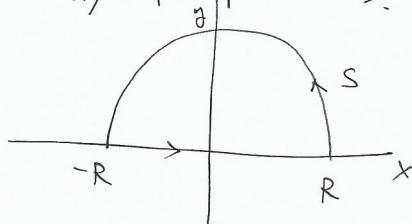
$$\text{Res} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left| \frac{1}{z - \sqrt{2} - 1} \right|_{z=\sqrt{2}-1} = -\frac{1}{2}$$

$$\therefore 2\pi i (-\frac{1}{2})(-\frac{2}{i}) = \underline{\underline{2\pi}},$$

$$\int_{-\infty}^{\infty} f(x) dx : \text{improper integral (not finite interval)}$$

$$= \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx \Rightarrow (\text{Cauchy principal value}) \text{ of the integral.}$$

$$= 2\pi i \sum \text{Res } f(z).$$



$\oint_C f(z) dz$  around C.

since  $f(x)$  is rational,  $f(z)$  has finitely many poles in the upper half-plane.

$$\oint_C f(z) dz = \int_S f(z) dz + \int_{-R}^R f(x) dx = 2\pi i \sum \text{Res } f(z)$$

$\underline{z = Re^{i\theta}} \quad 0 \leq \theta \leq \pi.$

Ex 2.

$$\text{first } \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2 \int_0^{\infty} \frac{dx}{1+x^4} \cdot \text{even function}$$

$$f(z) = \frac{1}{1+z^4}$$

4 simple poles.

$$z_1 = e^{i\pi/4}, z_2 = e^{3\pi/4i}, z_3 = e^{-3\pi/4i}, z_4 = e^{-\pi/4}$$

$$\begin{aligned} z^4 &= -1 \\ &= \cos \pi + i \sin \pi \\ &= \left( \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \end{aligned}$$

only  $z_1, z_2$  lie in the upper plane

$$\text{Res}_{z=z_1} f(z) = \left( \frac{1}{(1+z^4)} \right) \Big|_{z=z_1} = \frac{1}{4z^3} \Big|_{z=z_1} = \frac{1}{4} e^{-3\pi i/4} = -\frac{1}{4} e^{\pi i/4}$$

$$\text{Res}_{z=z_2} f(z) = \left. \frac{1}{4z^3} \right|_{z=z_2} = \frac{1}{4} e^{-9\pi i/4} = \frac{1}{4} e^{-\pi i/4}$$

$$2\pi i \left( -\frac{1}{4} e^{\pi i/4} + \frac{1}{4} e^{-\pi i/4} \right) = -\frac{2\pi i}{4} \left( 2 \sin \frac{\pi}{4} \right) = \pi \sin \frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Fourier integrals

$$\int_{-\infty}^{\infty} f(x) \cos sx dx \quad \int_{-\infty}^{\infty} f(x) \sin sx dx \quad ?$$

$$\oint_C f(z) e^{izs} dz \quad (s : \text{real positive})$$

$$\int_{-\infty}^{\infty} f(x) e^{ixs} dx = (2\pi i) \sum \text{Res} [f(z) e^{izs}] \rightarrow \boxed{\text{Real} + i \text{IM}}$$

where sum the residue of  $f(z)e^{izs}$  at its poles in the upper half-plane

$$\int_{-\infty}^{\infty} f(x) \cos sx dx + i \int_{-\infty}^{\infty} f(x) \sin sx dx = -2\pi \sum \text{Im Res} [f(z) e^{izs}] + 2\pi \sum \text{Re Res} [f(z) e^{izs}]$$

$$\therefore \int_{-\infty}^{\infty} f(x) \cos sx dx = -2\pi \sum \text{Im Res} [f(z) e^{izs}]$$

$$\int_{-\infty}^{\infty} f(x) \sin sx dx = 2\pi \sum \text{Re Res} [f(z) e^{izs}]$$

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2} dx = \frac{\pi}{k} e^{-ks}$$

$$\int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2} dx = 0$$

$e^{isx} \frac{1}{k^2+z^2}$  only one pole in the upper half-plane  $z = k$

$$\text{Res}_{z=i\bar{k}} \frac{e^{isx}}{z^2+k^2} = \lim_{z \rightarrow i\bar{k}} \frac{e^{isx}}{2z} = \frac{e^{-sk}}{2i\bar{k}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2} dx = 2\pi i \cdot \frac{e^{-ks}}{2i\bar{k}} = \frac{\pi}{k} e^{-ks}$$

Since  $e^{isx} = \cos sx + i \sin sx$

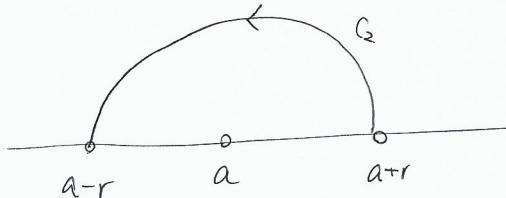
$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2} dx = \frac{\pi}{k} e^{-ks}$$

$$\int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2} dx = 0$$

Theorem 1.

Simple poles on the Real axis

$f(z)$  has a simple pole at  $z=a$  on the real axis



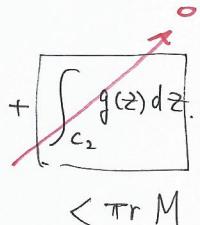
$$\lim_{r \rightarrow 0} \int_{C_2} f(z) dz = \pi i \text{Res}_{z=a} f(z)$$

by definition of a simple pole

$$f(z) = \frac{b_1}{z-a} + g(z) \quad b_1 = \text{Res}_{z=a} f(z).$$

$$C_2 : z = a + r e^{i\theta} \quad 0 \leq \theta \leq \pi$$

$$\int_{C_2} f(z) dz = \int_0^\pi \frac{b_1}{a + r e^{i\theta}} \cdot ir e^{i\theta} d\theta + \int_{C_2} g(z) dz = b_1 \pi i +$$



$$< \pi r M$$

Pr. v.  $\int_{-\infty}^{\infty} f(x) dx = \frac{2\pi i \sum \text{Res } f(z)}{\text{all poles in the upper half-plane}} + \frac{\pi i \sum \text{Res } f(z)}{\text{all poles on the real axis}}$

$$\text{Ex 4. } \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = \int_{-\infty}^{\infty} \frac{dx}{(x-1)(x-2)(x+i)(x-i)}$$

$z=1, z=2$  on the real axis

$z=i$  upper half-plane

$$\text{Res}_{(-1) \cdot 2} \frac{1}{(z-1)(z-2)} = -\frac{1}{2}$$

$$\text{Res}_{z=2} \frac{1}{(z-1)(z-2)} = \frac{1}{5}$$

$$\text{Res}_{z=i} \frac{1}{(z-1)(z-2)} = \frac{1}{6+2i} = \frac{3-i}{30}$$

$$\therefore 2\pi i \left( \frac{3-i}{30} \right) + \pi i \left( -\frac{1}{2} + \frac{1}{5} \right) = \frac{\pi}{10}.$$