
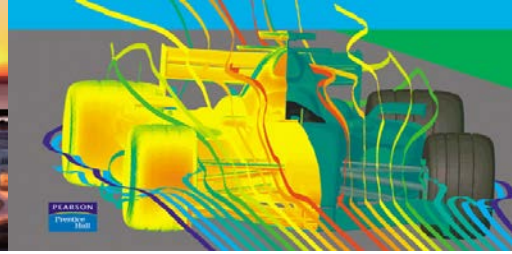
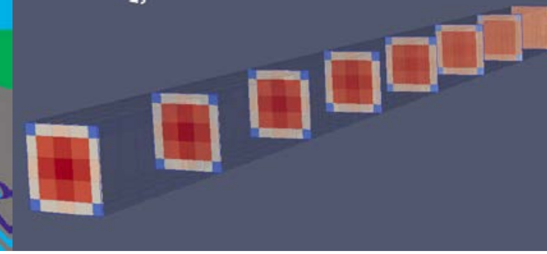
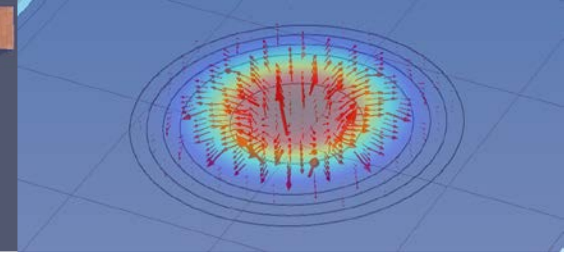

COMPUTATIONAL NUCLEAR THERMAL HYDRAULICS

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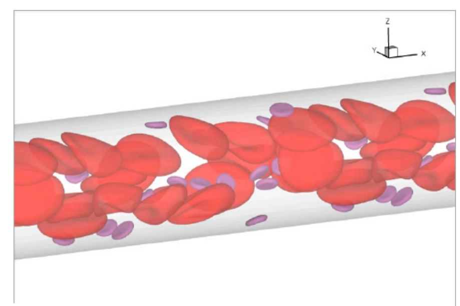
CHAPTER2.
CONSERVATION LAWS OF FLUID MOTION AND BOUNDARY CONDITIONS

2.1 Governing Equations of Fluid Flow and Heat Transfer

- The governing equations of fluid flow represent mathematical statements of the **conservation laws of physics**.

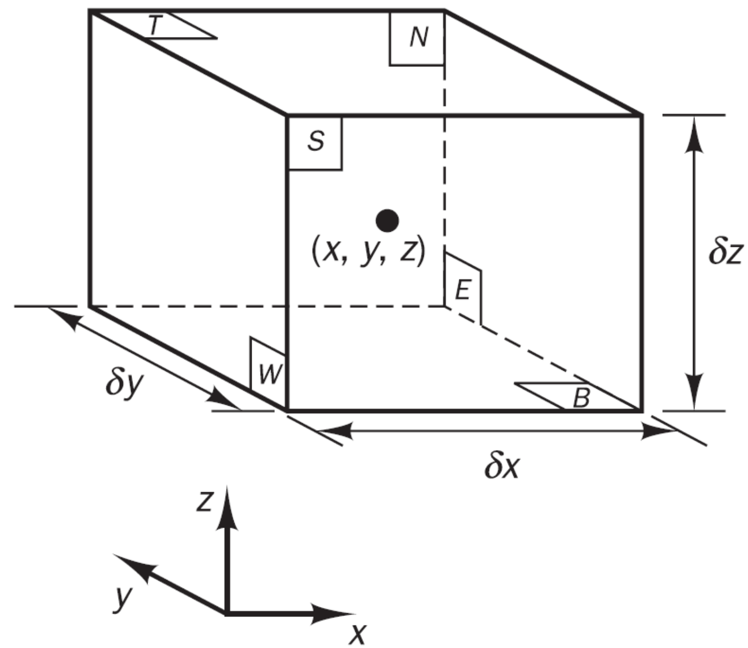
  - The mass of fluid is conserved.
  - The rate of change of momentum equals the sum of the forces on a fluid particle
    - Newton's second law
  - The rate of change of energy is equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle
    - First law of thermodynamics

  - Continuum assumption
    - Fluid flows at macroscopic length scales > 1 μm
    - The molecular structure and motions may be ignored.
    - Macroscopic properties
      - Velocity, pressure, density, temperature
      - Averaged over suitably large numbers of molecules
    - Fluid particle
      - The smallest possible element of fluid whose macroscopic properties are not influenced by individual molecules.



2.1 Governing Equations of Fluid Flow and Heat Transfer

- Control volume
  - Six faces: N, S, E, W, T, B
  - The center of the element: (x, y, z)



- Properties at the volume center

$$\rho = \rho\,(x, y, z, t)$$

$$p = p\,(x, y, z, t)$$

$$T = T\,(x, y, z, t)$$

$$\boldsymbol{u} = \boldsymbol{u}\,(x, y, z, t)$$

- Fluid properties at faces are approximated by means of the two terms of the Taylor series.
  - The pressure at the W and E faces

$$p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x \quad p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x$$

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.1 Mass Conservation in Three Dimensions

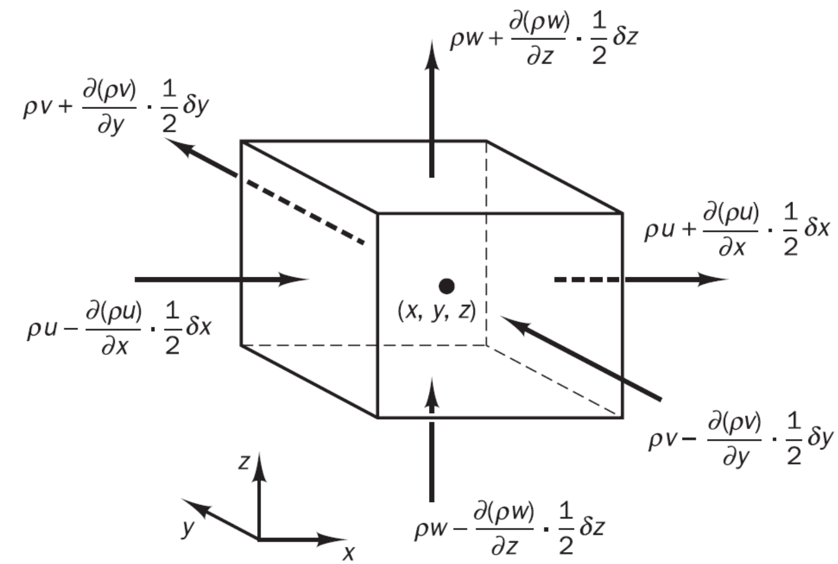
$$\begin{pmatrix}\text{Rate of increase}\\ \text{of mass in}\\ \text{fluid element}\end{pmatrix} = \begin{pmatrix}\text{Net rate of flow}\\ \text{of mass into}\\ \text{fluid element}\end{pmatrix}$$

- Rate of increase of mass

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z$$

- Net rate of flow of mass into the element

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\,\delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\,\delta z$$
$$+ \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z$$
$$+ \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y$$



Mass conservation/ Continuity Eq.

2.1.1 Mass Conservation in Three Dimensions

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0}$$

- For an incompressible fluid, the density ρ is constant.

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.2 Rates of change following a fluid particle and for a fluid element

- Lagrangian approach/ changes of properties of a fluid particle
- Total or substantial derivative of ϕ
- ϕ: Function of the position (x,y,z), property per unit mass

$$\frac{D\phi}{Dt} = \boxed{}$$

- A fluid particle follows the flow, so

$$dx/dt = u$$
$$dy/dt = v$$
$$dz/dt = w$$

- Hence, the substantive derivative of ϕ is given by

$$\boxed{}$$

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.2 Rates of change following a fluid particle and for a fluid element

- $\dfrac{D\phi}{Dt}$ defines the rate of change of property ϕ per unit mass.

- $\rho\dfrac{D\phi}{Dt}$ The rate of change of property ϕ per unit volume

$$\boxed{\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + \mathbf{u}\ .\ \mathrm{grad}\ \phi}$$

$$\boxed{\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial\phi}{\partial t} + \mathbf{u}\ .\ \mathrm{grad}\ \phi\right)}$$

- Eulerian approach/ changes of properties in a fluid element
  - Far more common than Lagrangian approach
  - Develop equations for collections of fluid elements making up a region fixed in space

2.1.2 Rates of change following a fluid particle and for a fluid element

- LHS of the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u})$$

- The generalization of these terms for an arbitrary conserved property

$$\frac{\partial (\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) \qquad \begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ per unit volume}\end{pmatrix} + \begin{pmatrix}\text{Net rate of flow of } \phi\\ \text{out of fluid element}\\ \text{per unit volume}\end{pmatrix}$$

$$\frac{\partial (\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \rho\left[\frac{\partial \phi}{\partial t} + \mathbf{u}\ .\ \operatorname{grad}\ \phi\right] + \phi\left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u})\right]$$

$$\boxed{\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial \phi}{\partial t} + \mathbf{u}\ .\ \operatorname{grad}\ \phi\right)} \qquad \boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0}$$

Rate of increase of ϕ of fluid element	+	Net rate of flow of ϕ out of fluid element	=	Rate of increase of ϕ for a fluid particle

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.2 Rates of change following a fluid particle and for a fluid element

- Relevant entries of ϕ for momentum and energy equations

x-momentum	u	$\rho \dfrac{Du}{Dt}$	$\dfrac{\partial(\rho u)}{\partial t} + \mathrm{div}(\rho u \mathbf{u})$
y-momentum	v	$\rho \dfrac{Dv}{Dt}$	$\dfrac{\partial(\rho v)}{\partial t} + \mathrm{div}(\rho v \mathbf{u})$
z-momentum	w	$\rho \dfrac{Dw}{Dt}$	$\dfrac{\partial(\rho w)}{\partial t} + \mathrm{div}(\rho w \mathbf{u})$
energy	E	$\rho \dfrac{DE}{Dt}$	$\dfrac{\partial(\rho E)}{\partial t} + \mathrm{div}(\rho E \mathbf{u})$

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.3 Momentum equation in three dimensions

- Newton's second law

Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

$$\rho\frac{Du}{Dt} \qquad \rho\frac{Dv}{Dt} \qquad \rho\frac{Dw}{Dt}$$

Surface forces

- Pressure force (p)
- Viscous force (τ)
- Gravity force

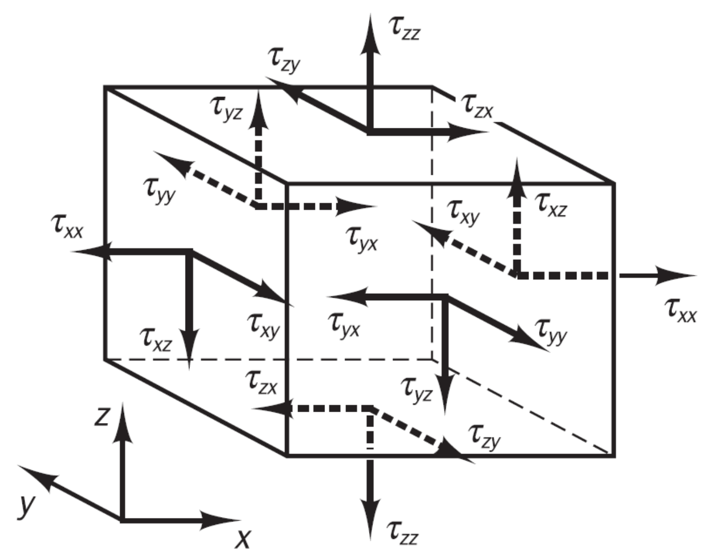
Body forces

- Centrifugal force
- Coriolis force
- Electromagnetic force
- Gravity force

Pressure = normal stress = p

Viscous stress = τ

τ_{ij} : stress component acts in the *j*-direction on a surface normal to *i*-direction



2.1.3 Momentum equation in three dimensions

- x-component of the forces due to pressure and viscous stress
  - On the pair of faces (E,W)

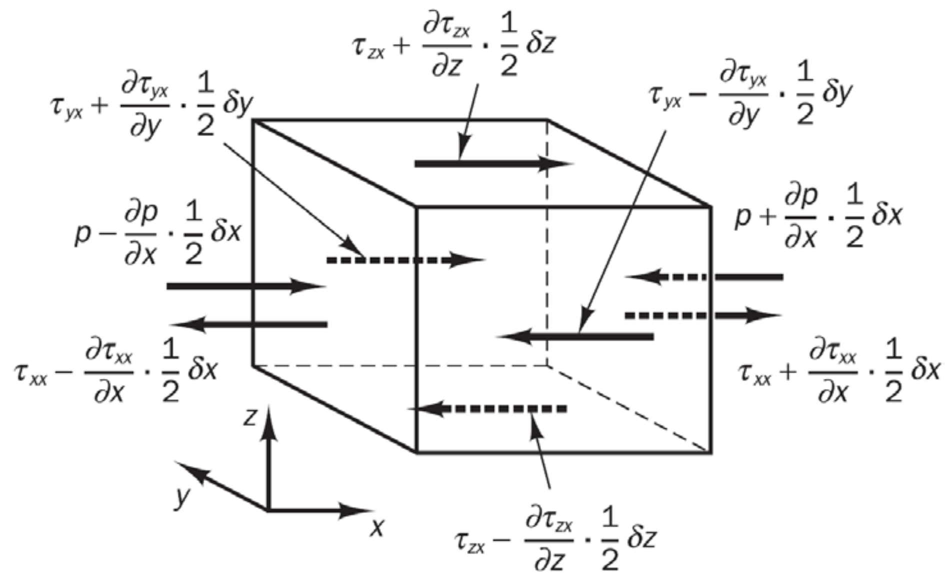
$$\left[\left(p-\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)-\left(\tau_{xx}-\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z+\left[-\left(p+\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)+\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z=\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\right)\delta x\,\delta y\,\delta z$$

  - On the pair of faces (N,S)

$$-\left(\tau_{yx}-\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z=\frac{\partial \tau_{yx}}{\partial y}\delta x\,\delta y\,\delta z$$

  - On the pair of faces (T,B)

$$-\left(\tau_{zx}-\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y+\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y=\frac{\partial \tau_{zx}}{\partial z}\delta x\,\delta y\,\delta z$$



2.1.3 Momentum equation in three dimensions

- *x*-component of the forces due to pressure and viscous stress
  - Total surface force per unit volume

$$\boxed{\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}}$$

- *x*-component of the momentum equation

Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

2.1.3 Momentum equation in three dimensions

Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

- x-component of the momentum equation

$$\boxed{\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+S_{Mx}}$$

- y-component of the momentum equation

$$\boxed{\rho\frac{Dv}{Dt}=\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial(-p+\tau_{yy})}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+S_{My}}$$

- z-component of the momentum equation

$$\boxed{\rho\frac{Dw}{Dt}=\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}+S_{Mz}}$$

Body force

$$S_{Mx}=0$$

$$S_{My}=0$$

$$S_{Mz}=-\rho g$$

2.1.4 Energy equation in three dimensions

- The first law of thermodynamics

$$\rho\frac{DE}{Dt}$$

$$\boxed{\begin{matrix}\text{Rate of increase} \\ \text{of energy of} \\ \text{fluid particle}\end{matrix} = \begin{matrix}\text{Net rate of} \\ \text{heat added to} \\ \text{fluid particle}\end{matrix} + \boxed{\begin{matrix}\text{Net rate of work} \\ \text{done on} \\ \text{fluid particle}\end{matrix}}}$$

- Rate of work done by surface forces

- In x-direction,

$$\boxed{\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}} \rightarrow \boxed{\left[\frac{\partial(u(-p+\tau_{xx}))}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\right]\delta x\,\delta y\,\delta z}$$

- In y and z-directions,

$$\boxed{\left[\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v(-p+\tau_{yy}))}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}\right]\delta x\,\delta y\,\delta z} \quad \boxed{\left[\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w(-p+\tau_{zz}))}{\partial z}\right]\delta x\,\delta y\,\delta z}$$

2.1.4 Energy equation in three dimensions

- Total rate of work done on the fluid particle by surface stresses

$$\left[\frac{\partial(u(-p+\tau_{xx}))}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\right]\delta x\delta y\delta z$$

$$+\left[\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v(-p+\tau_{yy}))}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}\right]\delta x\delta y\delta z$$

$$+\left[\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w(-p+\tau_{zz}))}{\partial z}\right]\delta x\delta y\delta z$$

$$-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}=\boxed{\quad}$$

$$\boxed{\begin{aligned}&[-\operatorname{div}(p\mathbf{u})]+\left[\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}\right.\\&\left.+\frac{\partial(v\tau_{zy})}{\partial z}+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}\right]\end{aligned}}$$

2.1.4 Energy equation in three dimensions

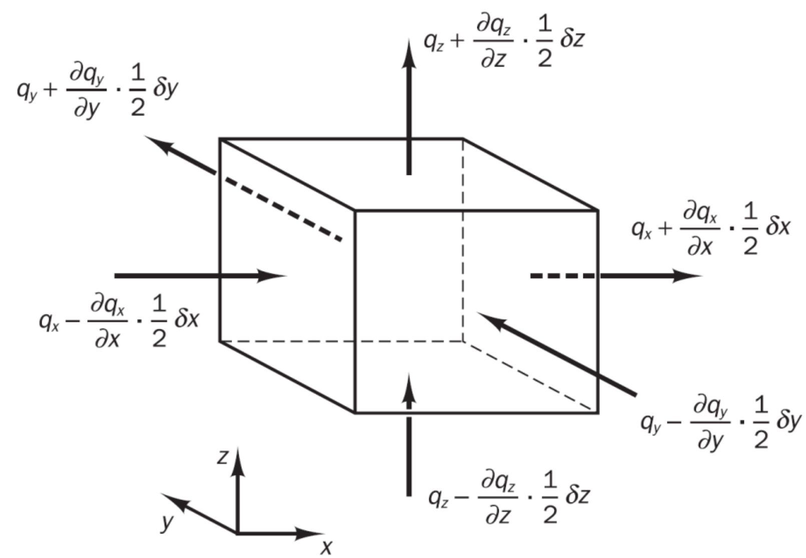
Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle

- Net rate of heat transfer to the fluid particle
  - In x-direction,

$$\left[\left(q_x - \frac{\partial q_x}{\partial x}\frac{1}{2}\delta x\right) - \left(q_x + \frac{\partial q_x}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z = \boxed{}$$

  - In y and z-directions,

$$-\frac{\partial q_y}{\partial y}\delta x\,\delta y\,\delta z \qquad -\frac{\partial q_z}{\partial z}\delta x\,\delta y\,\delta z$$



2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.4 Energy equation in three dimensions

Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle

- Total rate of heat added to the fluid particle per unit volume

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}=$$

- Fourier's law of heat conduction

$\mathbf{q}=-k\ \mathrm{grad}\ T$ in vector form

$$\boxed{-\mathrm{div}\ \mathbf{q}=\mathrm{div}(k\ \mathrm{grad}\ T)}$$

2.1.4 Energy equation in three dimensions

Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle	+ Energy Source

$$\rho\frac{DE}{Dt} \qquad \mathrm{div}(k\ \mathrm{grad}\ T) \qquad [-\mathrm{div}(p\mathbf{u})] + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right]$$

- Energy equation
  - E: Sum of internal energy and kinetic energy $\quad E = i + \frac{1}{2}(u^2 + v^2 + w^2)$

$$\boxed{\begin{aligned}\rho\frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) &+ \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x}\right. \\ &\left. + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] \\ &+ \mathrm{div}(k\ \mathrm{grad}\ T) + S_E\end{aligned}}$$

2.1.4 Energy equation in three dimensions

- Kinetic energy equation

$$\rho \frac{Du}{Dt} = \frac{\partial(-p+\tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \qquad \times u$$

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p+\tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My} \qquad \times v$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p+\tau_{zz})}{\partial z} + S_{Mz} \qquad \times w$$

$$\boxed{\begin{aligned}\rho \frac{D[\frac{1}{2}(u^2+v^2+w^2)]}{Dt} = -\mathbf{u} \,.\, \operatorname{grad} p &+ u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \\ &+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \\ &+ w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \mathbf{u} \,.\, \mathbf{S}_M\end{aligned}}$$

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.4 Energy equation in three dimensions

- Total energy equation – kinetic energy equation

$$\rho \frac{DE}{Dt} = -\operatorname{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] + \operatorname{div}(k \operatorname{grad} T) + S_E$$

$$E = i + \tfrac{1}{2}(u^2 + v^2 + w^2)$$

$$\rho \frac{D[\frac{1}{2}(u^2 + v^2 + w^2)]}{Dt} = -\mathbf{u} \cdot \operatorname{grad} p + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \mathbf{u} \cdot \mathbf{S}_M$$

- Internal energy equation

$$\boxed{\rho \frac{Di}{Dt} = -p \operatorname{div} \mathbf{u} + \operatorname{div}(k \operatorname{grad} T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i}$$

2.1 Governing Equations of Fluid Flow and Heat Transfer

2.1.4 Energy equation in three dimensions

- For the special case of an incompressible fluid, temperature equation

$$i = cT$$

$$\boxed{\begin{aligned}\rho c\frac{DT}{Dt} = \operatorname{div}(k \operatorname{grad} T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} \\ + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i\end{aligned}}$$

2.1.4 Energy equation in three dimensions

- Enthalpy equation

$h = \boxed{\qquad\qquad} \quad h_0 = \boxed{\qquad\qquad}$ Total enthalpy

$$h_0 = i + p/\rho + \tfrac{1}{2}(u^2 + v^2 + w^2) = E + p/\rho$$

- Total enthalpy equation

$$\rho \frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] + \mathrm{div}(k \text{ grad } T) + S_E$$

$$\rho \frac{DE}{Dt} = \frac{\partial(\rho E)}{\partial t} + \mathrm{div}(\rho E\mathbf{u}) = \frac{\partial(\rho h_0)}{\partial t} - \frac{\partial p}{\partial t} + \mathrm{div}(\rho h_0 \mathbf{u}) - \mathrm{div}(p\mathbf{u})$$

$$\frac{\partial(\rho h_0)}{\partial t} + \mathrm{div}(\rho h_0 \mathbf{u}) = \mathrm{div}(k \text{ grad } T) + \frac{\partial p}{\partial t} + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] + S_h$$

2.2 Equations of State

- Thermodynamic variables

$$\rho,\ p,\ i \text{ and } T$$

  - Assumption of thermodynamic equilibrium

- Equations of the state
  - Relate two state variables to the other variables

$$p = p(\rho, T) \qquad i = i(\rho, T)$$

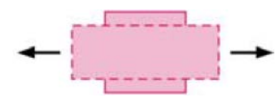
- Compressible fluids
  - EOS provides the linkage between the energy equation and other governing equations.

- Incompressible fluids
  - No linkage between the energy equation and the others.
  - The flow field can be solved by considering mass and momentum equations.

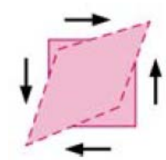
2.3 Navier-Stokes Equations for a Newtonian Fluid

- Viscous stresses τ_{ij} in momentum and energy equations
  - Viscous stresses can be expressed as functions of the local deformation rate (or strain rate).
  - In 3D flows the local rate of deformation is composed of
    - the linear deformation rate
    - the volumetric deformation rate.
  - All gases and many liquids are isotropic.

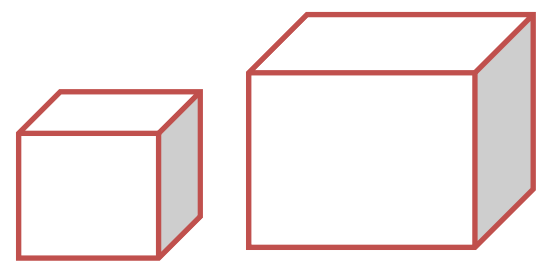
- The rate of linear deformation of a fluid element
  - Nine components in 3D
  - Linear elongating deformation



  - Shearing linear deformation components



- The rate of volume deformation of a fluid element

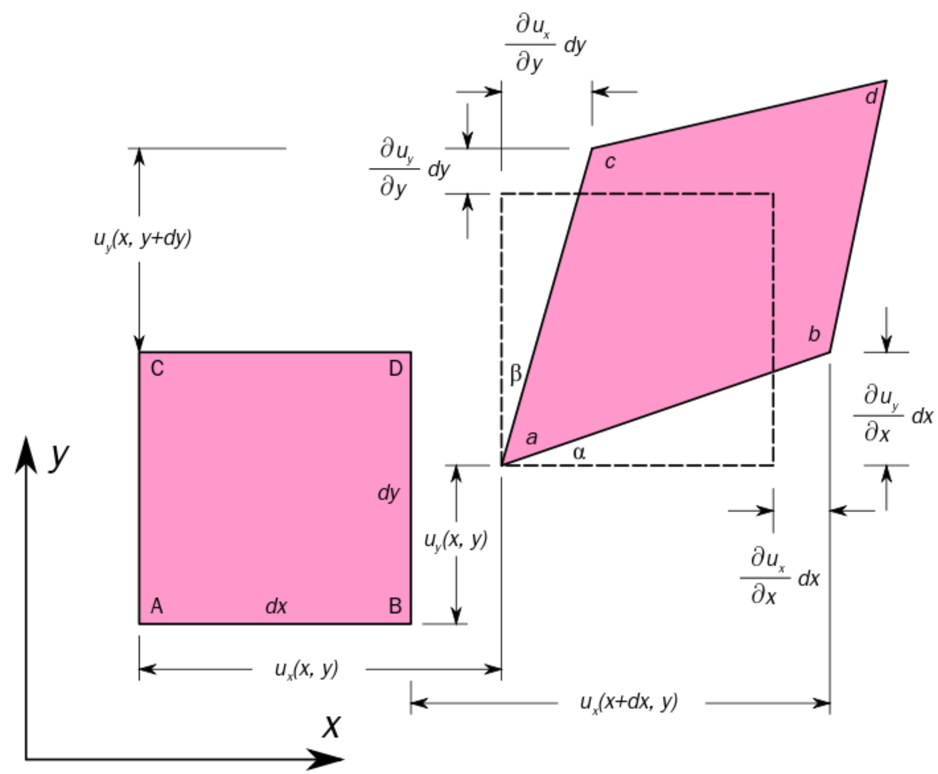


2.3 Navier-Stokes Equations for a Newtonian Fluid

- Viscous stresses τ_{ij} in momentum and energy equations

$$\tan\alpha = \frac{\dfrac{\partial u_y}{\partial x}dx}{dx + \dfrac{\partial u_x}{\partial x}dx} = \frac{\dfrac{\partial u_y}{\partial x}}{1 + \dfrac{\partial u_x}{\partial x}}$$

$$\tan\beta = \frac{\dfrac{\partial u_x}{\partial y}dy}{dy + \dfrac{\partial u_y}{\partial y}dy} = \frac{\dfrac{\partial u_x}{\partial y}}{1 + \dfrac{\partial u_y}{\partial y}}$$



$$\tan\alpha \approx \frac{\partial u_y}{\partial x} \qquad \tan\beta \approx \frac{\partial u_x}{\partial y}$$

$$\gamma_{xy} = \alpha + \beta$$

The rate at which
two sides close toward each other

$$s_{xy} = \frac{1}{2}\gamma_{xy}$$

2.3 Navier-Stokes Equations for a Newtonian Fluid

- Newtonian fluid
  - Viscous stresses are proportional to the rates of deformation.
  - Two constants of proportionality
    - Dynamic viscosity (μ): to relate stresses to linear deformations
    - Second viscosity (λ): to relate stresses to volumetric deformation

$$e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div } \mathbf{u}$$

- Viscous stress components

$$s_{xx} = \frac{\partial u}{\partial x} \quad s_{yy} = \frac{\partial v}{\partial y} \quad s_{zz} = \frac{\partial w}{\partial z} \qquad s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda \text{ div } \mathbf{u} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda \text{ div } \mathbf{u} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda \text{ div } \mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

  - Second viscosity

2.3 Navier-Stokes Equations for a Newtonian Fluid

- Momentum equations

$$\rho\frac{Du}{Dt}=\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+S_{Mx}$$

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}+\lambda\ \mathrm{div}\ \mathbf{u}\qquad \tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\qquad \tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}+\lambda\ \mathrm{div}\ \mathbf{u}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+S_{My}$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}+\lambda\ \mathrm{div}\ \mathbf{u}\right]+S_{Mz}$$

2.3 Navier-Stokes Equations for a Newtonian Fluid

- Rearrangement

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda \operatorname{div}\mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}(\lambda \operatorname{div}\mathbf{u})\right]$$

- N.-S. equations can be written as follows with modified source terms; $S_M = S_M + [s_M]$

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\operatorname{div}(\mu \operatorname{grad} u)+S_{Mx}$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\operatorname{div}(\mu \operatorname{grad} v)+S_{My}$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\operatorname{div}(\mu \operatorname{grad} w)+S_{Mz}$$

2.3 Navier-Stokes Equations for a Newtonian Fluid

- For incompressible fluids with constant μ

$$[s_{Mx}] = \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}(\lambda \operatorname{div} \mathbf{u})\right]$$

$$= \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right] = \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)\right]$$

$$= \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = 0$$

Internal energy equation

$$\rho \frac{Di}{Dt} = -p \text{ div } \mathbf{u} + \text{div}(k \text{ grad } T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$
$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z}$$
$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda \text{ div } \mathbf{u} \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda \text{ div } \mathbf{u} \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda \text{ div } \mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

- Dissipation function Φ
  - Always positive
  - Source of internal energy due to deformation work on the fluid particle.
  - Mechanical energy is converted into internal energy or heat.

$$\Phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} + \lambda(\text{div } \mathbf{u})^2$$

2.4 Conservative form of the governing equations of fluid flow

Mass $$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0$$

x-momentum $$\frac{\partial(\rho u)}{\partial t} + div(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + div(\mu \, grad \, u) + S_{Mx}$$

y-momentum $$\frac{\partial(\rho v)}{\partial t} + div(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + div(\mu \, grad \, v) + S_{My}$$

z-momentum $$\frac{\partial(\rho w)}{\partial t} + div(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + div(\mu \, grad \, w) + S_{Mz}$$

Internal energy $$\frac{\partial(\rho i)}{\partial t} + div(\rho i \mathbf{u}) = -p \, div \, \mathbf{u} + div(k \, grad \, T) + \Phi + S_i$$

\+ EOS $$u, v, w, p, i, \rho, T$$

This system is mathematically closed!

2.5 Differential and integral forms of the general transport equations

- General form of fluid flow equations

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \operatorname{div}(\Gamma \operatorname{grad} \phi) + S_\phi$$

Rate of increase of ϕ of fluid element	+	Net rate of flow of ϕ out of fluid element	=	Rate of increase of ϕ due to diffusion	+	Rate of increase of ϕ due to sources
Temporal term		Convective term		Diffusive term		Source term

- By setting ϕ, Γ, S_ϕ,

$$\phi = 1, u, v, w, i$$
$$\Gamma = 0, \mu, k$$
$$S_\phi = 0, (S_{Mx} - \partial p/\partial x), \ldots,$$

$$\frac{\partial \rho}{\partial t} + div(\rho\mathbf{u}) = 0$$

Differential form

2.5 Differential and integral forms of the general transport equations

- Starting point for computational procedures in FVM
  - Integration of the general form over a 3D control volume (CV)

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t} dV + \int_{CV} \operatorname{div}(\rho\phi\mathbf{u})dV = \int_{CV} \operatorname{div}(\Gamma \operatorname{grad} \phi)dV + \int_{CV} S_\phi dV$$

  - Gauss's divergence theorem
    - Volume integral ⇔ surface integral

$$\int_{CV} \operatorname{div}(\mathbf{a})dV = \int_A \mathbf{n} \,.\, \mathbf{a} dA$$

    - $\mathbf{n} \cdot \mathbf{a}$**:** component of vector $\mathbf{a}$ in the direction of the vector $\mathbf{n}$ normal to surface element dA

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_A \mathbf{n} \,.\, (\rho\phi\mathbf{u})dA = \int_A \mathbf{n} \,.\, (\Gamma \operatorname{grad} \phi)dA + \int_{CV} S_\phi dV$$

A special case of the Reynold' transport theorem

2.5 Differential and integral forms of the general transport equations

- Starting point for computational procedures in FVM

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_{A} \mathbf{n} \, . \, (\rho\phi\mathbf{u})\mathrm{d}A = \int_{A} \mathbf{n} \, . \, (\Gamma \operatorname{grad} \phi)\mathrm{d}A + \int_{CV} S_{\phi}\mathrm{d}V$$

- In time-dependent problems
  - Integrate with respect to time *t* over a small interval Δt

$$\int_{\Delta t} \frac{\partial}{\partial t}\left(\int_{CV} \rho\phi \mathrm{d}V\right)\mathrm{d}t + \int_{\Delta t}\int_{A} \mathbf{n} \, . \, (\rho\phi\mathbf{u})\mathrm{d}A\mathrm{d}t = \int_{\Delta t}\int_{A} \mathbf{n} \, . \, (\Gamma \operatorname{grad} \phi)\mathrm{d}A\mathrm{d}t + \int_{\Delta t}\int_{CV} S_{\phi}\mathrm{d}V\mathrm{d}t$$

2.10 Auxiliary conditions for viscous fluid flow equations

- Initial and boundary conditions for compressible viscous flow
  - Initial conditions for unsteady flows
    - Everywhere in the solution region, ρ, $\mathbf{u}$ and T must be given at time t=0.
  - Boundary conditions
    - On solid Walls
      - No-slip condition: $\mathbf{u} = \mathbf{u}_w$
      - Fixed temperature $T = T_w$
      - Fixed heat flux $k\dfrac{\partial T}{\partial n} = -q_w$
    - On fluid boundaries
      - Inlet $\rho, \mathbf{u}$ and T
      - Outlet

$$-p + \mu\frac{\partial u_n}{\partial n} = F_n \qquad \mu\frac{\partial u_t}{\partial n} = F_t$$

    - Outflow boundaries
      - Far from solid objects in an external flow
      - Commonly, no change in any of the velocity components in the direction across the boundary
      - Open boundary

$$-p = F_n \qquad 0 = F_t$$

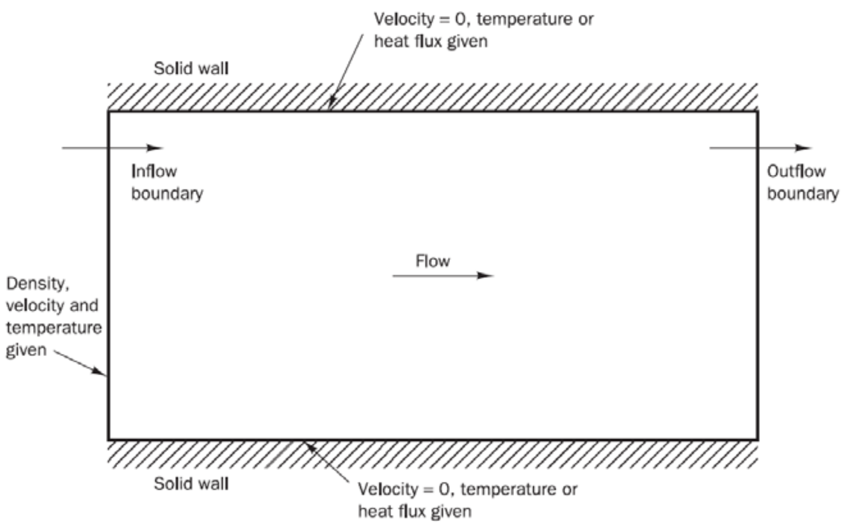
2.10 Auxiliary conditions for viscous fluid flow equations

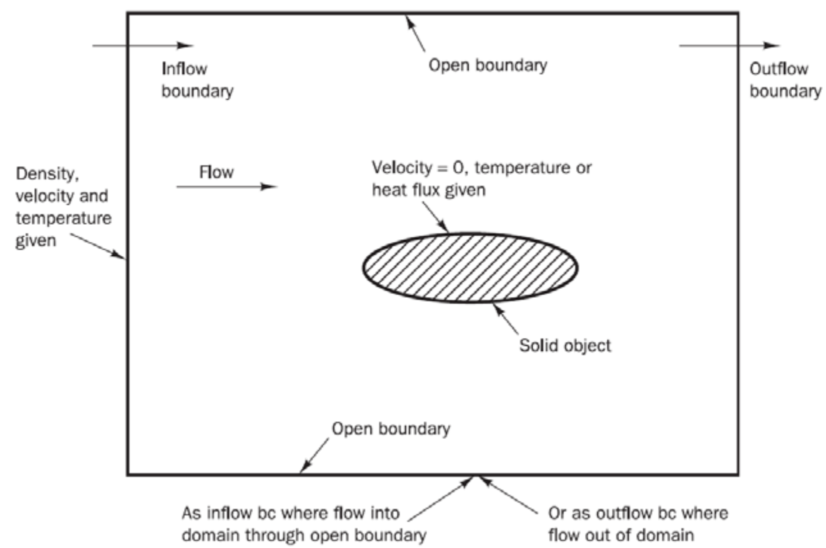
- Initial and boundary conditions for compressible viscous flow
  - Symmetry boundary condition

$$\partial\phi/\partial n = 0$$

  - Cyclic (periodic boundary condition)

$$\phi_1 = \phi_2$$





2.10 Auxiliary conditions for viscous fluid flow equations

❖ Initial and boundary conditions for compressible viscous flow

- Symmetry boundary condition

$$\partial\phi/\partial n = 0$$

- Cyclic (periodic boundary condition)

$$\phi_1 = \phi_2$$