Ch.8 Potential flow and CFD

CFD: computational fluid dynanamics 전산유체역학

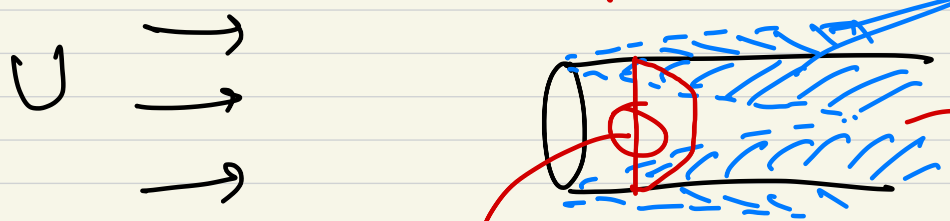
= inviscid irrotational flow

$\mu = 0$ $\qquad \underline{\omega} = \nabla \times \underline{V} = 0$

8.1 Introduction

inviscid external flow

boundary layer



fully viscous flow

$\mu \neq 0$ inviscid internal flow

$\mu \dfrac{\partial u}{\partial y} = \tau = 0$, $\quad \dfrac{\partial u}{\partial y} = 0$

- Frictionless irrotational flow (potential flow)
  (Frictionless = inviscid)

N-S eqs : $\rho \dfrac{d\underline{V}}{dt} = \rho \underline{g} - \nabla p + \cancel{\nabla \cdot \underline{\tau}}^{\,0}$

$$\rightarrow \rho \frac{\partial \underline{V}}{\partial t} + \underbrace{\rho (\underline{V} \cdot \nabla) \underline{V}}_{} = \rho \underline{g} - \nabla p$$

$$= \rho \nabla \left(\frac{V^2}{2}\right) + \rho \underbrace{(\nabla \times \underline{V})}_{\underline{\omega}} \times \underline{V}$$

$(\because \underline{\tau} = 0)$

$$\rightarrow \frac{\partial \underline{V}}{\partial t} + \nabla \left(\frac{V^2}{2}\right) + \underline{\omega} \times \underline{V} + \frac{1}{\rho} \nabla p - \underline{g} = 0$$: frictionless flow

Let $d\underline{r}$ be an arbitrary displacement vector.

$$\left[\frac{\partial \underline{V}}{\partial t} + \nabla \left(\frac{V^2}{2}\right) + \underline{\omega} \times \underline{V} + \frac{1}{\rho} \nabla p - \underline{g}\right] \cdot d\underline{r} = 0$$

$\boxed{(\underline{\omega} \times \underline{V}) \cdot d\underline{r} = 0}$ if

① $\underline{V} = 0$ trivial

② $\underline{\omega} = 0 = \nabla \times \underline{V}$: irrotational flow

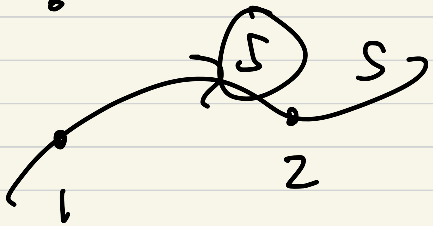
③ $\underline{\omega} \times \underline{V} \perp d\underline{r}$

④ $\underline{V} \parallel d\underline{r}$: streamline

⑤ $\underline{\omega} \parallel d\underline{r}$: vortex line with $\underline{g} = -g\hat{k}$ $\quad \cancel{\frac{d}{dr}}\nabla\left(\frac{V^2}{2}\right)\cdot \cancel{d\underline{r}}$ $\quad \frac{1}{\rho}\nabla p\cdot d\underline{r}$ $\quad \underline{g}\cdot d\underline{r}$

$$\rightarrow \boxed{\frac{\partial \underline{V}}{\partial t}\cdot d\underline{r} + d\left(\frac{V^2}{2}\right) + \frac{1}{\rho}dp + g\,dz = 0}.$$

④ : $1 \rightarrow 2$ along the streamline



$$\boxed{\int_1^2 \frac{\partial V}{\partial t}ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0}$$

frictionless flow along the streamline

② : $\underline{\omega} = \nabla\times\underline{V} = 0$ (irrotational flow)

$\hookrightarrow \underline{V} = \nabla\phi \qquad \phi$: velocity potential

potential line : line of constant ϕ

streamline : " " " ψ (stream ft.)

$$\frac{\partial \underline{V}}{\partial t}\cdot d\underline{r} = \frac{\partial}{\partial t}(\nabla\phi)\cdot d\underline{r} = \nabla\left(\frac{\partial\phi}{\partial t}\right)\cdot d\underline{r} = d\left(\frac{\partial\phi}{\partial t}\right)$$

$$\rightarrow \boxed{\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}|\nabla\phi|^2 + gz = \text{const.}} \quad \leftarrow (1)$$

unsteady irrotational Bernoulli eq.

If incomp (ρ = const),

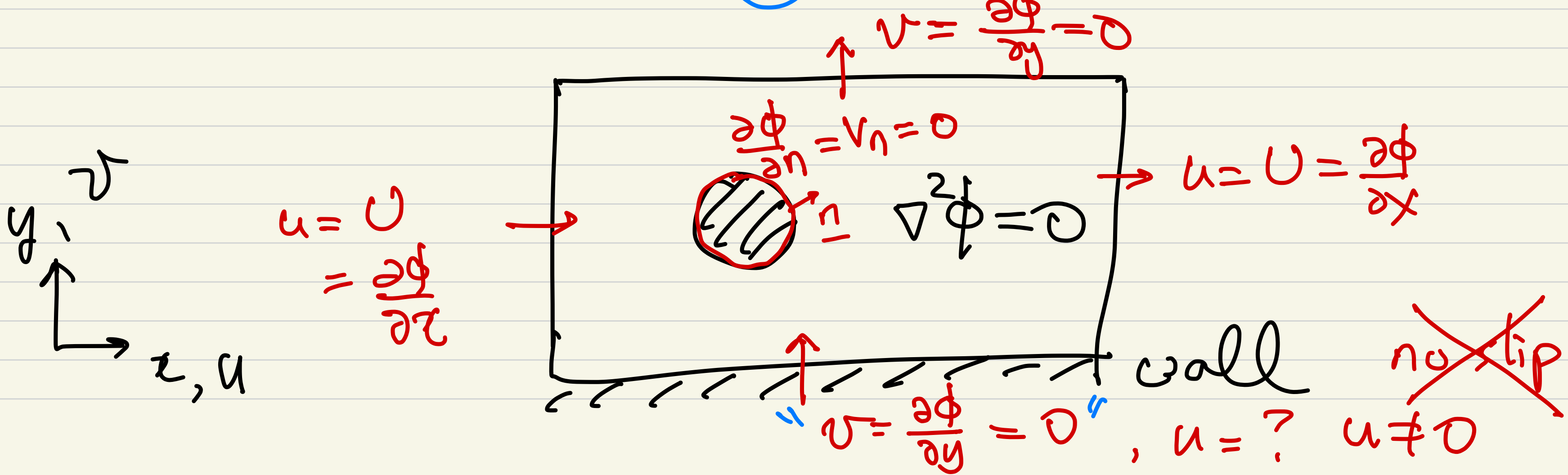
$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + gz = \text{const}.$$

Continuity $\nabla \cdot \underline{V} = 0 \rightarrow \nabla \cdot (\nabla \phi) = \boxed{\nabla^2 \phi = 0} \quad - (2)$

Laplace eq.

<u>Potential theory</u>

Solve (2) to get $\phi \rightarrow \underline{V} = \nabla \phi \rightarrow$ solve (1) to get p.



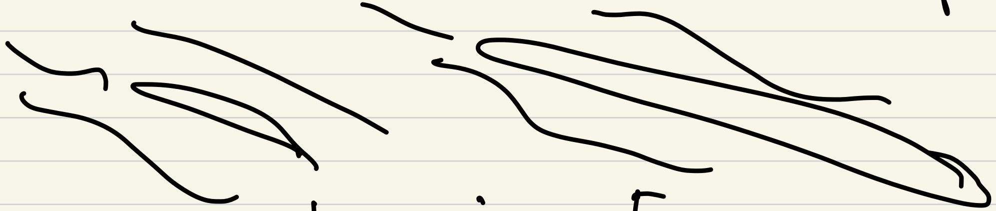
ϕ

$\phi + c$

much easier to solve $\nabla^2\phi = 0$ than to solve N-S eqs.

no parameter in governing eq. like Re, Fr, Ma, ...

→ Inviscid flows are kinematically similar without additional parameters.



1 : 5

- Stream ft. ψ

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

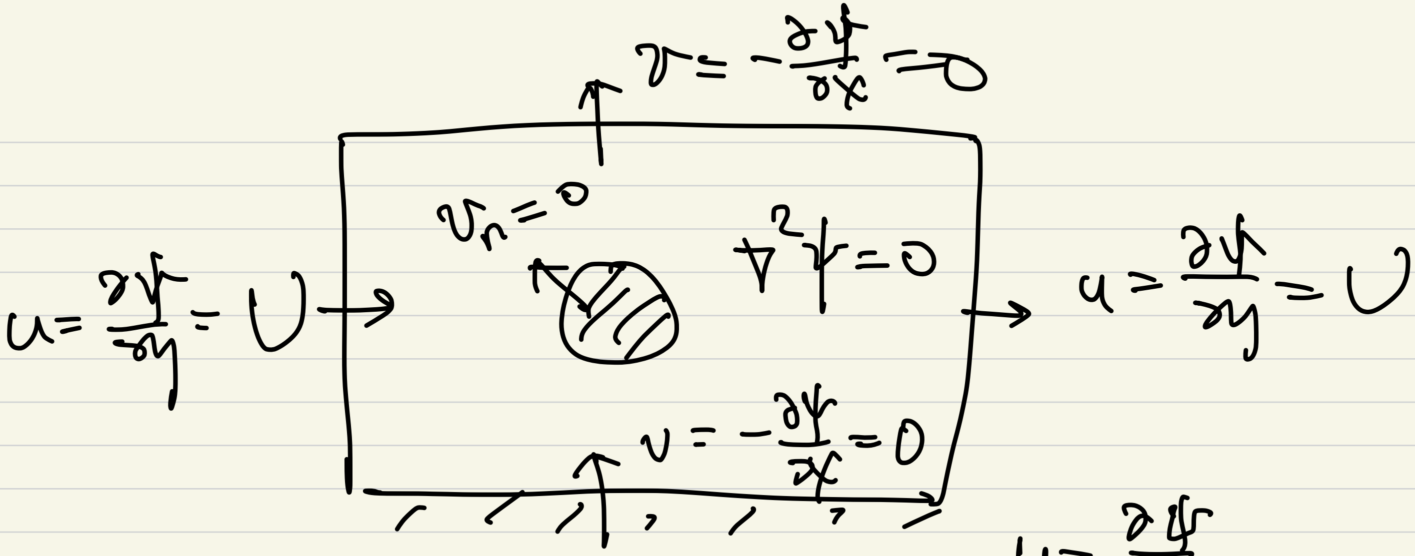
$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \qquad \text{2D} \qquad \nabla \cdot \underline{V} = 0$$

$$\underline{\omega} = \nabla \times \underline{V} \qquad \text{2D} \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$= -\nabla^2\psi$$

$$\rightarrow \boxed{\nabla^2\psi = -\omega_z}$$

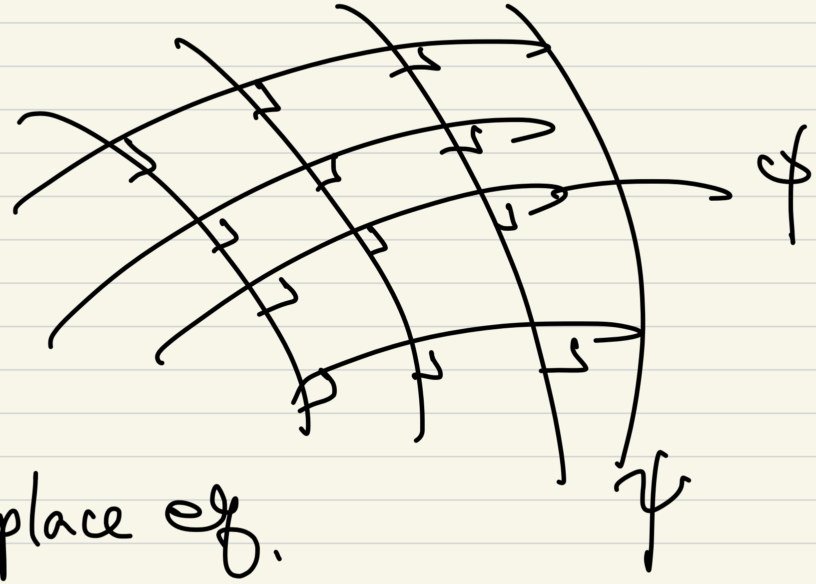
$$\therefore \boxed{\nabla^2\psi = 0} \text{ for irrotational flow } \underline{\omega} = 0$$



Solve $\nabla^2 \psi = 0$ to get ψ. $\rightarrow$ $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ $\rightarrow$ p from ①

- Orthogonality bet. potential lines and streamlines
  $\phi = c_1$ $\qquad \psi = c_2$

- Plane polar coord. (r, θ)



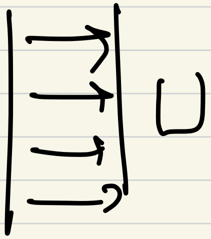
$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \quad \text{Laplace eq.}$$

8.2 Elementary plane flow solutions

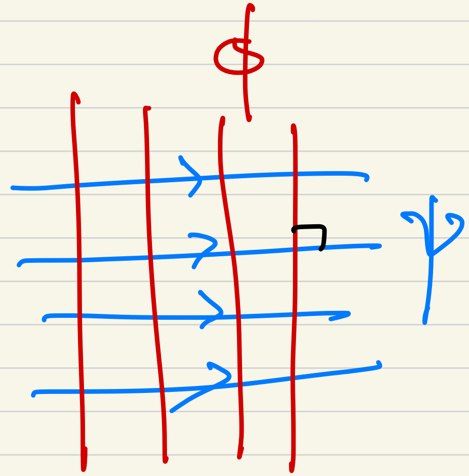
- uniform stream : $u = U,\ v = 0$



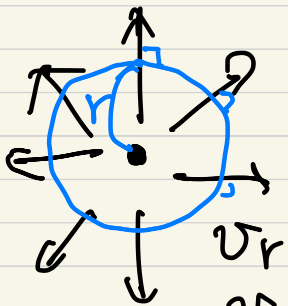
$$u = U = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\left.\right\} \quad \phi = Ux, \quad \psi = Uy$$

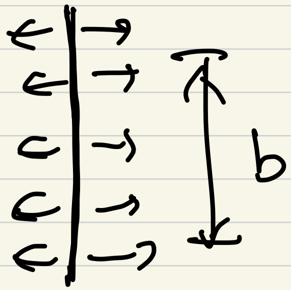


- Line source/sink at the origin



$v_r \neq 0$

$v_\theta = 0$



$Q = v_r \cdot 2\pi r \cdot b$

$m = \frac{Q}{2\pi b}$: source strength

$$v_r = \frac{Q}{2\pi r b} = \frac{m}{r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

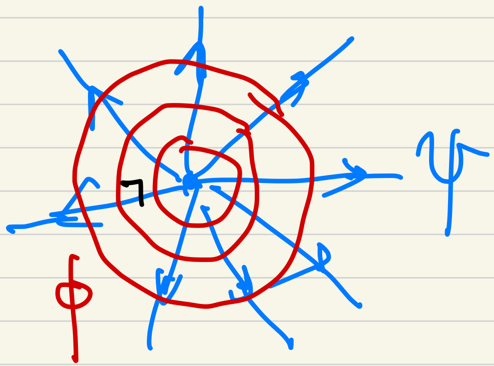
$$v_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$\rightarrow \psi = m\theta$

$\phi = m \ln r$

$m > 0$ source

$m < 0$ sink



• Line "irrotational" vortex (free vortex)

$$v_\theta \neq 0 \;\rightarrow\; v_\theta = f(r)$$

$$v_r = 0$$

$$\nabla \times \underline{V} = 0 \;\rightarrow\; \omega_z = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)}_{=0 \;\rightarrow\; r v_\theta = \text{const}} - \frac{1}{r}\cancelto{0}{\frac{\partial v_r}{\partial \theta}} = 0$$

$$\Rightarrow \boxed{v_\theta = \frac{k}{r}}$$

k : vortex strength

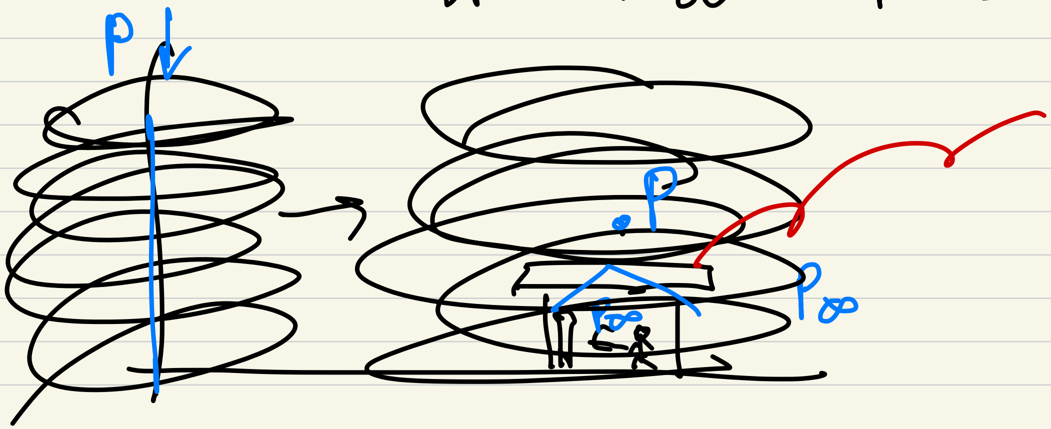
as $r \to 0$, $v_\theta \to \infty$: pressure @ vortex center is **lowest**.

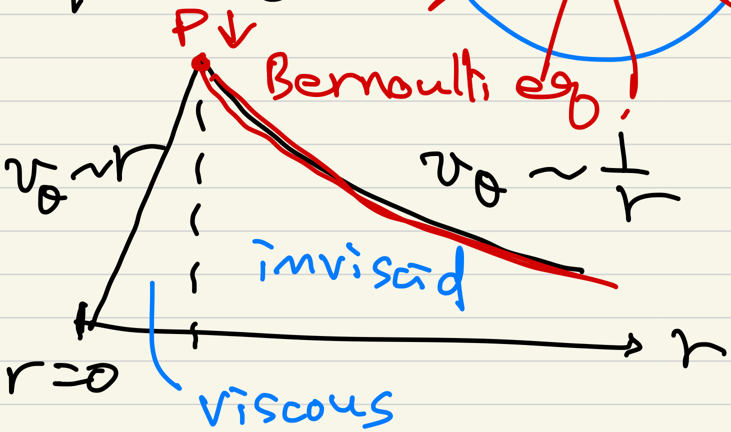
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = 0$$

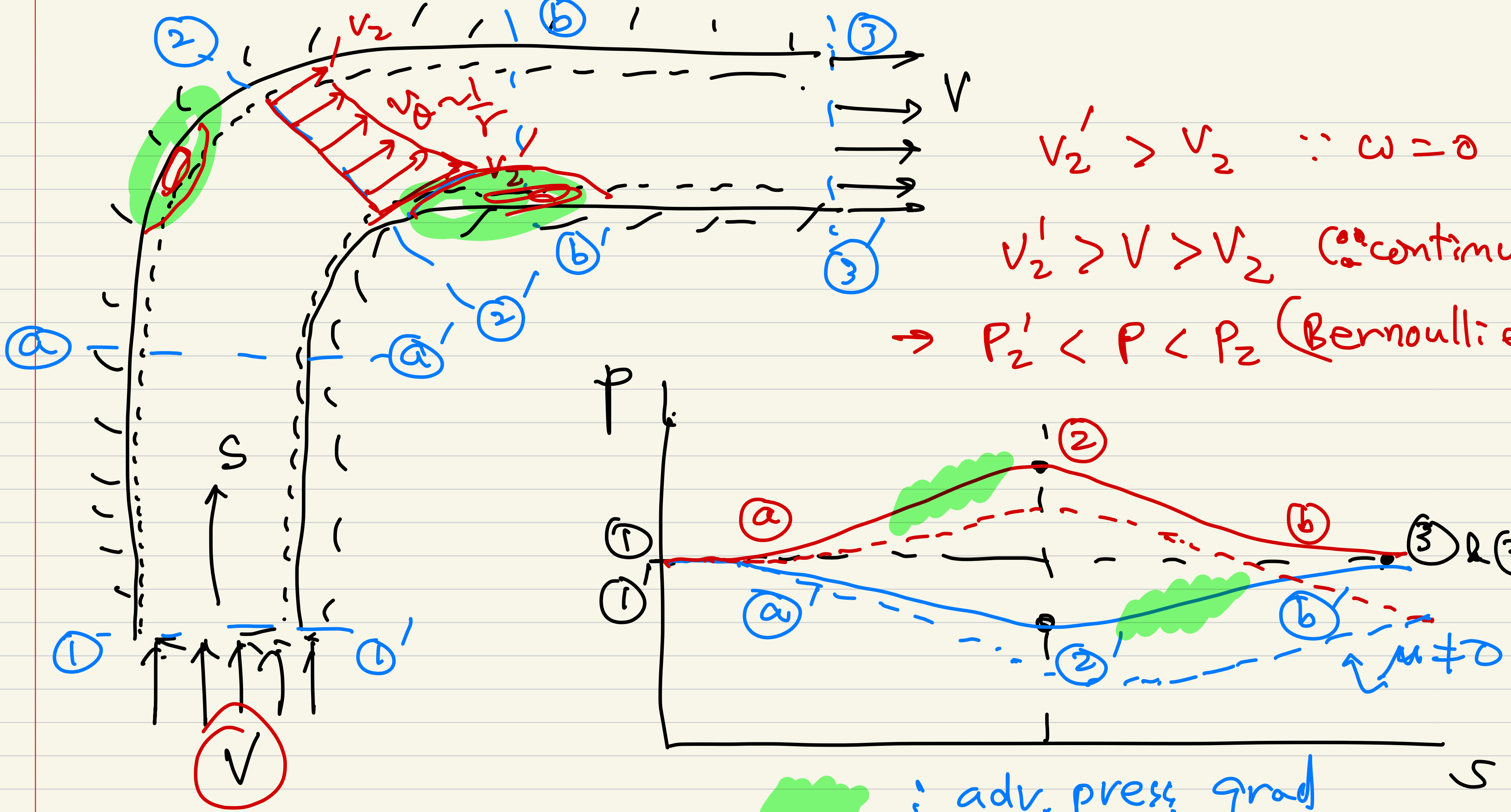
$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{k}{r}$$

$$\psi = -k \ln r$$

$$\phi = k\theta$$







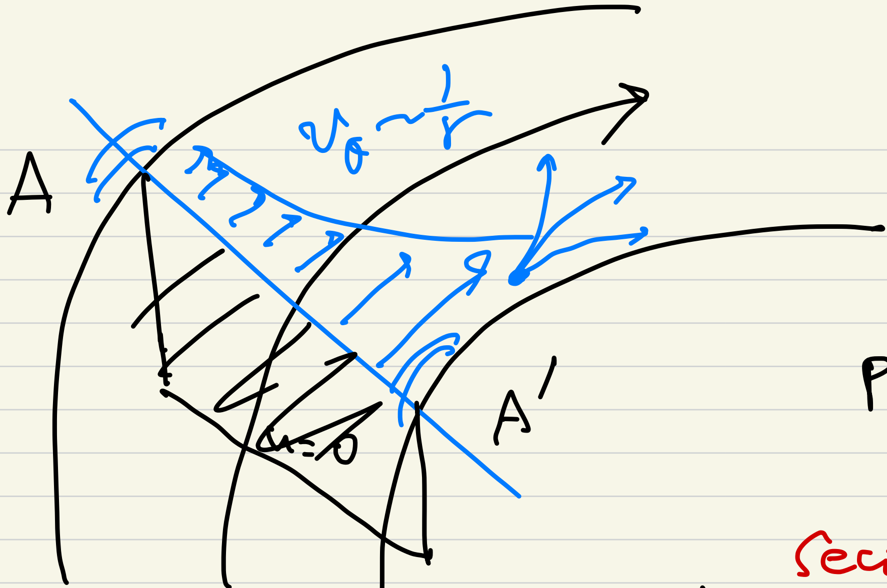
$V_2' > V_2 \quad \because \omega = 0$

$V_2' > V > V_2$ ($\because$ continuity)

$\rightarrow P_2' < P < P_2$ (Bernoulli eq)

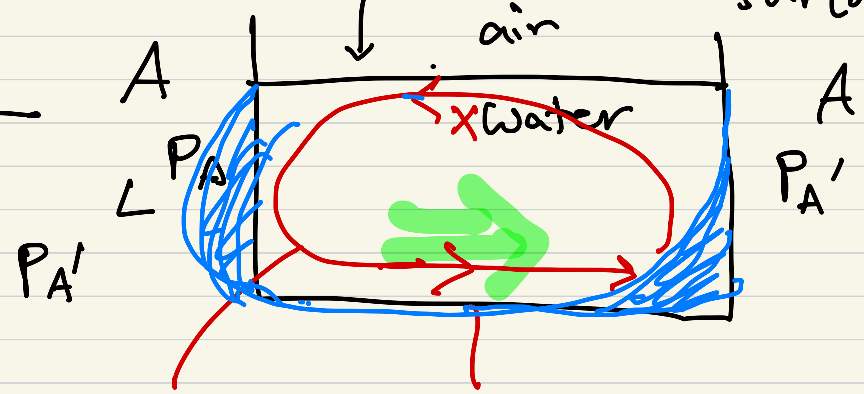
: adv. press grad

$\rightarrow$ possible for flow sep.



river

$\tau = \mu \frac{\partial u}{\partial y} \simeq 0$ shear-free surface



secondary flow $u=0$

$P_A > P_{A'}$