Chapter 6: Basic Plasticity

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- The procedure for computing the internal forces, $\mathbf{q}_i$, from a set of nodal displacements, $\mathbf{p}$, is as follows:

  1. Compute the **strain** from:
     $$\varepsilon = \frac{l_n^2 - l_0^2}{2l_0^2} = \mathbf{b}_1^T \mathbf{p} + \frac{1}{2\alpha_0^2} \mathbf{p}^T \mathbf{A} \mathbf{p}$$
  2. Compute the **stress**, σ_G (here, constant over the element), for now assuming a linear material response from: $\sigma_G = E\varepsilon$
  3. Compute the **internal forces**, $\mathbf{q}_i$, from [eq. 3.63] above.

[Review of chapter 3]

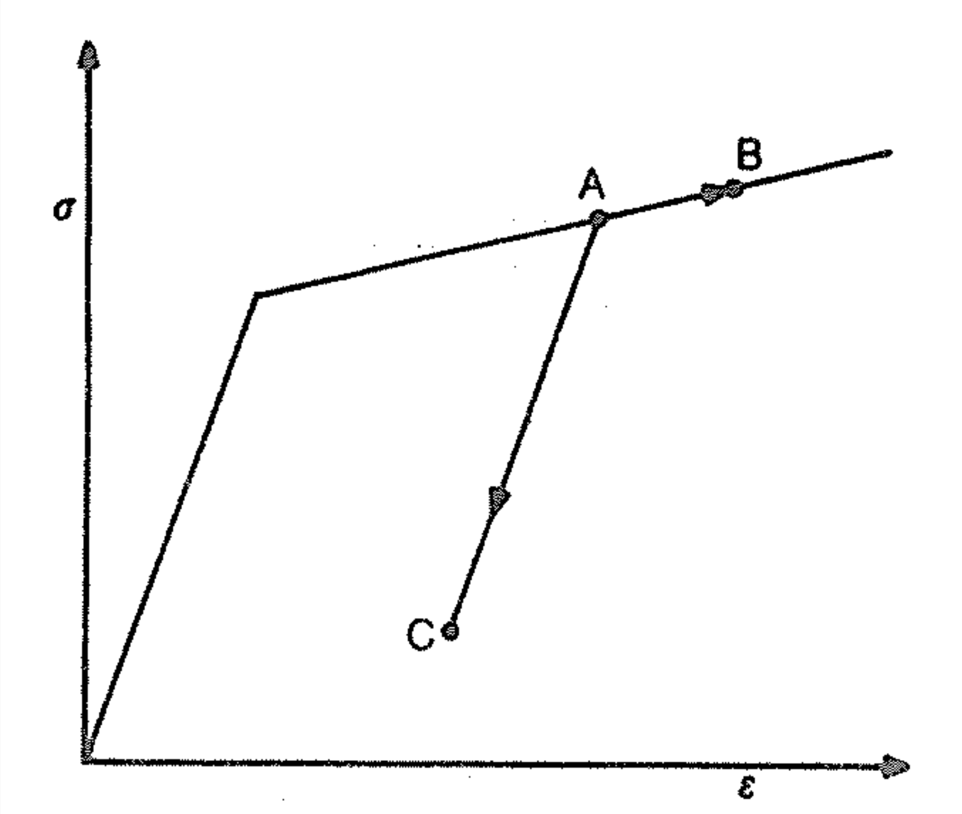
- We have only considered linear or nonlinear elasticity when calculating stress from strain. Iterative (Newton Raphson based) solution method with small strain increment with linear elasticity can be simply put as:

$$\mathbf{K}_t \Delta \mathbf{p} = \mathbf{q}_e - \underbrace{\left(\int_V \mathbf{b}\boldsymbol{\sigma} \, dV \right)}_{\mathbf{q}_i} = -\mathbf{g}(\boldsymbol{\sigma})$$

- The main objective of the present chapter is to concentrate on a '**numerical solution**' of **plasticity**.

- Because plastic flow rules are incremental in nature, elasto-plastic problems should strictly be solved using small equilibrium steps: **strain increments**

- When **plastic deformation** occurs due to a strain increment, the **flow rule** must always be satisfied, and the stress state must be on the **yield surface** if time in-dependent plasticity is employed. C.f. visco-plasticity



[Fig 6.1 One-dimensional stress-strain relationship]

- Equilibrium point **A** may continue to point **B** or unloaded elastically to **C.**
  - path **AB**: **elasto-plastic tangent stiffness** should be used.
  - path **AC**: **elastic tangent stiffness** should be used.

- Roles of plasticity algorithms of a finite element code are:
  - The formation of the **standard tangent modular matrix** for use in the incremental tangent stiffness matrix of the structure or for use with the stress/strain law
  - The formation of **'consistent' tangent modular matrix** for Newton-Raphson iterations.
  - The integration of the stress/strain laws to update **stress**.

- Structural tangent stiffness matrix

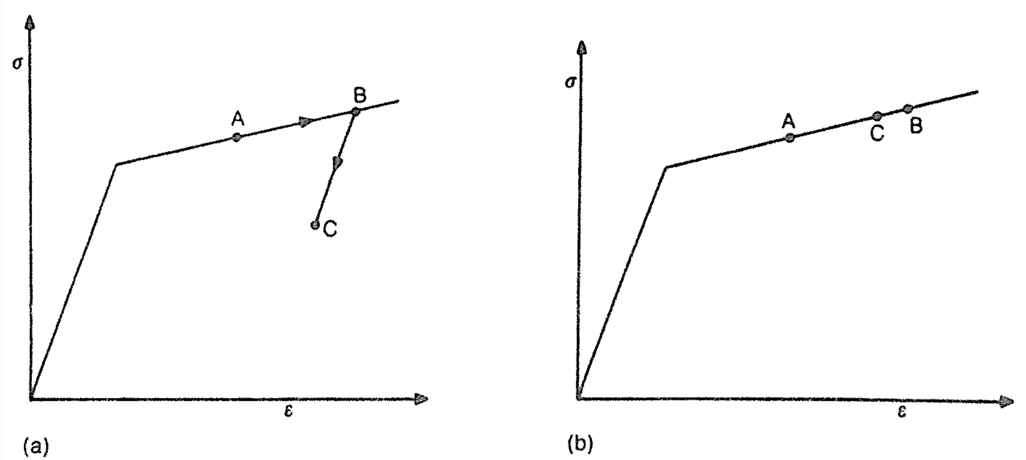
$$\mathbf{K}_t = \int \mathbf{B}^T \mathbf{C}_t \mathbf{B} dV + (initial\ stress\ matrix)$$ [eq. 6.1]

where $\mathbf{C}_t = \dfrac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : standard\ tangential\ modular\ matrix$ [eq. 6.2]

- The initial stress matrix is related to geometrical non-linearity, and will be ignored in this chapter.

- **'Consistent' tangent modular matrix** is the matrix which is consistent with certain numerical form of stress update algorithm, which can decrease the number of iterations.

- **The present chapter only concentrates on**
  - Von Mises yield function vs. other anisotropic yield functions
  - Isotropic hardening vs. kinematic hardening/mixed (combined) hardening
  - Associated flow rule vs. non-associated flow rule
  - Time in-dependent plasticity vs. visco-plasticity (time dependent)



[Fig 6.2 One-dimensional illustration of alternative updating strategies]
(a) strategy A (b) strategy B

- **Stress update can be done in two ways:**
  - Strategy **A** : iterative strains
  - Strategy **B** : incremental strains

- Strategy **A** : iterative strains

$$\delta\mathbf{p} = -\mathbf{K}_t^{-1}\mathbf{g}(\boldsymbol{\sigma}_{n-1}) \Rightarrow \delta\boldsymbol{\varepsilon} = fn\left(\delta\mathbf{p}\right) \Rightarrow \delta\boldsymbol{\sigma} = fn(\boldsymbol{\sigma}_{n-1}, \delta\boldsymbol{\varepsilon}) \Rightarrow \boldsymbol{\sigma}_n = \boldsymbol{\sigma}_{n-1} + \delta\boldsymbol{\sigma}$$

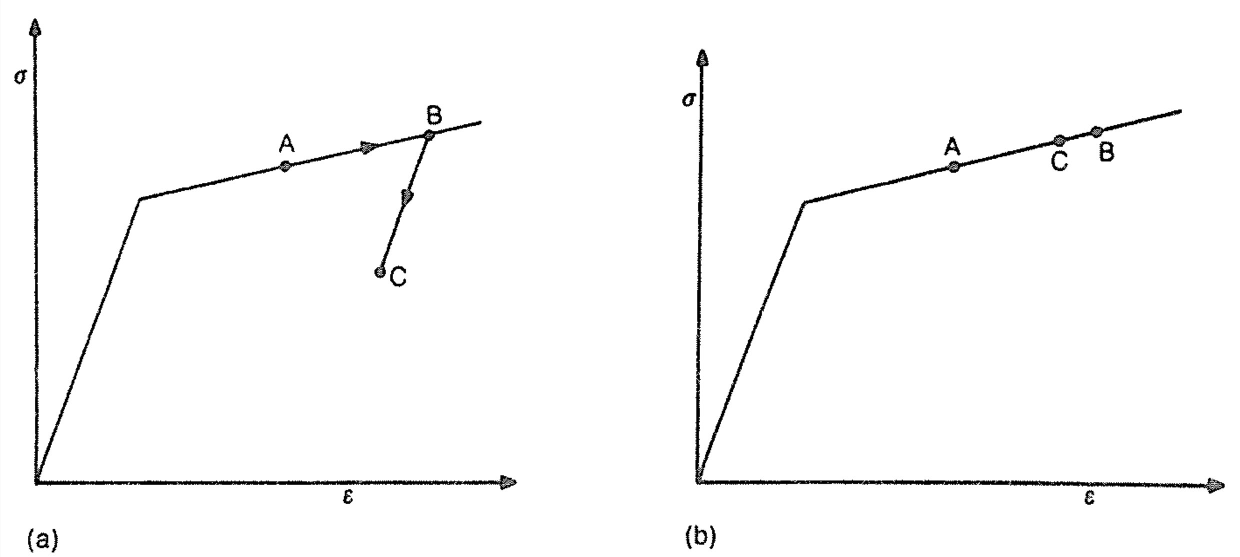
$: \boldsymbol{\sigma}$ *consistently accumulated by* $\delta\boldsymbol{\sigma}$ *at each iteration*

- Strategy **B** : incremental strains

$$\delta\mathbf{p} = -\mathbf{K}_t^{-1}\mathbf{g}(\boldsymbol{\sigma}_{n-1}) \Rightarrow \Delta\mathbf{p}_n = \Delta\mathbf{p}_{n-1} + \delta\mathbf{p} \Rightarrow \Delta\boldsymbol{\varepsilon}_n = fn\left(\Delta\mathbf{p}_n\right)$$

$$\Rightarrow \Delta\boldsymbol{\sigma}_n = fn(\boldsymbol{\sigma}_o, \Delta\boldsymbol{\varepsilon}_n) \Rightarrow \boldsymbol{\sigma}_n = \boldsymbol{\sigma}_o + \Delta\boldsymbol{\sigma}_n$$

$: \Delta\boldsymbol{\sigma}$ *changes at each iteration, but* $\boldsymbol{\sigma}_o$ *remains*



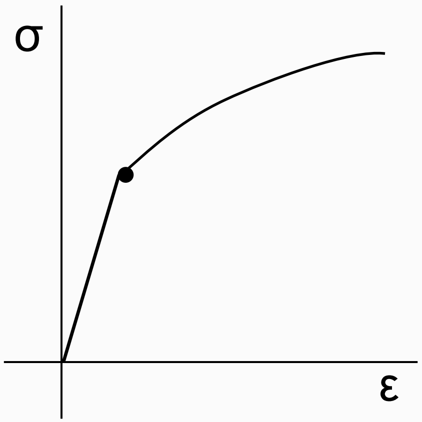
[Fig 6.2 One-dimensional illustration of alternative updating strategies]
(a) strategy A (b) strategy B

- Strategy A is not recommended as it may lead to ‘spurious unloading’.
  - $\delta \mathbf{p} = -\mathbf{K}_t^{-1} \mathbf{g}(\boldsymbol{\sigma}_{n-1})$ might result in negative iterative displacement.

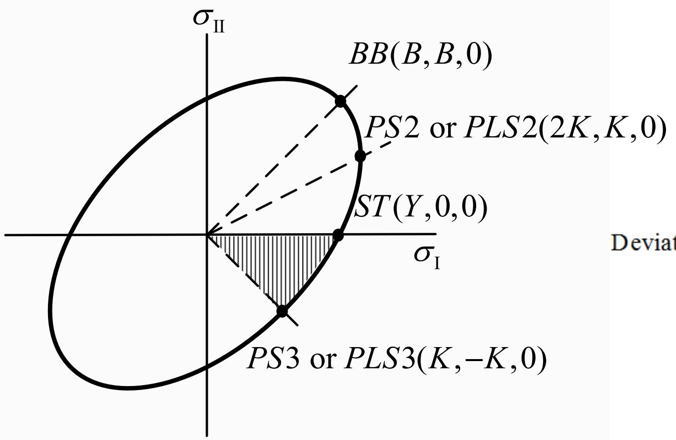
- Since $\boldsymbol{\sigma}_o$ in strategy B will be always in equilibrium, it does not lead to ‘spurious unloading’.

For isotropic case,

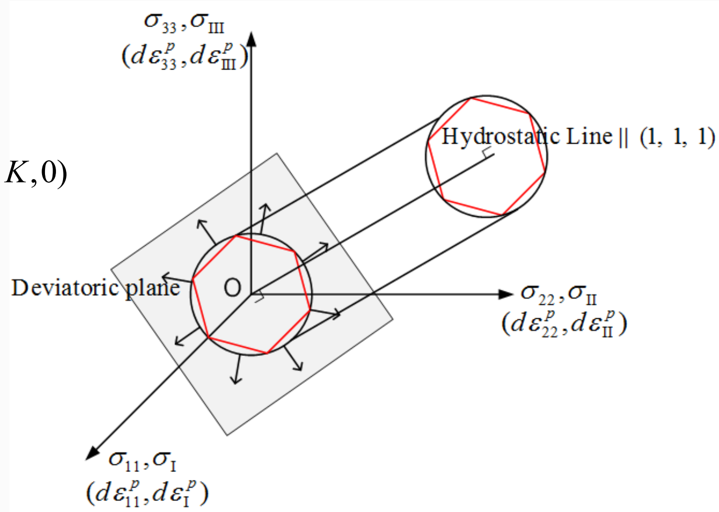
1-D : point



2-D : line



3-D : surface



Isotropic generalization

$$f(\sigma_{ij})$$

$$= f(\sigma_I, \sigma_{II}, \sigma_{III}, \tilde{\mathbf{n}}_I, \tilde{\mathbf{n}}_{II}, \tilde{\mathbf{n}}_{III}) \longleftarrow \text{General case}$$

$$= \boxed{f(\sigma_I, \sigma_{II}, \sigma_{III})} \;\; \textit{as a symmetric function} \longleftarrow \text{Isotropic case}$$

$$= \boxed{f(I_1, I_2, I_3)} \longleftarrow \text{Denote with invariants}$$

where $\sigma_I, \sigma_{II}, \sigma_{III}$: *pricipal stresses*

$\tilde{\mathbf{n}}_I, \tilde{\mathbf{n}}_{II}, \tilde{\mathbf{n}}_{III}$: *principal directions*

I_1, I_2, I_3 : the three invariants of $\boldsymbol{\sigma}$

Hydrostatic & deviatoric stresses

- Decomposition of the stress into two components: the hydrostatic and deviatoric components

$$\sigma_{ij} = S_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad \text{where } \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\boldsymbol{\sigma} = \mathbf{S} + \frac{1}{3}trace(\boldsymbol{\sigma})\mathbf{I}$$

$\mathbf{S}$: deviatric stress

$$(\sigma_{ij}) = \begin{pmatrix} \frac{2\sigma_{11}-\sigma_{22}-\sigma_{33}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \frac{2\sigma_{22}-\sigma_{33}-\sigma_{11}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \frac{2\sigma_{33}-\sigma_{11}-\sigma_{22}}{3} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \sigma_{11}+\sigma_{22}+\sigma_{33} & 0 & 0 \\ 0 & \sigma_{11}+\sigma_{22}+\sigma_{33} & 0 \\ 0 & 0 & \sigma_{11}+\sigma_{22}+\sigma_{33} \end{pmatrix}$$

Similarly,

Deviatoric strains

$$d\boldsymbol{\varepsilon}^p = d\mathbf{e}^p + \frac{1}{3}trace(d\boldsymbol{\varepsilon}^p)\mathbf{I}$$

$d\boldsymbol{\varepsilon}^p$: (plastic) strain increment

$d\mathbf{e}^p$: deviatoric (plastic) strain increment

$$trace(d\mathbf{e}^p) = de^p_{kk} = 0 \quad (for\ crystaline\ materials)$$

Yield surface for incompressible case

- For crystal materials, plastic deformation is incompressible and the hydrostatic stress does not affect the plastic deformation since the plastic deformation is incurred by shear stress.

$$d\varepsilon^p{}_{kk} = 0 \quad for\ arbitrary\ \sigma_{kk}$$

$$\therefore dW^p = S_{ij} de^p{}_{ij}$$

$$f = f(\sigma_{ij}) \text{ is independent of } \sigma_{kk}$$

$$f = f(\sigma_{ij}) = f(S_{ij}, \sigma_{kk}) \Rightarrow f = f(S_{ij})$$

Isotropic & Incompressible yield surface

$f(\sigma_{ij}) \leftarrow$ General case

$= f(S_{ij}) \leftarrow$ incompressible case

$= f(S_I, S_{II}, S_{III}, n_I^S, n_{II}^S, n_{III}^S) \leftarrow$ Isotropic case

$= \boxed{f(S_I, S_{II}, S_{III})}$ as a symmetric function

$= f(J_1, J_2, J_3) \leftarrow$ Denote with invariants

$= \boxed{f(J_2, J_3)} \leftarrow$ Since $J_1 = 0$

where J_1, J_2, J_3 are invariants of deviatoric tensor

$J_1 = trace(\mathbf{S}) = 0$

$J_2 = \frac{1}{2}\left(trace(\mathbf{S})^2 - trace(\mathbf{S}^2) \right)$

$J_3 = \det(\mathbf{S})$

Von Mises isotropic yield function (or J_2 plasticity)

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{S}) = f(J_2, J_3) = J_2 = \frac{1}{2}\left|\boldsymbol{S}\right|^2 = \frac{1}{2}S_{ij}S_{ij} = \frac{1}{2}(S_I^2 + S_{II}^2 + S_{III}^2) = Const.$$

$$\boxed{f(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{S}) = \sqrt{\alpha S_{ij}S_{ij}} = c}$$

$$\boxed{\sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6\sigma_{12}{}^2 + 6\sigma_{23}{}^2 + 6\sigma_{31}{}^2\right]}(=Y) = \bar{\sigma}}$$

$$f = f(\sigma_{ij}) = f(S_{ij})$$

$$f = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \bar{\sigma}$$

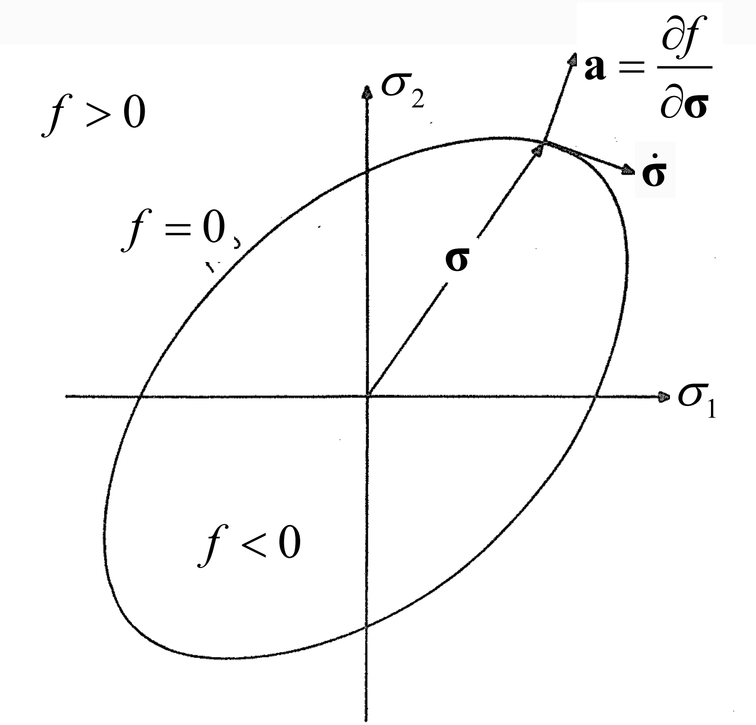
$$\Leftrightarrow f = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \sqrt{\frac{3}{2}(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij})(\sigma_{ij} - \frac{1}{3}\sigma_{pp}\delta_{ij})} = \bar{\sigma}$$

$$\Leftrightarrow \sqrt{\frac{3}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}\sigma_{pp}\right)} = \bar{\sigma}$$

$$\Leftrightarrow \sqrt{\frac{3}{2}\left(\sigma_{11}{}^2 + \sigma_{22}{}^2 + \sigma_{33}{}^2 + 2\sigma_{12}{}^2 + 2\sigma_{23}{}^2 + 2\sigma_{31}{}^2 - 1/3(\sigma_{11} + \sigma_{22} + \sigma_{33})^2\right)} = \bar{\sigma}$$

$$\therefore \sqrt{\frac{1}{2}\left\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6\sigma_{12}{}^2 + 6\sigma_{23}{}^2 + 6\sigma_{31}{}^2\right\}} = \bar{\sigma}$$

$$f = \left(\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2\right)^{1/2} - \sigma_0 = \sigma_e - \sigma_0 \qquad \text{[eq. 6.3]}$$



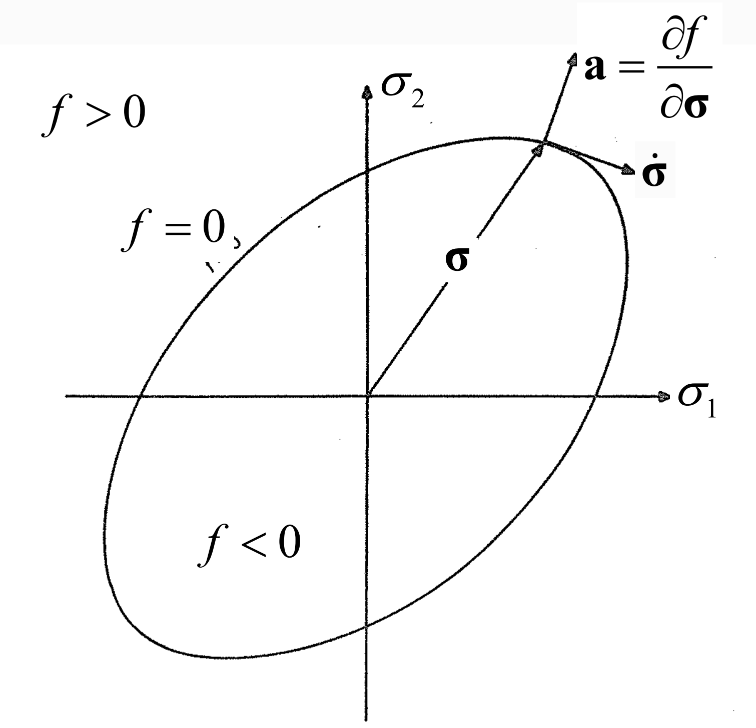
[Fig 6.3 The von Mises yield criterion under plane-stress condition]

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right) = \dot{\lambda}\mathbf{a} = \begin{pmatrix} \dot{\varepsilon}_{px} \\ \dot{\varepsilon}_{py} \\ \dot{\varepsilon}_{pxy} \end{pmatrix} = \frac{\dot{\lambda}}{2\sigma_e}\begin{pmatrix} 2\sigma_x - \sigma_y \\ 2\sigma_y - \sigma_x \\ 6\tau_{xy} \end{pmatrix} \qquad \text{[eq. 6.4]}$$

$\dot{\lambda}$: *plastic strain − rate multiplier*

$$\dot{\boldsymbol{\sigma}} = \begin{pmatrix} \dot{\boldsymbol{\sigma}}_x \\ \dot{\boldsymbol{\sigma}}_y \\ \dot{\boldsymbol{\sigma}}_{xy} \end{pmatrix} = [\mathbf{C}]\left(\begin{pmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\varepsilon}_{xy} \end{pmatrix} - \begin{pmatrix} \dot{\varepsilon}_{px} \\ \dot{\varepsilon}_{py} \\ \dot{\varepsilon}_{pxy} \end{pmatrix}\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}}_t - \dot{\boldsymbol{\varepsilon}}_p\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}}_t - \dot{\lambda}\mathbf{a}\right) \qquad \text{[eq. 6.5]}$$

$$\mathbf{C} = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \qquad \text{[eq. 6.6]}$$



[Fig 6.3 The von Mises yield criterion under plane-stress condition]

$$\dot{\boldsymbol{\sigma}} = \begin{pmatrix} \dot{\boldsymbol{\sigma}}_x \\ \dot{\boldsymbol{\sigma}}_y \\ \dot{\boldsymbol{\sigma}}_{xy} \end{pmatrix} = [\mathbf{C}]\left(\begin{pmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\varepsilon}_{xy} \end{pmatrix} - \begin{pmatrix} \dot{\varepsilon}_{px} \\ \dot{\varepsilon}_{py} \\ \dot{\varepsilon}_{pxy} \end{pmatrix}\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}}_t - \dot{\boldsymbol{\varepsilon}}_p\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}}_t - \dot{\lambda}\mathbf{a}\right)$$

[eq. 6.5]

- $\dot{\lambda}$ should be always positive.
- For plastic flow to occur, **the stresses must remain on the yield surface or consistency condition**

$$\dot{f} = \frac{\partial f^T}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} = \mathbf{a}^T\dot{\boldsymbol{\sigma}} = \mathbf{a} : \dot{\boldsymbol{\sigma}} = 0 \qquad \text{[eq. 6.7]}$$

- By combining eq.6.7 and eq.6.5,

$$\dot{\lambda} = \frac{\mathbf{a}^T\mathbf{C}\dot{\boldsymbol{\varepsilon}}}{\mathbf{a}^T\mathbf{C}\mathbf{a}} = \frac{\mathbf{a} : \mathbf{C} : \dot{\boldsymbol{\varepsilon}}}{\mathbf{a} : \mathbf{C} : \mathbf{a}} \qquad \text{[eq. 6.8]}$$

- Consequently,

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}_t\dot{\boldsymbol{\varepsilon}} = \mathbf{C}\left(\mathbf{I} - \frac{\mathbf{a}\mathbf{a}^T\mathbf{C}}{\mathbf{a}^T\mathbf{C}\mathbf{a}}\right)\dot{\boldsymbol{\varepsilon}}$$

$$= \left(\mathbf{C} - \frac{1}{\mathbf{a} : \mathbf{C} : \mathbf{a}}(\mathbf{C} : \mathbf{a}) \otimes (\mathbf{C} : \mathbf{a})\right) : \dot{\boldsymbol{\varepsilon}} \qquad \text{[eq. 6.9]}$$

6.3.1 Non-associative plasticity

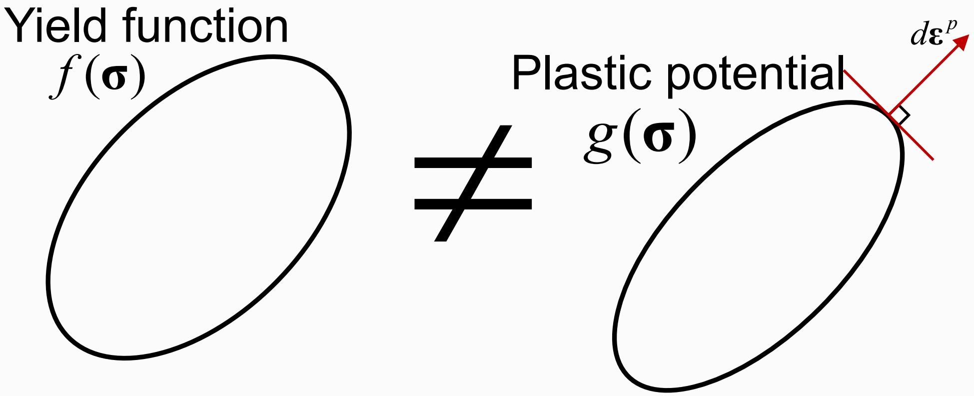
- If a separate plastic potential g exists,

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\frac{\partial g}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\mathbf{b}$$

$$\Rightarrow \dot{\lambda} = \frac{\mathbf{a}:\mathbf{C}:\dot{\boldsymbol{\varepsilon}}}{\mathbf{a}:\mathbf{C}:\mathbf{b}}$$

$$\Rightarrow \mathbf{C}_t = \mathbf{C}\left(\mathbf{I} - \frac{\mathbf{b}\mathbf{a}^T\mathbf{C}}{\mathbf{a}^T\mathbf{C}\mathbf{b}}\right)$$

$\mathbf{C}_t$ is generally nonsymmetric.



6.4.1 Isotropic strain hardening

- Hardening can be introduced by changing the fixed yield stress, σ_0 , in eq.6.3 to a variable stress, $\sigma_0(\varepsilon_{ps})$, so that

$$f = \sigma_e - \sigma_0(\varepsilon_{ps})$$ [eq. 6.10]

- The variable yield stress is now a function of the equivalent plastic strain:

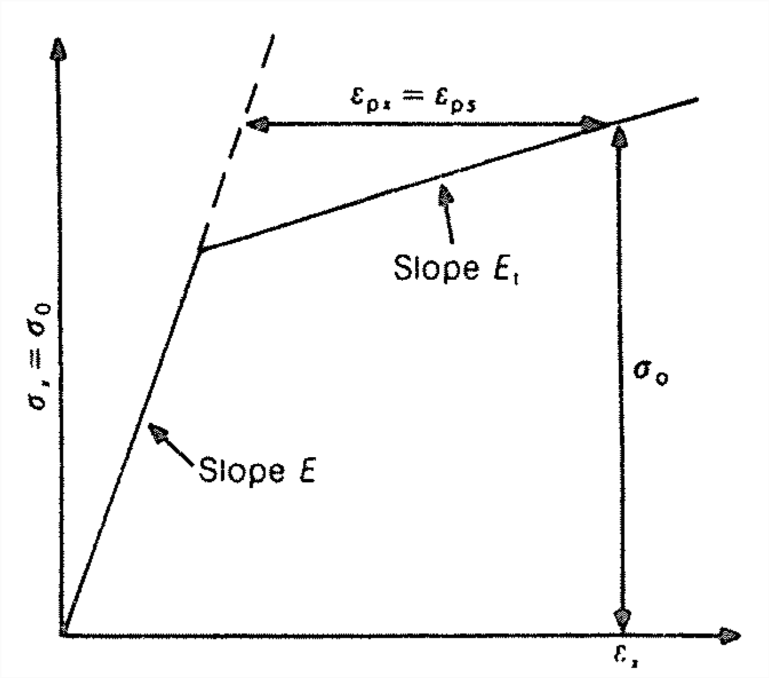
$$\varepsilon_{ps} = \int \dot{\varepsilon}_{ps} dt$$ [eq. 6.11]

where $$\dot{\varepsilon}_{ps} = \frac{2}{\sqrt{3}} \left(\dot{\varepsilon}_{px}^2 + \dot{\varepsilon}_{py}^2 + \dot{\varepsilon}_{px}\dot{\varepsilon}_{py} + \frac{1}{4}\dot{\gamma}_{pxy}^2 \right)^{1/2}$$ [eq. 6.12]

  - Under uniaxial tension σ_x

$$\dot{\varepsilon}_{py} = \dot{\varepsilon}_{pz} = -\frac{1}{2}\dot{\varepsilon}_{px} \quad \Rightarrow \quad \dot{\varepsilon}_{ps} = \dot{\varepsilon}_{px}$$

  - The relationship between σ_0 and ε_{ps} can be taken from the uniaxial stress/plastic strain relationship.



[Fig 6.5 One-dimensional stress/strain relationship with linear hardening]

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right) = \dot{\lambda}\mathbf{a} = \begin{pmatrix} \dot{\varepsilon}_{px} \\ \dot{\varepsilon}_{py} \\ \dot{\varepsilon}_{pxy} \end{pmatrix} = \frac{\dot{\lambda}}{2\sigma_e}\begin{pmatrix} 2\sigma_x - \sigma_y \\ 2\sigma_y - \sigma_x \\ 6\tau_{xy} \end{pmatrix} \quad \text{[eq. 6.4]}$$

$$\dot{\varepsilon}_{ps} = \frac{2}{\sqrt{3}}\left(\dot{\varepsilon}_{px}^2 + \dot{\varepsilon}_{py}^2 + \dot{\varepsilon}_{px}\dot{\varepsilon}_{py} + \frac{1}{4}\dot{\gamma}_{pxy}^2\right)^{1/2} \quad \text{[eq. 6.12]}$$

- To obtain E_t , $\dfrac{\partial \sigma_0}{\partial \varepsilon_{ps}}$ is required.

$$H' = \frac{\partial \sigma_0}{\partial \varepsilon_{ps}} = \frac{\partial \sigma_x}{\partial \varepsilon_{px}} = \frac{E_t}{1 - E_t / E} \quad \text{[eq. 6.13]}$$

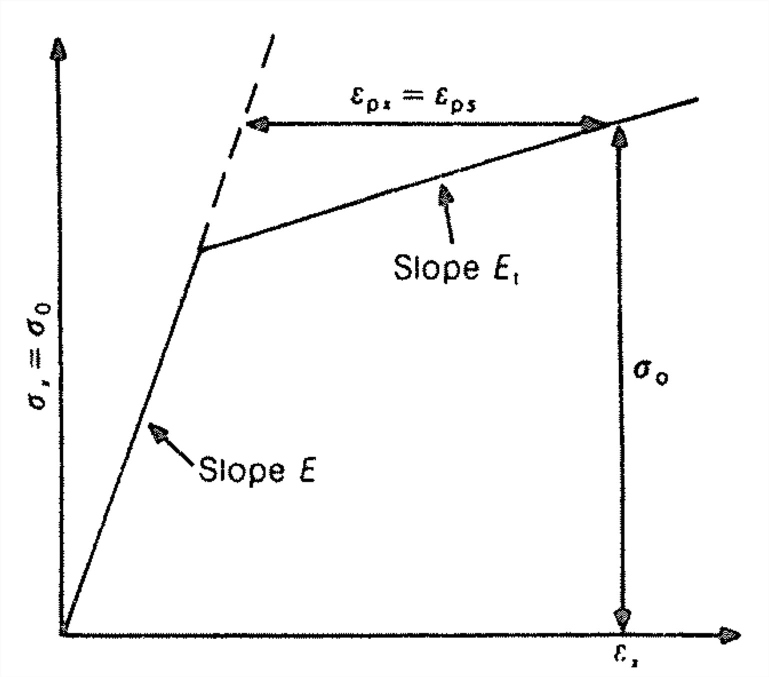
- Once hardening is introduced, the tangency condition of eq.6.7 is modified to:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial \varepsilon_{ps}}\dot{\varepsilon}_{ps} = \mathbf{a}^T\dot{\boldsymbol{\sigma}} - H'\dot{\varepsilon}_{ps} \quad \text{[eq. 6.14]}$$

- Substitution form eq.6.4 into eq.6.12 gives:

$$\dot{\varepsilon}_{ps} = \dot{\lambda} = B(\boldsymbol{\sigma})\dot{\lambda} \quad \text{[eq. 6.15]}$$

  - For the present von Mises yield criterion, $B(\boldsymbol{\sigma}) = 1$ but for other criteria, maybe not.



[Fig 6.5 One-dimensional stress/strain relationship with linear hardening]

$$\dot{f} = \frac{\partial f^T}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial \varepsilon_{ps}}\dot{\varepsilon}_{ps} = \mathbf{a}^T\dot{\boldsymbol{\sigma}} - H'\dot{\varepsilon}_{ps} \qquad \text{[eq. 6.14]}$$

$$\dot{\boldsymbol{\sigma}} = \begin{pmatrix} \dot{\boldsymbol{\sigma}}_x \\ \dot{\boldsymbol{\sigma}}_y \\ \dot{\boldsymbol{\sigma}}_{xy} \end{pmatrix} = [\mathbf{C}]\left(\begin{pmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\varepsilon}_{xy} \end{pmatrix} - \begin{pmatrix} \dot{\varepsilon}_{px} \\ \dot{\varepsilon}_{py} \\ \dot{\varepsilon}_{pxy} \end{pmatrix}\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}}_t - \dot{\boldsymbol{\varepsilon}}_p\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}}_t - \dot{\lambda}\mathbf{a}\right) \qquad \text{[eq. 6.5]}$$

- Substituting from eq.6.15 into eq.6.14 gives

$$\dot{f} = \mathbf{a}^T\dot{\boldsymbol{\sigma}} - H'B\dot{\lambda} = \mathbf{a}^T\dot{\boldsymbol{\sigma}} - A\dot{\lambda} \qquad \text{[eq. 6.16]}$$

- Substituting from eq.6.5 into eq.6.16 give

$$\dot{\lambda} = \frac{\mathbf{a}^T\mathbf{C}\dot{\mathbf{a}}}{\mathbf{a}^T\mathbf{C}\mathbf{a} + A'} = \frac{\mathbf{a}:\mathbf{C}:\dot{\mathbf{a}}}{\mathbf{a}:\mathbf{C}:\mathbf{a} + A'} \qquad \text{[eq. 6.17]}$$

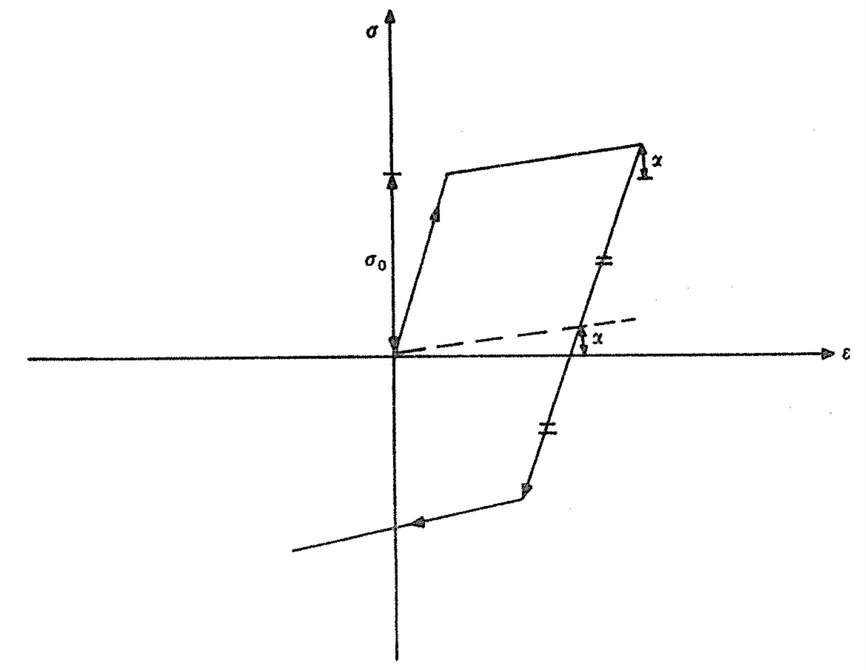
$$\Rightarrow \dot{\boldsymbol{\sigma}} = \mathbf{C}_t\dot{\boldsymbol{\varepsilon}} = \mathbf{C}\left(\mathbf{I} - \frac{\mathbf{a}\mathbf{a}^T\mathbf{C}}{\mathbf{a}^T\mathbf{C}\mathbf{a} + A'}\right)\dot{\boldsymbol{\varepsilon}}$$

$$= \left(\mathbf{C} - \frac{1}{\mathbf{a}:\mathbf{C}:\mathbf{a} + A'}(\mathbf{C}:\mathbf{a}) \otimes (\mathbf{C}:\mathbf{a})\right):\dot{\boldsymbol{\varepsilon}} \qquad \text{[eq. 6.18]}$$

6.4.2 Isotropic work hardening

Reading assignment

6.4.3 Kinematic hardening



[Fig 6.6 One-dimensional illustration of kinematic hardening]

- Isotropic hardening cannot represent the asymmetry in tension and compression after plastic loading: **Bauschinger effect**
- For certain problems (ex: low-cycle fatigue, cup drawing, …), **Bauschinger effect** may be significant.
- Yielding in tension has lowered the compressive stress by kinematic shfit

$$(\sigma - \alpha) = \pm \sigma_0$$ [eq. 6.25] $\alpha : kinematic\ shift$

- For the general three-dimensional case, the von Mises yield criterion is

$$
\begin{aligned}
f &= \sigma_e - \sigma_0 = \sqrt{3} J_2^{1/2} - \sigma_0 \\
&= \frac{1}{\sqrt{2}} \left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \right]^{1/2} - \sigma_0 \\
&= \sqrt{3} \left[\frac{1}{2} \left(s_x^2 + s_y^2 + s_z^2 \right) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right]^{1/2} - \sigma_0 \\
&= \sqrt{\frac{3}{2}} \left(\mathbf{s}^T \mathbf{L} \mathbf{s} \right)^{1/2} - \sigma_0 = \sqrt{\frac{3}{2}} \left(\mathbf{s}_2 : \mathbf{s}_2 \right)^{1/2} - \sigma_0
\end{aligned}
$$

[eq. 6.26]

where $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ [eq. 6.27] and $\mathbf{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix}$ [eq. 6.28]

Von Mises isotropic yield criterion

From normality rule,

$$d\varepsilon_{ij}^{p} = AS_{ij}$$

From work equivalence principle,

$$AS_{ij}S_{ij} = S_{ij}d\varepsilon_{ij}^{p} = \bar{\sigma}d\bar{\varepsilon} = \frac{A}{\alpha}\bar{\sigma}^{2} \quad \rightarrow \quad A = \frac{\alpha d\bar{\varepsilon}}{\bar{\sigma}}$$

From above equations,

$$d\varepsilon_{ij}^{p} = AS_{ij} = \alpha d\bar{\varepsilon}\frac{S_{ij}}{\bar{\sigma}}$$

In addition,

$$d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p} = AS_{ij}d\varepsilon_{ij}^{p} = A\bar{\sigma}d\bar{\varepsilon} = \alpha d\bar{\varepsilon}^{2}$$

The von Mises yield surface, which is incompressible, isotropic and symmetric for tension and compression, is a sphere in the eight-dimensional deviatoric space as shown in Eq. (12.15). Therefore, the plastic strain increment is figuratively proportional to the deviatoric stress by the normality rule; i.e., $d\varepsilon_{ij}^{p} = AS_{ij}$ with a proportional constant A. Then,

$$AS_{ij}S_{ij} = S_{ij}d\varepsilon_{ij}^{p} = \bar{\sigma}d\bar{\varepsilon} = \frac{A}{\alpha}\bar{\sigma}^{2}$$

considering Eqs. (12.15) and (13.19). Therefore, $A = \frac{\alpha d\bar{\varepsilon}}{\bar{\sigma}}$ so that

$$d\varepsilon_{ij}^{p} = \alpha d\bar{\varepsilon}\frac{S_{ij}}{\bar{\sigma}} \tag{13.30}$$

Furthermore, since $d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p} = AS_{ij}d\varepsilon_{ij}^{p} = A\bar{\sigma}d\bar{\varepsilon} = \alpha d\bar{\varepsilon}^{2}$ so that

$$d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha}d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p}} \tag{13.31}$$

with which the plastic strain increment surface is also a sphere in the deviatoric space.

$$f(\boldsymbol{\sigma}) = \bar{\sigma}(\boldsymbol{\sigma}) = \bar{\sigma}(\mathbf{S}) = \sqrt{\alpha S_{ij}S_{ij}} = c \tag{12.15}$$

$$\boxed{\therefore d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha}d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p}}}$$

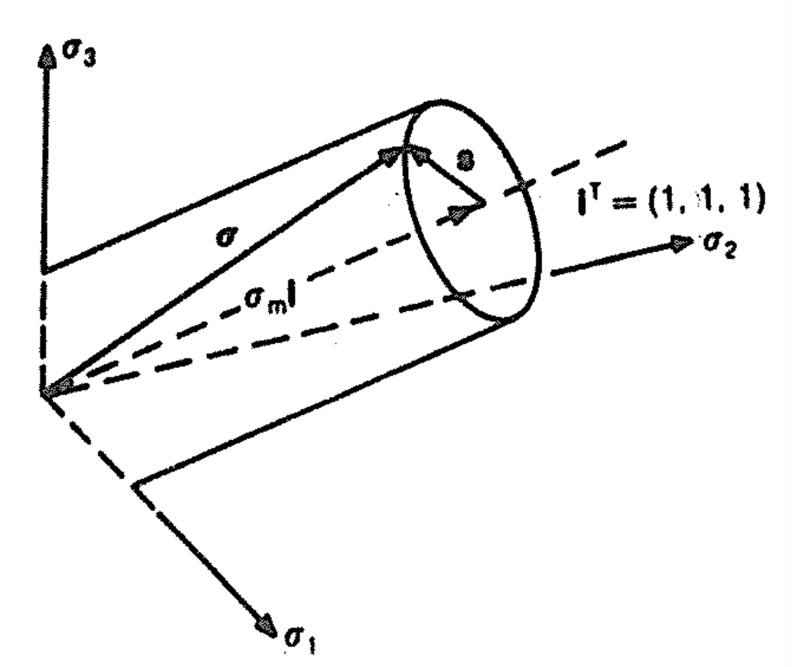
Von Mises isotropic yield criterion

Reference state : simple tension

$$\boxed{\therefore d\bar{\varepsilon} = \sqrt{\frac{1}{\alpha} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}}} \qquad \alpha = \frac{3}{2}$$

$$\sigma_{ij} = \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow S_{ij} = \begin{pmatrix} \frac{2}{3}Y & 0 & 0 \\ 0 & -\frac{1}{3}Y & 0 \\ 0 & 0 & -\frac{1}{3}Y \end{pmatrix} \sim d\varepsilon_{ij}^{p} \sim \begin{pmatrix} d\varepsilon_{11}^{ST,p} & 0 & 0 \\ 0 & -\frac{1}{2}d\varepsilon_{11}^{ST,p} & 0 \\ 0 & 0 & -\frac{1}{2}d\varepsilon_{11}^{ST,p} \end{pmatrix}$$

$$\boxed{d\bar{\varepsilon} = \sqrt{\frac{2}{3}(1+\frac{1}{4}+\frac{1}{4})} d\varepsilon_{11}^{ST,p} = d\varepsilon_{11}^{ST,p}}$$



[Fig 6.7 von Mises yield criterion in three-dimensional principal stress space]

$$\boldsymbol{\varepsilon}_2 = \varepsilon_m \mathbf{I} + \boldsymbol{\varepsilon}_{2d} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_{xx} - \varepsilon_m & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{yy} - \varepsilon_m & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{zz} - \varepsilon_m \end{bmatrix} \quad \text{[eq. 4.19]}$$

- The equivalent plastic strain rate is given by

$$\dot{\varepsilon}_{ps} = \frac{2}{\sqrt{3}}\left(\dot{\varepsilon}_{px}^2 + \dot{\varepsilon}_{py}^2 + \dot{\varepsilon}_{pz}^2 + \frac{1}{2}\left(\dot{\gamma}_{pxy}^2 + \dot{\gamma}_{pyz}^2 + \dot{\gamma}_{pxz}^2\right)\right)^{1/2}$$

$$= \sqrt{\frac{2}{3}}\left(\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p\right)^{1/2} = \sqrt{\frac{2}{3}}\left(\dot{\mathbf{e}}_p : \dot{\mathbf{e}}_p\right)^{1/2} \quad \text{[eq. 6.29]}$$

$$\dot{\mathbf{e}}_p : \textit{deviatoric plastic strain (refer to eq.4.19)}$$

- Only elastic strain is related to stress by

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix}$$

[eq. 6.30]

$$\boldsymbol{\sigma} = \mathbf{C}_2 \boldsymbol{\varepsilon} \quad \text{or} \quad \boldsymbol{\sigma}_2 = \mathbf{C}_4 : \boldsymbol{\varepsilon}_2 \quad \text{[eq. 6.31]}$$

$$f = \sigma_e - \sigma_0 = \sqrt{3} J_2^{1/2} - \sigma_0$$
$$= \frac{1}{\sqrt{2}} \left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \right]^{1/2} - \sigma_0$$

[eq. 6.26]

- Differentiating eq.6.26 leads to

$$\mathbf{a}^T = \frac{\partial f^T}{\partial \boldsymbol{\sigma}} = \frac{1}{2\sigma_e} \left\{ \left(2\sigma_x - \sigma_y - \sigma_z\right), \left(2\sigma_y - \sigma_x - \sigma_z\right), \left(2\sigma_z - \sigma_x - \sigma_y\right), 6\tau_{xy}, 6\tau_{yz}, 6\tau_{zx} \right\}$$
$$= \frac{3}{2\sigma_e} \left\{ s_x, s_y, s_z, 2\tau_{xy}, 2\tau_{yz}, 2\tau_{zx} \right\} = \frac{3}{2} (\mathbf{Ls})^T = \frac{\partial f^T}{\partial \mathbf{s}}$$

[eq. 6.32]

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \right) = \dot{\lambda} \mathbf{a} = \begin{pmatrix} \dot{\varepsilon}_{px} \\ \dot{\varepsilon}_{py} \\ \dot{\varepsilon}_{pxy} \end{pmatrix} = \frac{\dot{\lambda}}{2\sigma_e} \begin{pmatrix} 2\sigma_x - \sigma_y \\ 2\sigma_y - \sigma_x \\ 6\tau_{xy} \end{pmatrix}$$

[eq. 6.4]

$$\dot{\varepsilon}_{ps} = \frac{2}{\sqrt{3}} \left(\dot{\varepsilon}_{px}^2 + \dot{\varepsilon}_{py}^2 + \dot{\varepsilon}_{pz}^2 + \frac{1}{2} \left(\dot{\gamma}_{pxy}^2 + \dot{\gamma}_{pyz}^2 + \dot{\gamma}_{pxz}^2 \right) \right)^{1/2}$$
$$= \sqrt{\frac{2}{3}} \left(\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p \right)^{1/2} = \sqrt{\frac{2}{3}} \left(\dot{\mathbf{e}}_p : \dot{\mathbf{e}}_p \right)^{1/2}$$

[eq. 6.29]

- Or, using the tensor(indicial) form,

$$\mathbf{a} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial \mathbf{s}} = \frac{3}{2\sigma_e} \mathbf{s}$$

[eq. 6.33]

- As in eq.6.4, $\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda} \mathbf{a}$ so that in eq.6.29

$$\dot{\varepsilon}_{ps} = \sqrt{\frac{2}{3}} \dot{\lambda} \left(\mathbf{a}^T \mathbf{L}^{-1} \mathbf{a} \right)^{1/2} = \sqrt{\frac{2}{3}} \dot{\lambda} \left(\mathbf{a}_2 : \mathbf{a}_2 \right)^{1/2} = \sqrt{\frac{3}{2}} \frac{\dot{\lambda}}{\sigma_e} \left(\mathbf{s}^T \mathbf{L} \mathbf{s} \right)^{1/2} = \dot{\lambda}$$

[eq. 6.34]

6.5.1 Splitting the update into volumetric and deviatoric parts

- For the 'radial return' method in section 6.6.7, it is useful to split the stress update into volumetric and deviatoric components.

- For von Mises yield criterion,

$$\mathbf{Ca} = \frac{\sqrt{3}\mu}{\sqrt{J_2}}\mathbf{s} = \frac{3\mu}{\sigma_e}\mathbf{s} = 2\mu\mathbf{L}^{-1}\mathbf{a} \qquad \text{[eq. 6.35]}$$

$$\mathbf{a}^T\mathbf{Ca} = 3\mu \qquad \text{[eq. 6.36]}$$

- Substitution into eq.6.18 gives

$$\boxed{\begin{aligned}\dot{\boldsymbol{\sigma}} &= \mathbf{C}_t\dot{\boldsymbol{\varepsilon}} = \mathbf{C}\left(\mathbf{I} - \frac{\mathbf{aa}^T\mathbf{C}}{\mathbf{a}^T\mathbf{Ca} + A'}\right)\dot{\boldsymbol{\varepsilon}} \\ &= \left(\mathbf{C} - \frac{1}{\mathbf{a}:\mathbf{C}:\mathbf{a} + A'}(\mathbf{C}:\mathbf{a})\otimes(\mathbf{C}:\mathbf{a})\right):\dot{\boldsymbol{\varepsilon}}\end{aligned}} \qquad \text{[eq. 6.18]}$$

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{C} - \frac{3\mu}{\sigma_e^2\left(1 + \frac{A'}{3\mu}\right)}\mathbf{ss}^T\right)\dot{\boldsymbol{\varepsilon}} = \left(\mathbf{C} - \frac{3\mu}{\sigma_e^2\left(1 + \frac{A'}{3\mu}\right)}\mathbf{s}\otimes\mathbf{s}\right):\dot{\boldsymbol{\varepsilon}} \qquad \text{[eq. 6.37]}$$

- In addition, the total strain rate, $\dot{\boldsymbol{\varepsilon}}$, can be split into

$$\dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_m \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \dot{\mathbf{e}} = \dot{\varepsilon}_m \mathbf{j} + \dot{\mathbf{e}} \qquad \text{[eq. 6.38]}$$

where $\dot{\varepsilon}_m = \left(\dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z \right) / 3 = \frac{1}{3} \mathbf{j}^T \dot{\boldsymbol{\varepsilon}}$

6.5.2 Using tensor notation

$$\dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_m \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \dot{\mathbf{e}} = \dot{\varepsilon}_m \mathbf{j} + \dot{\mathbf{e}}$$ [eq. 6.38]

Voigt

$$\Longleftrightarrow \quad \dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_m \mathbf{1} + \dot{\mathbf{e}}$$ [eq. 6.40]

indicial

$$\mathbf{C}\dot{\boldsymbol{\varepsilon}} = 3k\dot{\varepsilon}_m \mathbf{j} + 2\mu \mathbf{L}^{-1}\dot{\mathbf{e}}$$ [eq. 6.41a]

Voigt

$$\Longleftrightarrow \quad \mathbf{C} : \dot{\boldsymbol{\varepsilon}} = 3k\mathbf{1} + 2\mu\dot{\mathbf{e}}$$ [eq. 6.41b]

indicial

$k : bulk\ modulus$

$\mu : shear\ modulus$

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{C} - \frac{3\mu}{\sigma_e^2 \left(1 + \frac{A'}{3\mu} \right)} \mathbf{s}\mathbf{s}^T \right) \dot{\boldsymbol{\varepsilon}} = \left(\mathbf{C} - \frac{3\mu}{\sigma_e^2 \left(1 + \frac{A'}{3\mu} \right)} \mathbf{s} \otimes \mathbf{s} \right) : \dot{\boldsymbol{\varepsilon}}$$ [eq. 6.37]

Voigt — indicial

$$\mathbf{s}^T \mathbf{j} = 0$$ [eq. 6.42]

$$\Longrightarrow \quad \dot{\boldsymbol{\sigma}} = \dot{\sigma}_m \mathbf{j} + \dot{\mathbf{s}} = 3k\dot{\varepsilon}_m \mathbf{j} + 2\mu \left(\mathbf{L}^{-1} - \frac{3}{2\sigma_e^2 \left(1 + \frac{A'}{3\mu} \right)} \mathbf{s}\mathbf{s}^T \right) \dot{\mathbf{e}} = \mathbf{C}_t \dot{\boldsymbol{\varepsilon}}$$ [eq. 6.43]

indicial

$$\dot{\boldsymbol{\sigma}} = \dot{\sigma}_m \mathbf{j} + \dot{\mathbf{s}} = 3k\dot{\varepsilon}_m \mathbf{j} + 2\mu \left(\mathbf{L}^{-1} - \frac{3}{2\sigma_e^2 \left(1 + \frac{A'}{3\mu}\right)} \mathbf{s}\mathbf{s}^T \right) \dot{\mathbf{e}} = \mathbf{C}_t \dot{\boldsymbol{\varepsilon}}$$ [eq. 6.43]

Voigt

$$\Leftrightarrow \quad \dot{\boldsymbol{\sigma}} = \dot{\sigma}_m \mathbf{1} + \dot{\mathbf{s}} = 3k\dot{\varepsilon}_m \mathbf{1} + 2\mu \left(\mathbf{I} - \frac{3}{2\sigma_e^2 \left(1 + \frac{A'}{3\mu}\right)} \mathbf{s} \otimes \mathbf{s} \right) : \dot{\mathbf{e}} = \mathbf{C}_t : \dot{\boldsymbol{\varepsilon}}$$ [eq. 6.44]

indicial

$$\mathbf{1} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j : 2^{nd} \ \textit{order unit tensor}$$

$$\mathbf{I} = \frac{1}{2}\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right) \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l : \textit{symmetric } 4^{th} \ \textit{order unit tensor}$$

$$\mathbf{C}_t = \left(k - \frac{2\mu}{3}\right)(\mathbf{1} \otimes \mathbf{1}) + 2\mu\left(\mathbf{I} - \frac{3}{2\sigma_e^2\left(1 + \frac{A'}{3\mu}\right)}\mathbf{s} \otimes \mathbf{s}\right)$$ [eq. 6.45a]

Voigt

$$\Longleftrightarrow \quad \mathbf{C}_t = \left(k - \frac{2\mu}{3}\right)\mathbf{j}\mathbf{j}^T + 2\mu\left(\mathbf{L}^{-1} - \frac{3}{2\sigma_e^2\left(1 + \frac{A'}{3\mu}\right)}\mathbf{s}\mathbf{s}^T\right)$$ [eq. 6.45b]

indicial


Thank you!