Ch. 6 Scales of turbulent motion

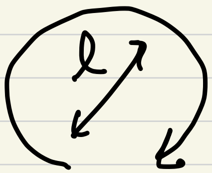
6.1 Energy cascade and Kolmogorov hypotheses

High Re #, $Re = UL/\nu$

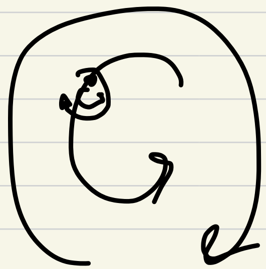
U: char. vel. scale
L: " length "

- Energy cascade

Richardson: turbulence is composed of eddies of different sizes.



$u(\ell)$ $\qquad \tau(\ell) = \ell / u(\ell)$



eddy $\neq$ v

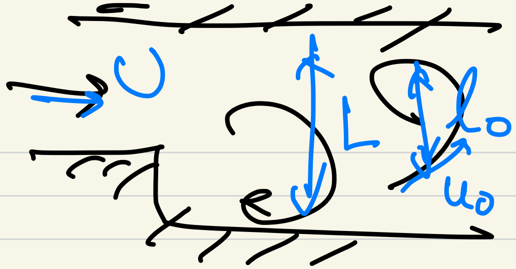
✓ large eddies contain smaller eddies

✓ Eddies in the largest size range $\ell_0 \sim L$

$$u_0 = u(\ell_0) \sim u' \equiv \left(\tfrac{2}{3}k\right)^{\frac{1}{2}} \sim U \qquad k = \tfrac{1}{2}u_i u_i$$

$$Re_0 = \frac{u_0 \ell_0}{\nu} \sim Re \gg 1$$

↳ direct effect of viscosity is negligibly small.



largest eddies $\xrightarrow[\text{energy}]{\text{transfer}}$ smaller eddies $\xrightarrow[\text{energy}]{\text{transfer}}$ smaller eddies - - - - → eddies are so small and stable

↳ unstable break up

energy cascade

eddies are so small and stable → viscosity is effective → dissipation

⇒ Dissipation is determined by the transfer of energy from the largest eddies.
↳ energy of u_o^2

→ largest eddies have energy of u_o^2
time scale of $\tau_o = l_o/u_o$

$\rightarrow$ rate of energy transfer $\sim u_0^2/\tau_0 = u_0^3/l_0$

$\rightarrow$ $\boxed{\varepsilon \sim u_0^3/l_0}$ indep of ν!

dissipation

$\varepsilon = 2\nu\langle s_{ij}s_{ij}\rangle$

⊙ Kolmogorov hypothesis

what is the size of the smallest eddies that are responsible for dissipating the energy?

As l decreases, do $u(l)$ and $\tau(l)$ increase, decrease or remain the same?

Kolmogorov (1941)'s hypothesis or theory $\Rightarrow$ $u(l)$ and $\tau(l) \downarrow$ as $l \downarrow$

- Kolmogorov hypothesis of local isotropy

At sufficiently high Re number, the small-scale turbulent motions ($l \ll l_0$) are statistically isotropic.

$l > l_{EI}$: anisotropic large eddies

$l < l_{EI}$: isotropic small eddies

$l_{EI} \simeq \frac{1}{6} l_0$

- Kolmogorov's first similarity hypothesis

In every turb. flow at sufficiently high Re #, the statistics of small-scale motions ($l < l_{EI}$) have a universal form that is uniquely determined by ν and ε.

$l < l_{EI}$: $\boxed{\text{universal equilibrium range}}$

time scale $l/u(l) < l_0/u_0$,

$\therefore$ small eddies adapt quickly to maintain a dynamic equilibrium with energy transfer from large eddies.

ε and ν $\rightarrow$ $\boxed{\text{Kolmogorov scales}}$ $\rightarrow$ smallest dissipative eddies

$$\begin{cases} \eta = (\nu^3/\varepsilon)^{1/4} \\ u_\eta = (\varepsilon\nu)^{1/4} \\ \tau_\eta = (\nu/\varepsilon)^{1/2} \end{cases} \rightarrow \frac{u_\eta \eta}{\nu} = 1 \;:\; \text{smallest dissipative eddies}$$

- On the small scales, all high-Re-# turbulent vel. fields are statistically similar; i.e. they are statistically identical when they are scaled by the Kolmogorov scales.

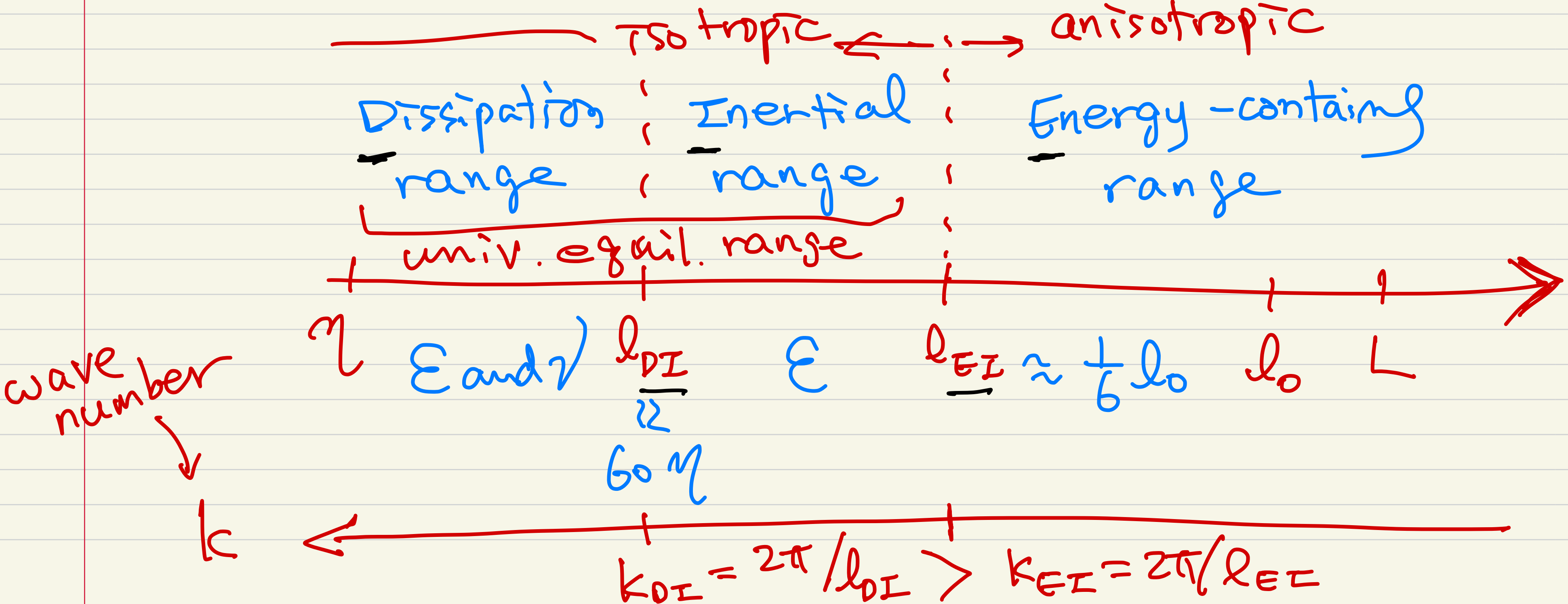
- $\varepsilon \sim u_0^3 / l_0 \Rightarrow \dfrac{\eta}{l_0} \sim Re^{-\frac{3}{4}}, \quad \dfrac{u_\eta}{u_0} \sim Re^{-\frac{1}{4}}, \quad \dfrac{\tau_\eta}{\tau_0} \sim Re^{-\frac{1}{2}}$

At high Re #, u_η and τ_η are smaller than u_0 and τ_0.

- Kolmogorov's second similarity hypothesis

In every turb. flow at sufficiently high Re #, the statistics of the motions of scale ℓ in the range $\boxed{\eta \ll \ell \ll \ell_0}$ have a universal form that is uniquely determined by ε, indep. of ν.

because the eddies in this range are much bigger than the dissipative eddies.



In inertial subrange, motions are determined by inertial effects (viscous effects are negligible)

- Given an eddy size l in the inertial subrange $\sim \varepsilon \sim u_0^3/l_0$, $\varepsilon = [L^2 T^{-3}]$

$$u(l) = (\varepsilon l)^{\frac{1}{3}} = u_\eta (l/\eta)^{\frac{1}{3}} \sim u_0 (l/l_0)^{\frac{1}{3}}$$

$$\tau(l) = (l^2/\varepsilon)^{\frac{1}{3}} = \tau_\eta (l/\eta)^{\frac{2}{3}} \sim \tau_0 (l/l_0)^{2/3}$$

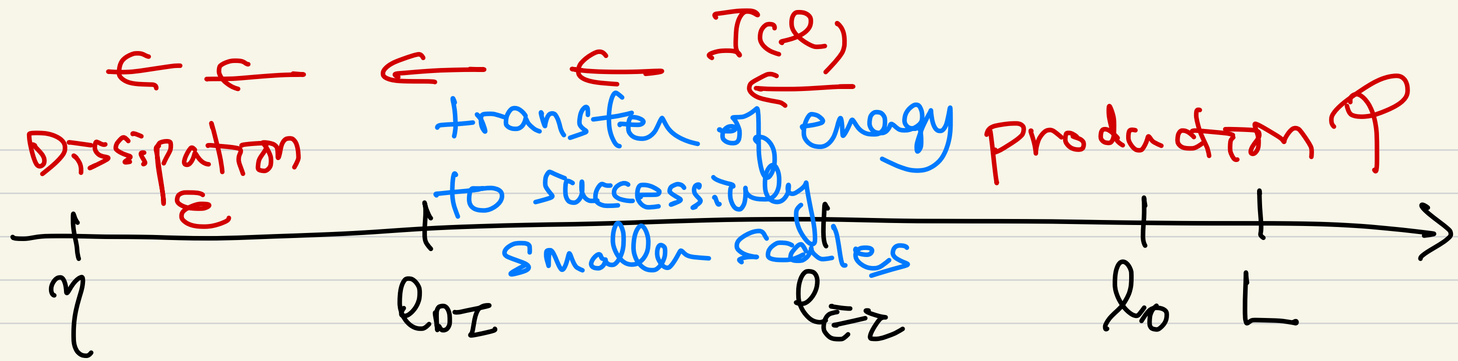
As l decrease, $u(l)$ and $\tau(l)$ also decrease.

$$\left(\begin{array}{lll} \eta = (\nu^3/\varepsilon)^{1/4}, & u_\eta = (\varepsilon\nu)^{1/4}, & \tau_\eta = (\nu/\varepsilon)^{1/2} \\ \eta/l_0 \sim Re^{-\frac{3}{4}}, & u_\eta/u_0 \sim Re^{-1/4}, & \tau_\eta/\tau_0 \sim Re^{-1/2} \end{array} \right)$$

$\mathcal{T}(l)$: energy transferred from eddies larger than l to those smaller than l.

$$\mathcal{T}(l) \sim u(l)^2/\tau(l) = \varepsilon, \quad \text{indep. of } l$$

$$\mathcal{T}_{EI} = \mathcal{T}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI} = \mathcal{T}(l_{DI}) = \varepsilon$$



⑤ Energy spectrum $E(k)$

length scale ℓ $\rightarrow$ wavenumber $k = 2\pi/\ell$

energy in (k_a, k_b) : $k(k_a, k_b) = \int_{k_a}^{k_b} E(k)\,dk$ §6.5

dissipation rate ε in (k_a, k_b) : $\varepsilon(k_a, k_b) = \int_{k_a}^{k_b} 2\nu k^2 E(k)\,dk$

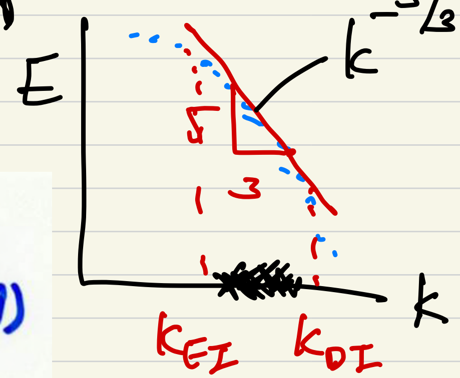
In the universal equil. range $(k > k_{EI} = 2\pi/\ell_{EI})$,
spectrum is a universal ft. of ε and ν.

In the inertial subrange $(k_{EI} < k < k_{DI} = 2\pi/\ell_{DI})$,
spectrum is a universal ft. of ε only.

$$\rightarrow \boxed{E(k) = f(k, \varepsilon) = C\,\varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}}$$ (inertial subrange)

k: wavenumber

C: universal constant



$k_{DI} = \dfrac{2\pi}{l_{DI}} \uparrow$ as $Re \uparrow$

Chapman (1979)

E_{uu}

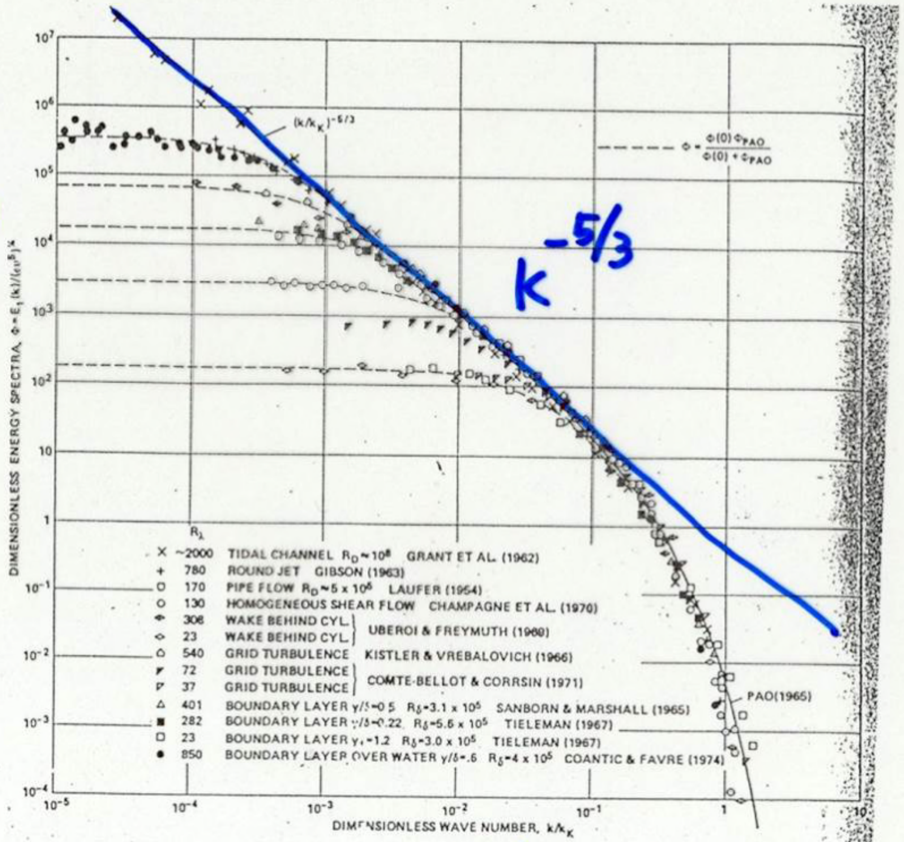


Fig. 13 Streamwise turbulence energy spectra for various flows.

K

$$\varepsilon : 2\nu k^2 E(k) \sim k^{\frac{1}{3}}$$

($E(k) \sim k^{-5/3}$)

↳ bulk of energy is in the large scales

" " dissipation is in the small scales