• Work

$$\mathbf{F} \cdot \mathbf{ds} = F \cos\theta ds$$

Here, θ is the angle between F and ds.

Force vector:

$$\mathbf{F}^T = \begin{pmatrix} F_x & F_y & F_z \end{pmatrix}$$

Displacement vector:

$$\mathbf{ds}^T = \begin{pmatrix} dx & dy & dz \end{pmatrix}$$

•Inner product means…

- •Inner product of two vector quantities results in a scalar, that is, the work is a scalar quantity.
- •No work is done when the direction of the displacement is perpendicular to that of the force direction.

• General Work

$$\int \mathbf{F} \cdot \mathbf{ds}$$

- We use general work when force varies with a point of application.
- There are two kinds of work.
  - Conservative: work done by external force is stored in the form of potential energy, and recoverable.

    (ex. gravitational potential energy, elastic potential energy)
  - Nonconservative: work done in system is not recoverable.

    (ex. sliding block with friction)

• Elastic Spring

$$\int \mathbf{F} \cdot \mathbf{ds} = \int_0^{\delta} F d\delta = U$$

- *F* (external force) remains in equilibrium with the internal tension (spring force).

- The potential energy appears as the shaded area in Fig. 2.17*b*.
- *U* (potential energy) is a function of elongation ***δ***.

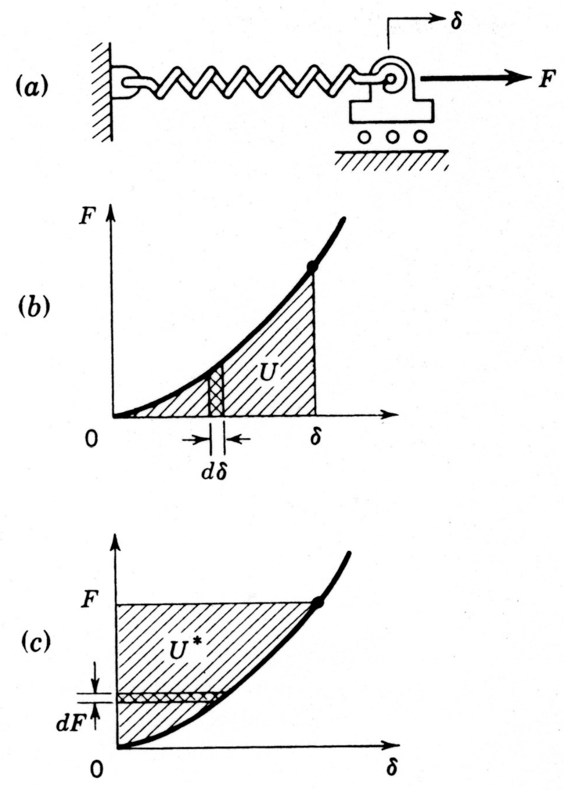


Fig. 2.17 Nonlinear Spring undergoes a gradual elongation.

• Application

Total work done by all the external loads

(at each point A_i , load is $\mathbf{P}_i$, and displacement is $\mathbf{s}_i$)

=

Total potential energy U

$$\sum_i \int_0^{\mathbf{s}_i} \mathbf{P}_i \cdot \mathbf{ds}_i = U$$

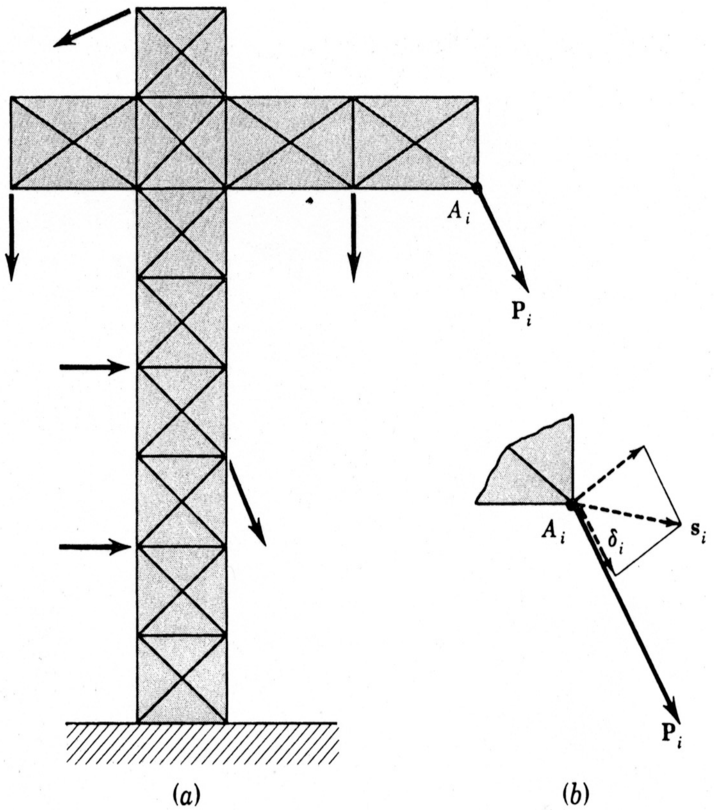


Fig. 2.18 General elastic structure.

(a) P_i at A_i
(b) s_i at A_i

• Complementary Work

$$\int \mathbf{s} \cdot \mathbf{dF} = \int_0^F \delta dF = U^*$$

When complementary work is done on this system, their internal force states are altered in such a way that they are capable of giving up equal amounts of complementary work when they are returned to their original force states. Under these circumstances the complementary work done on such system is said to be stored as complementary energy.

- This energy appears as the shaded area in Fig. 2.17*c*.
- *U**(complementary energy) is a function of the force *F*.

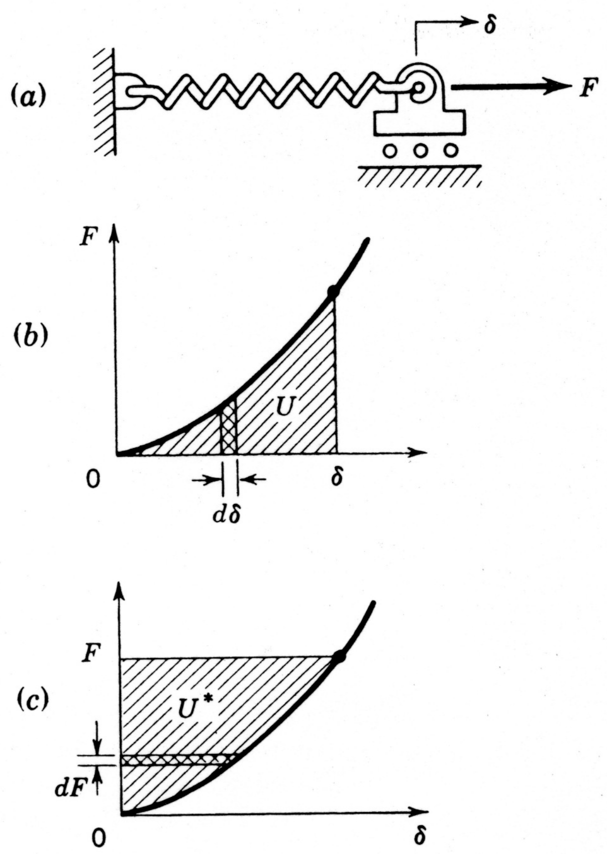


Fig. 2.17 Nonlinear Spring undergoes a gradual elongation.

• Application

Total complementary work done by all the external loads

(at each point A_i, load is $\mathbf{P}_i$, and displacement is $\mathbf{s}_i$)

=

Total complementary energy *U**

$$\sum_i \int_0^{\mathbf{P}_i} \mathbf{s}_i \cdot \mathbf{dP}_i = \sum_i \int_0^{P_i} \delta_i dP_i = U^*$$

where $\mathbf{s}_i$ can be decomposed into parallel and perpendicular to $\mathbf{P}_i$. The parallel component is δ_i . (Fig. 2.18*b*)

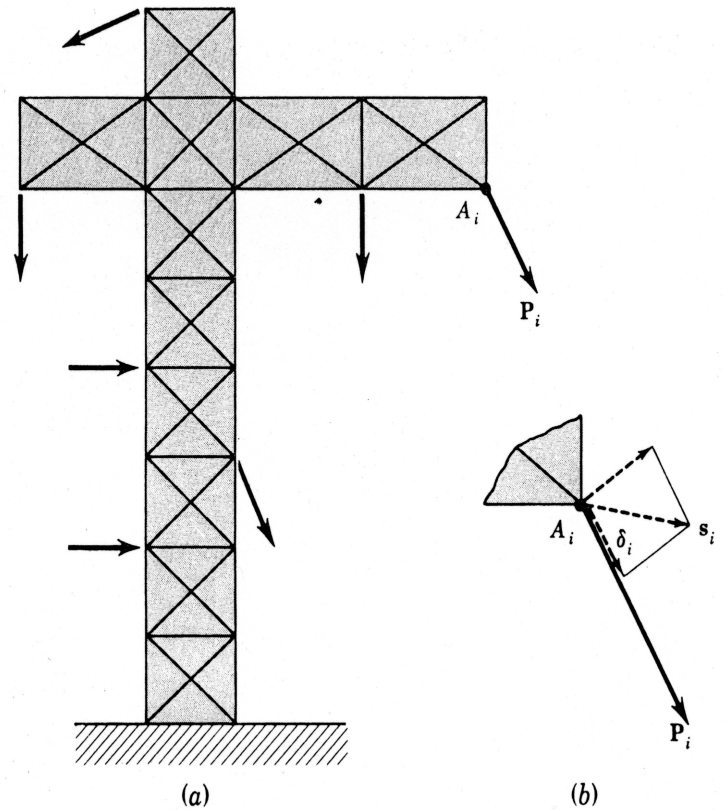


Fig. 2.18 General elastic structure.
(a) P_i at A_i
(b) s_i at A_i

• Castigliano's Theorem(1)

Now if the loads in Fig. 2.18*a* are gradually increased from zero so that the system passes through a succession of equilibrium states, the total complementary work done by all the external loads will equal the total complementary energy *U** stored in all the internal elastic members.

Let's consider a small increment ΔP_i

$$\delta_i \Delta P_i = \Delta U^*$$

or

$$\frac{\Delta U^*}{\Delta P_i} = \delta_i$$

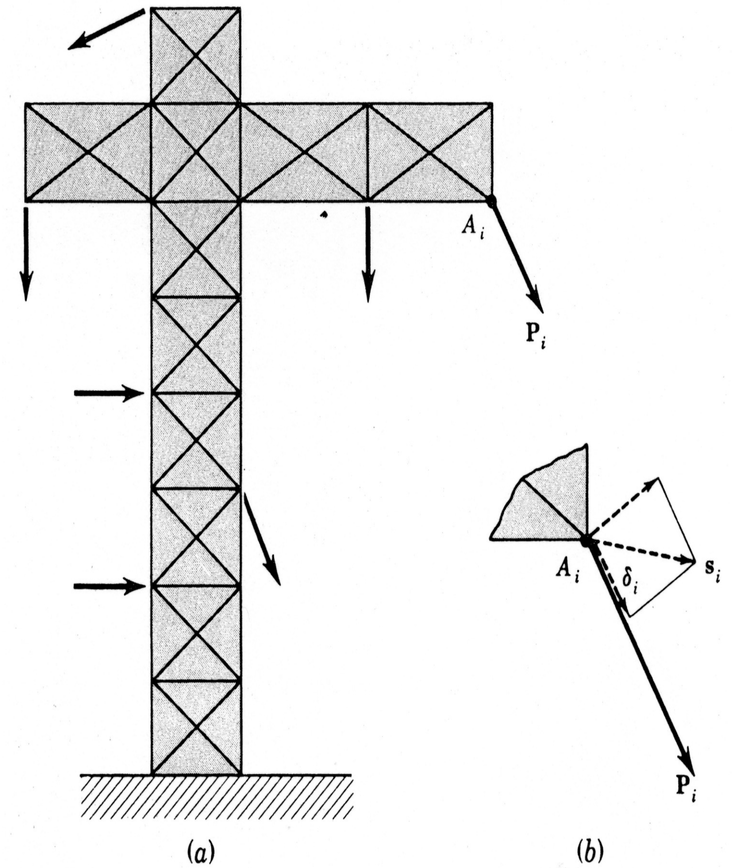


Fig. 2.18 General elastic structure.

(a) P_i at A_i
(b) S_i at A_i

•Castigliano's Theorem(2)

In the limit as $\Delta P_i \rightarrow 0$ this approaches a derivative which we indicates as a partial derivative since all the other loads were held fixed.

$$\frac{\partial U^*}{\partial P_i} = \delta_i$$

This result is a form of Castigliano's theorem.

The theorem can be extended to include moment loads.

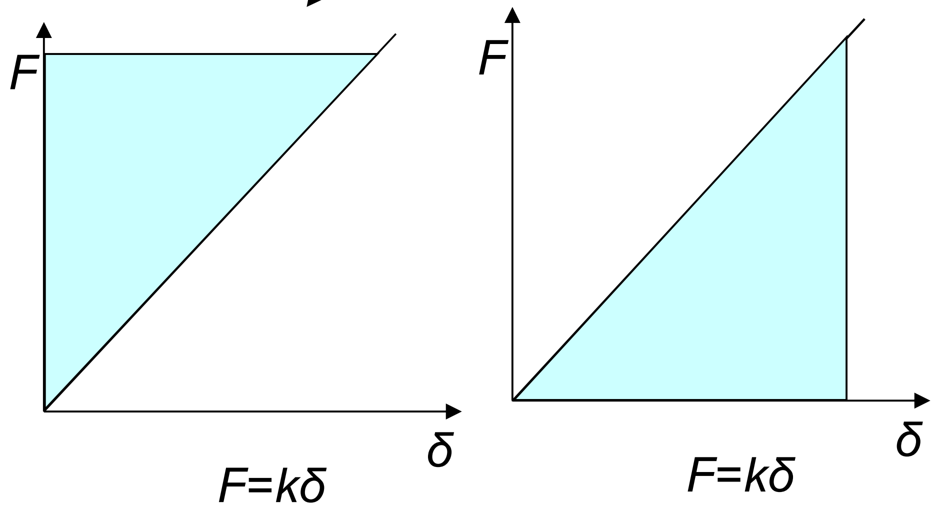
$$\frac{\partial U^*}{\partial M_i} = \phi_i$$

where M_i is moment loads, and ϕ_i is an angle of rotation.

•Castigliano's Theorem in Linear System

In nonlinear system $U^* \neq U$

But, in linear system, $U^* = U$.



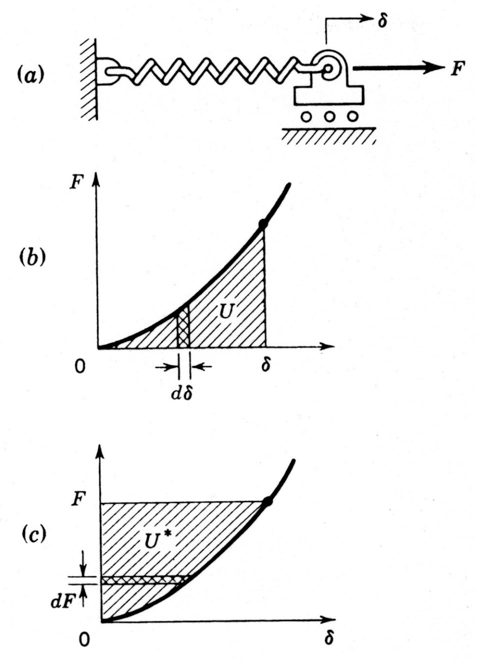


Fig. 2.17 Nonlinear Spring undergoes a gradual elongation.

•Example: Linear Spring(1)

$$F = k\delta$$

where *k* is spring constant

$$U = \frac{1}{2}k\delta^2, U^* = \frac{F^2}{2k}$$

$$U = U^*$$

$$U = \frac{1}{2}k\delta^2 = \frac{1}{2}F\delta = \frac{F^2}{2k}$$

•Example: Linear Spring(2)

For the linear uniaxial member in Figs. 2.4 and 2.5

$$k = \frac{EA}{L}$$

$$U = \frac{EA}{2L}\delta^2 = \frac{P^2 L}{2EA}$$

Finally, the in-line deflection δ_i at any loading point A_i is obtained by differentiation with respect to the load

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i}\left(\frac{P^2 L}{2EA}\right) = \frac{PL}{EA}$$

Example 2.11

Consider the system of two springs shown in Fig. 2.19. We shall use Castigliano's theorem to obtain the deflections δ_1 and δ_2 which are due to the external loads P_1 and P_2.

- To satisfy the equilibrium requirements the internal spring forces must be

$$F_1 = P_1 + P_2$$

$$F_2 = P_2$$

The total elastic energy, using $U = \frac{1}{2}k\delta^2 = \frac{1}{2}F\delta = \frac{F^2}{2k}$, is

$$U = U_1 + U_2 = \frac{(P_1 + P_2)^2}{2k_1} + \frac{P_2^{\,2}}{2k_2}$$

The deflections then follow the form of $\delta_i = \frac{\partial U}{\partial P_i}$

$$\delta_1 = \frac{\partial U}{\partial P_1} = \frac{P_1 + P_2}{k_1}$$

$$\delta_2 = \frac{\partial U}{\partial P_2} = \frac{P_1 + P_2}{k_1} + \frac{P_2}{k_2}$$

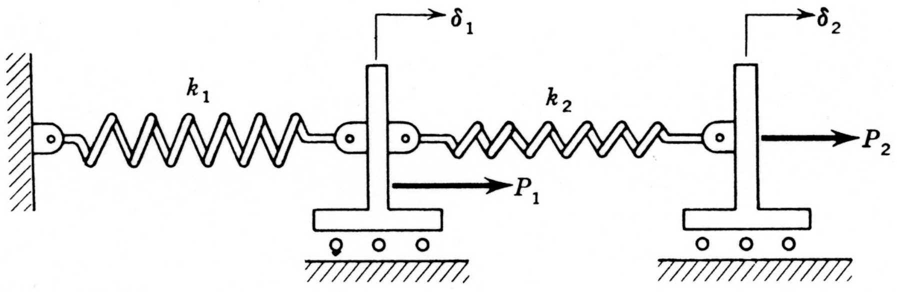


Fig. 2.19 Example 2.11.

Example 2.12

Let us consider again Example 2.4 (also Example 1.3), and determine the deflections using Castigliano's theorem.

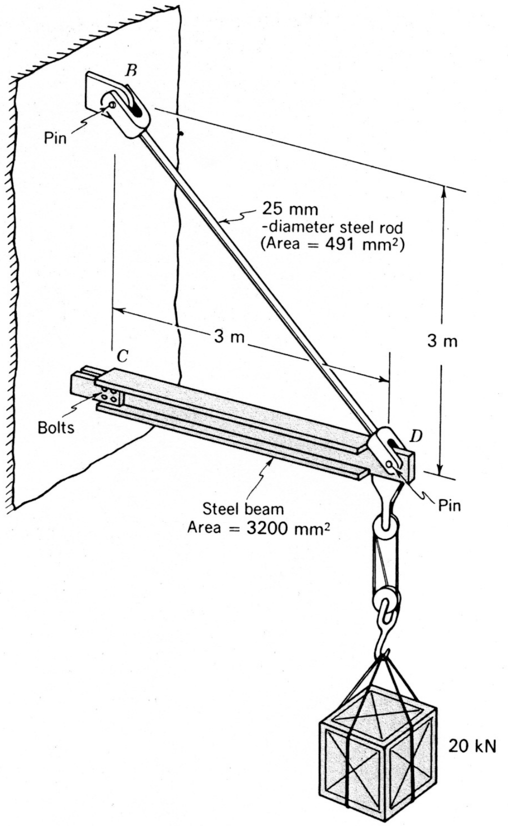


Fig. 2.6 Example 2.4.

- In Fig. 2.20 the isolated system from Example 2.4 is shown together with the applied loads.
  Because we will treat the members of the frame as springs, their "constants" are given.

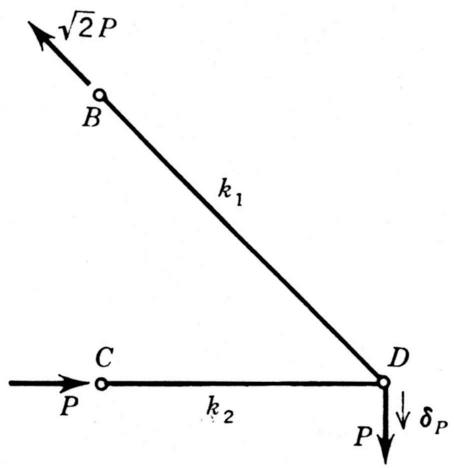


Fig. 2.20 Example 2.12.

$$k_1 = \frac{491\times10^{-6}\times205\times10^{6}}{3\sqrt{2}} = 23.73\text{MN/m}$$

$$k_2 = \frac{3.2\times10^{-3}\times205\times10^{6}}{3} = 218.67\text{MN/m}$$

We use the equilibrium requirements to express the member forces F_1 and F_2 in terms of The load *P* so that the total energy is

$$U = U_1 + U_2 = \frac{P_1^{\,2}}{2k_1} + \frac{P_2^{\,2}}{2k_2} = \frac{P^2}{k_1} + \frac{P^2}{2k_2}$$

We can calculate directly the deflection of point *D* from $\delta_i = \dfrac{\partial U}{\partial P_i}$

$$\delta_1 = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{P^2}{k_1} + \frac{P^2}{2k_2}\right) = 2P\left(\frac{1}{k_1} + \frac{1}{2k_2}\right)$$

$$\delta_P = 2 \times 20 \times 10^3 \times \left(0.0421 + 0.0023\right) \times 10^{-6} = 1.77\,\text{mm}$$

In order to calculate the horizontal deflection at point *D* using Castigliano's theorem, there must be a horizontal force at *D*. But the horizontal force at *D* is zero.

We can satisfy both requirements by applying a fictitious horizontal force *Q* and setting *Q* = 0.

Figure 2.21 shows the frame isolated with both *P* and *Q* applied.

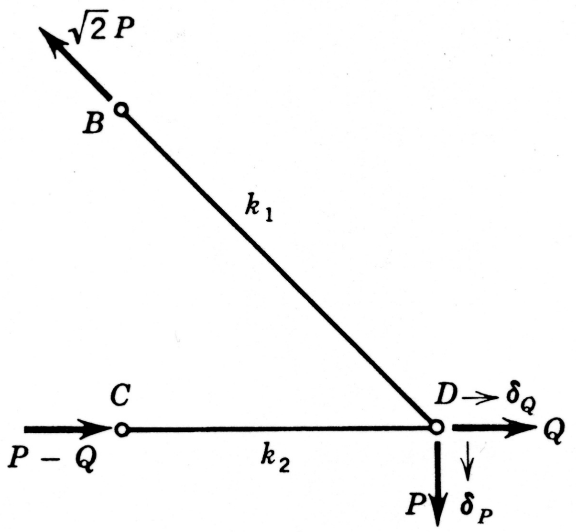


Fig. 2.21 Structure of Fig. 2.20 with fictitious load *Q* at *D*.

The total energy in terms of the loads *P* and *Q* is

$$U = \frac{P^2}{k_1} + \frac{1}{2k_2}(P-Q)^2$$

$$\delta_Q = \frac{\partial U}{\partial Q} = 0 - \frac{P-Q}{k_2} = \frac{-P}{k_2} = -0.0915\text{mm}$$

Example 2.13

Let us use Castigliano's theorem to determine deflections in the Truss problem that we considered in Example 2.5 and in the computer solution example of Sec. 2.5.

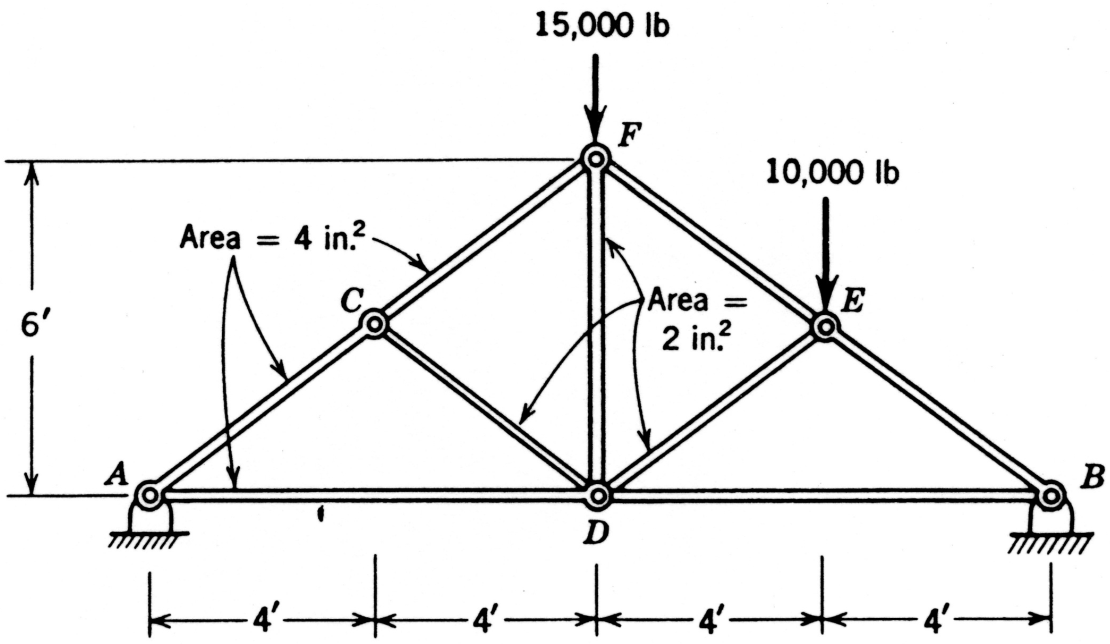


Fig. 2.16 Statically indeterminate version of truss in Example 2.5.

- If a truss is made of n axially loaded members,

$$U_i = \frac{F_i L_i}{2A_i E_i}$$ (energy stored in the ith member)

$$U = \sum_{i=1}^{n} U_i$$ (total energy in the system of n members)

The deflection at any external load *P* in the direction of *P*, is

$$\delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\sum_{i=1}^{n}\frac{F_i^2 L_i}{2A_i E_i} = \sum_{i=1}^{n}\frac{F_i L_i}{A_i E_i}\frac{\partial F_i}{\partial P} = \sum_{i=1}^{n} F_i \frac{L_i}{A_i E_i}\frac{\partial F_i}{\partial P} \quad \text{—— (d)}$$

We will number the members as shown in Fig. 2.22.

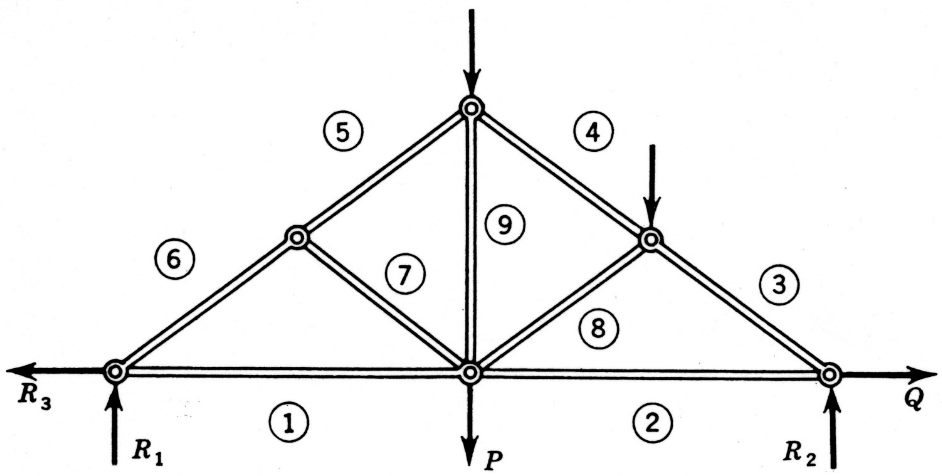


Fig. 2.22 Example 2.13.

In example 2.5 we solved for the forces F_i due to the actual applied loads.
We can now set up a system for evaluating (*d*).
The deflection at the joint at which the fictitious load *P* is applied, it appears that we need to find the forces F_i in each members as a function of the actual applied loads and in terms of *P*.
However, once the member forces are found, we set *P* = 0 in (*d*).
Therefore, we can use immediately the member forces F_i from the actual loads and the forces for a unit load at *P* to evaluate $\partial F_i / \partial P$.

In the Table 2.6 we have tabulated the individual quantities in (*d*) as well as their products.

Table 2.6 Truss solution by energy methods

i	F_i 10^3 lb	(L/AE)* in./lb	$\dfrac{\partial F_i}{\partial P}$	$\dfrac{\partial F_i}{\partial Q}$	$\left(\dfrac{FL}{AE}\dfrac{\partial F}{\partial P}\right)_i$†	$\left(\dfrac{FL}{AE}\dfrac{\partial F}{\partial Q}\right)_i$†
1	$+13.33+Q$	2.4×10^{-6}	$+2/3$	$+1$	21.36×10^{-3}	32.0×10^{-3}
2	$+20.0+Q$	2.4×10^{-6}	$+2/3$	$+1$	31.95×10^{-3}	48.0×10^{-3}
3	-25.0	1.5×10^{-6}	$-5/6$	0	31.26×10^{-3}	
4	-16.67	1.5×10^{-6}	$-5/6$	0	20.85×10^{-3}	
5	-16.67	1.5×10^{-6}	$-5/6$	0	20.85×10^{-3}	
6	-16.67	1.5×10^{-6}	$-5/6$	0	20.85×10^{-3}	
7	0	1.5×10^{-6}	0	0	0	
8	-8.33	3.0×10^{-6}	0	0	0	
9	$+5.0$	3.6×10^{-6}	$+1$	0	18.00×10^{-3}	
					$\Sigma=0.1651$ in. $=\delta_y$	$\Sigma=0.080$ in. $=\delta_x$

* Calculated for $E=10\times10^6$ lb/in.^{2}
† $Q=0$.

If we wish to solve for the deflection at *P*, we must reevaluate the products in row 1 and 2 of Table 2.6 with *Q*.

i	$\left(\frac{FL}{AE}\frac{\partial F}{\partial P}\right)_{i,\,Q\neq 0}$
1	$21.36 \times 10^{-3} + 1.6Q \times 10^{-6}$
2	$31.95 \times 10^{-3} + 1.6Q \times 10^{-6}$

The values for member 3 through 9 do not change since they carry no *Q*. Therefore

$$\delta_P = \frac{\partial U}{\partial P} = 0.1651 + 3.2Q \times 10^{-6} \qquad \text{here} \quad Q = -16.67 \times 10^{3}$$

$$\therefore \delta_P = 0.1651 - 0.0534 = 0.1117\text{in}$$