457.644 Advanced Bridge Engineering

Aerodynamic Design of Bridges

Part 6: Wind Tunnel Test

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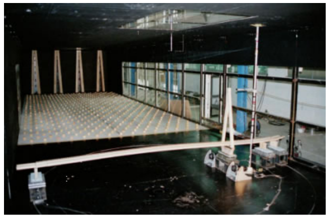
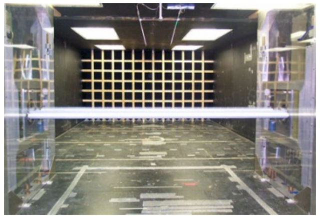
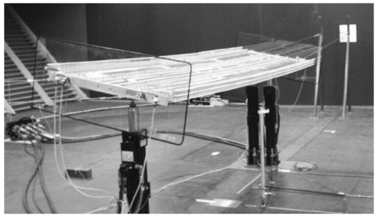
Wind Tunnel Test

Requirements

- Although the science of theoretical fluid mechanics is well developed and computational methods are experiencing rapid growth, it remains necessary to perform physical experiments to gain needed insights into many **complex effects associated with fluid flow**.
  - Bluff / complicate shape of bridge section
  - Change of flow according to the shape of structure is difficult to predict.

Application

- Wind tunnels can be used to study interference between flow and bridge model by **vibration test.**
- The bridge model can be instrumented to measure the lift and drag and other **aerodynamic parameters** for analysis.
- Wind tunnel test can be used to study and **design a bridge shape.**


Similitudes

Requirements

- In general, scaled model is used for wind tunnel test of bridge.

- Therefore, similitude laws for flow condition and aerodynamic behavior should be satisfied to reproduce the physical phenomenon of real bridge.
- Similitude laws is given by **dimensionless analysis**

Dimensional analysis

- **For concreteness, let it be assumed that the force *F* developed somewhere on a body immersed in a flowing fluid is a function only of the following six parameter**
  - Air density ρ, flow velocity U, some typical dimension D, some frequency f, fluid viscosity μ, gravitational acceleration g, then:

$$F = \rho^{\alpha} U^{\beta} D^{\gamma} f^{\delta} \mu^{\epsilon} g^{\zeta}$$

  - Each exponents is to be determined.

- **Writing the dimensional equivalent of each of the quantities with**
  - mass M, length L, and time T

$$\frac{ML}{T^2} = \left(\frac{M}{L^3}\right)^{\alpha} \left(\frac{L}{T}\right)^{\beta} (L)^{\gamma} \left(\frac{1}{T}\right)^{\delta} \left(\frac{M}{LT}\right)^{\epsilon} \left(\frac{L}{T^2}\right)^{\zeta}$$

Dimensional analysis

- **Equating corresponding exponents:**
  - *M:* $1 = \alpha + \epsilon$
  - *L:* $1 = -3\alpha + \beta + \gamma - \epsilon + \zeta$
  - *T:* $-2 = -\beta - \delta - \epsilon - 2\zeta$

- **These equations may now be solved for any three of the exponents in terms of the remaining three; for example,**
  - $\alpha = 1 - \epsilon$
  - $\beta = 2 - \epsilon - \delta - 2\zeta$
  - $\gamma = 2 - \epsilon + \delta + \zeta$

- **whence it is seen that**

$$F = \rho^{1-\epsilon} U^{2-\epsilon-\delta-2\zeta} D^{2-\epsilon+\delta+\zeta} f^{\delta} \mu^{\epsilon} g^{\zeta}$$

or

$$F = \rho U^2 D^2 \left(\frac{Df}{U}\right)^{\delta} \left(\frac{\mu}{\rho U D}\right)^{\epsilon} \left(\frac{Dg}{U^2}\right)^{\zeta}$$

Dimensional analysis

$$F = \rho U^2 D^2 \boxed{\left(\frac{Df}{U}\right)}^{\delta} \boxed{\left(\frac{\mu}{\rho U D}\right)}^{\epsilon} \boxed{\left(\frac{Dg}{U^2}\right)}^{\zeta}$$

①　②　③

- These dimensionless grouping of physical variables provide the natural groupings in which to develop physical relations. The reoccurrence of these dimensional groupings means they are given their own names.

Name	Definition	Interpretation
Reynolds number	$\mathrm{Re} = \dfrac{\rho U D}{\mu}$	$\dfrac{\text{inertial force}}{\text{viscous force}}$
Froude number	$\mathrm{Fr} = \dfrac{U^2}{Dg}$	$\dfrac{\text{inertial force}}{\text{gravitational force}}$
Cauchy number	$\mathrm{Ca} = \dfrac{\rho U^2}{E}$	$\dfrac{\text{inertial force}}{\text{elastic force}}$
Strouhal number	$\mathrm{St} = \dfrac{fD}{V}$	$\dfrac{\text{inertial (local) force}}{\text{inertial (convective) force}}$

Inertia Parameter

$$\frac{\rho_s}{\rho} = constant$$

- **The ratio between air density and structure density.**

- **From this parameter, the scale of mass and moment of inertia per unit length is obtained:**

$$\rho_s/\rho = (\rho_s D^3/D)/(\rho D^2) = m/(\rho D^2)$$

$$\rho_s/\rho = (\rho_s D^5/D)/(\rho D^4) = I_p/(\rho D^4)$$

- **In general situation, air density does not change according to prototype and model. Therefore, inertia parameter should be satisfied for all cases.**

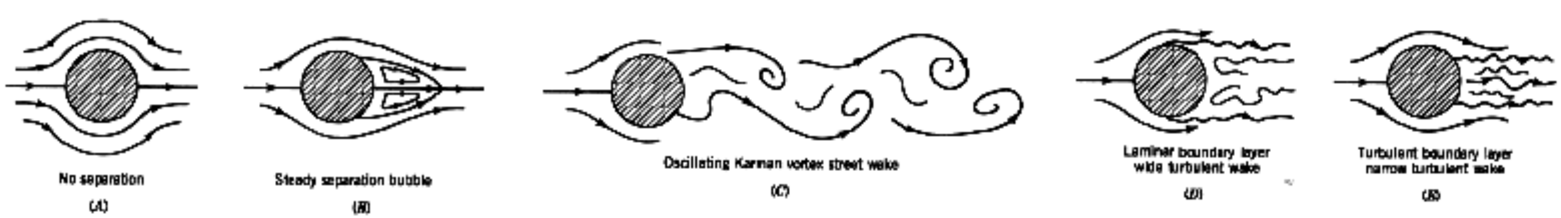
Reynolds Number

$$\mathrm{Re} = \frac{\rho U D}{\mu} = \frac{UD}{\nu} = \frac{\text{inertial force}}{\text{viscous force}}$$

- **The Reynolds number is the ratio of the inertial to the viscous forces.**

- **In a view of wind tunnel test for bridge, it is almost impossible to satisfy the Reynolds similitude.**
  - The scale of length is around 1/80~1/150, so wind velocity of model should be 80~150 times larger than wind velocity of prototype for Reynolds similitude.

- **This viscosity is associated with "separation point" of flow and general bridge section has a significant separation point. Therefore, in the wind tunnel test of bridge section, Reynolds similitude can be neglected.**
  - For the circular model, Reynolds number is most important factor.



Froude Number

$$\mathrm{Fr} = \frac{U^2}{Dg} = \frac{\text{inertial force}}{\text{gravitational force}}$$

- **The Froude numbers occurs in gravity driven flows and wave behavior on the surface of a fluid.**
  - Firstly, it is used in a ship hydrodynamics as an important parameter with respect to the ship's drag, or resistance, including the wave making resistance.

- **In the bridge aerodynamics, gravitational force is important for suspension bridge because of cable tension by dead load. Therefore, Froude similitude is considered for 3D wind tunnel test of suspension bridge.**
  - Not considered in 2D section model test.

Cauchy Number

$$\mathrm{Ca} = \frac{\rho U^2}{E} = \frac{\text{inertial force}}{\text{elastic force}}$$

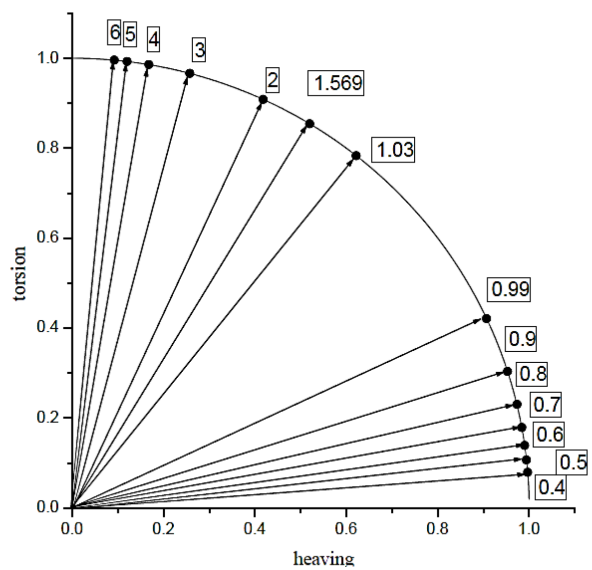
- **Cauchy similarity is associated with stiffness (elastic force of the structure). Therefore, in a vibration test, it is very important.**
  - Especially for the 2D section model test, stiffness is much dominant parameter than gravitational force for dynamic behavior.
- **It is often convenient to replace a Cauchy number by a reduced frequency.**
  - Natural frequency f is proportional to $\sqrt{E/\rho_s D^2}$, then

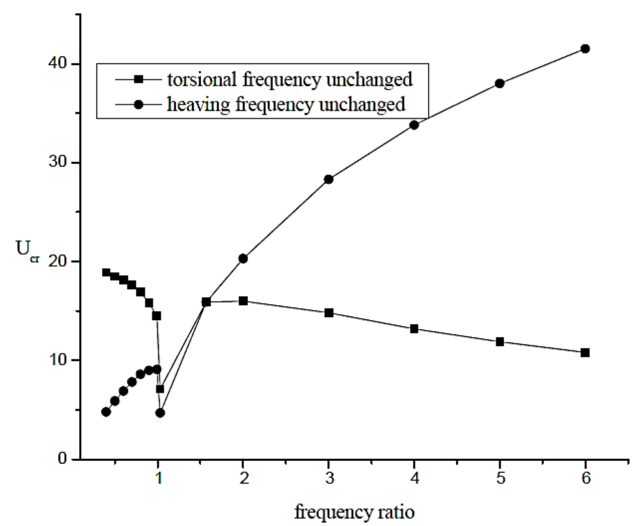
$$\frac{fD}{U} = K\sqrt{\frac{E}{\rho_s D^2}} \cdot \frac{D}{U} = K \underbrace{\boxed{\sqrt{\frac{E}{\rho U^2}}}}_{\text{Cauchy number}} \cdot \underbrace{\boxed{\sqrt{\frac{\rho}{\rho_s}}}}_{\text{Inertia parameter}}$$

  - Inertia parameter should be always satisfied so Cauchy similarity can be substituted by equality of reduced frequency (or reduced wind velocity).

Other Parameters for Wind Tunnel Test

- **Damping Ratio** (δ_s)
  - It affect such deflections so must remain the same in prototype and model
- **Frequency ratio** (f_v/f_a)
  - The ratio between vertical natural frequency and torsional natural frequency.
  - It is associated with **'coupled motion'** in an unstable oscillation driven by wind. For the onset of flutter, frequency ratio around 1.0 causes the flutter quiet easily. (e.g. Frequency ratio of Tacoma Narrow Bridge = 1.538)





YANG *et al,* 2009, Investigation on the flutter mechanism of thin plate sections, EACWE 5

Similitude Laws for Froude Number

Features

- For the 3D full scale model test
- Consideration of gravitational effect on structure and wake around model.

Length scale (λ_L) has to be determined according to size of wind tunnel

$$\frac{D_m}{D_p} = \lambda_L$$

From the inertia parameter, the scale of mass and moment of inertia is decided:

$$\left(\frac{m}{\rho D^2}\right)_m = \left(\frac{m}{\rho D^2}\right)_p \quad \rightarrow \quad \frac{m_m}{m_p} = \frac{\rho_m D_m^2}{\rho_p D_p^2} = \lambda_L^2$$

$$\left(\frac{I_p}{\rho D^4}\right)_m = \left(\frac{I_p}{\rho D^4}\right)_p \quad \rightarrow \quad \frac{I_m}{I_p} = \frac{\rho_m D_m^4}{\rho_p D_p^4} = \lambda_L^4$$

Similitude Laws for Froude Number

- By the Froude number, the scale of velocity (λ_U), time (λ_T) and frequency (λ_f) are obtained, respectively:

$$\left(\frac{U^2}{Dg}\right)_m = \left(\frac{U^2}{Dg}\right)_p \quad \rightarrow \quad \left(\frac{U_m^2}{U_p^2}\right) = \left(\frac{D_m g_m}{D_p g_p}\right) = \lambda_L$$

$$\therefore \lambda_U = \sqrt{\lambda_L}$$

$$\lambda_T = \lambda_L / \lambda_U = \sqrt{\lambda_L} \quad \text{and} \quad \lambda_f = \lambda_T^{-1} = 1/\sqrt{\lambda_L}$$

- Other parameters for full scaled model are calculated by dimensional consideration
  - Stiffness of girder or cable (EI, EA)

$$EI = \left(\frac{kg \cdot m \cdot s^{-2}}{m^2}\right)(\lambda_L^4) = \left(\frac{\lambda_L^3 \lambda_L \lambda_L^{-1}}{\lambda^2}\right)(\lambda_L^4) = \lambda_L^5$$

$$EA = \left(\frac{kg \cdot m \cdot s^{-2}}{m^2}\right)(\lambda_L^2) = \left(\frac{\lambda_L^3 \lambda_L \lambda_L^{-1}}{\lambda^2}\right)(\lambda_L^2) = \lambda_L^3$$

Similitude Laws for Cauchy Number

- **Features**
  - For the 2D section model test

- **Length, mass and moment of inertia is same with Froude similarity.**

$$\frac{D_m}{D_p} = \lambda_L$$

$$\left(\frac{m}{\rho D^2}\right)_m = \left(\frac{m}{\rho D^2}\right)_p \quad \rightarrow \quad \frac{m_m}{m_p} = \frac{\rho_m D_m^2}{\rho_p D_p^2} = \lambda_L^2$$

$$\left(\frac{I_p}{\rho D^4}\right)_m = \left(\frac{I_p}{\rho D^4}\right)_p \quad \rightarrow \quad \frac{I_m}{I_p} = \frac{\rho_m D_m^4}{\rho_p D_p^4} = \lambda_L^4$$

- **Frequency scale has to be decided according to the target wind velocity.**

$$\frac{f_m}{f_p} = \lambda_f \quad \text{(decided)}$$

Similitude Laws for Cauchy Number

- **By the Cauchy number, the scale of velocity (λ_U) and time (λ_T) are obtained, respectively:**

$$\left(\frac{fD}{U}\right)_m = \left(\frac{fD}{U}\right)_p \quad \rightarrow \quad \left(\frac{U_m}{U_p}\right) = \left(\frac{f_m D_m}{f_p D_p}\right) = \lambda_f \lambda_L$$

$$\therefore \lambda_U = \lambda_f \lambda_L$$

$$\lambda_T = \lambda_L / \lambda_U = 1/\lambda_f$$

- **Other parameters are calculated by dimensional consideration**
  - Stiffness of girder or cable (EI, EA)

$$EI = \left(\frac{kg \cdot m \cdot s^{-2}}{m^2}\right)(\lambda_L^4) = \left(\frac{\lambda_L^3 \lambda_L \lambda_f^2}{\lambda^2}\right)(\lambda_L^4) = \lambda_L^6 \lambda_f^2$$

$$EA = \left(\frac{kg \cdot m \cdot s^{-2}}{m^2}\right)(\lambda_L^2) = \left(\frac{\lambda_L^3 \lambda_L \lambda_f^2}{\lambda^2}\right)(\lambda_L^2) = \lambda_L^4 \lambda_f^2$$

  - These parameters are not important for 2D section model test

Summary of Similitude Laws for Each Parameter

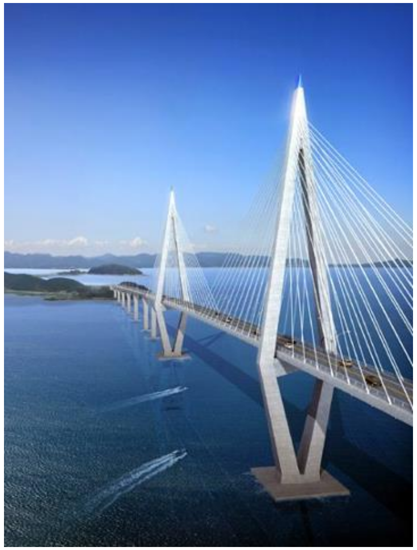
- **Inertia parameter is commonly used for both way.**

Parameter	Froude similitude	Cauchy similitude
Length	λ_L	λ_L
Time	$\sqrt{\lambda_L}$	$1/\lambda_f$
Wind velocity	$\sqrt{\lambda_L}$	$\lambda_f \lambda_L$
Mass per length	λ_L^2	λ_L^2
Moment of inertia per length	λ_L^4	λ_L^4
Natural frequency	$1/\sqrt{\lambda_L}$	Decided on λ_f
Bending stiffness (girder)	λ_L^5	$\lambda_L^6 \lambda_f^2$
Axial stiffness (cable)	λ_L^3	$\lambda_L^4 \lambda_f^2$
Damping ratio	1	1
Applicable for	3D full model	2D section model

Example

▶ **Target bridge: Younggang-Haejae**

Length	45.000m
Width	14.800m
Height	2.500m
Mass per length	26.650ton/m
Moment of inertia per length	485.040ton·m²/m
Vertical natural frequency	0.349Hz
Torsional natural frequency	1.422Hz
Frequency ratio	4.074



1. Determination of the scale of length (λ_L)
   - The length of the model is fixed by the specification of wind tunnel. Therefore, the scale of length scale is determined with the width according to appropriate aspect ratio (width/length).
   - If the model is too large, the aspect ratio is large so the model does not behave like a actual bridge.
   - Unless, target of moment of inertia is too small to satisfy.

$$\lambda_L = 1/50$$

Example

- **Target bridge: Younggang-Haejae**

2. Decide the mass of the model

$$m_m = m_p \lambda_L^2 = 26.650(ton/m)\times(1/50)^2 = 10.660(kg/m)$$

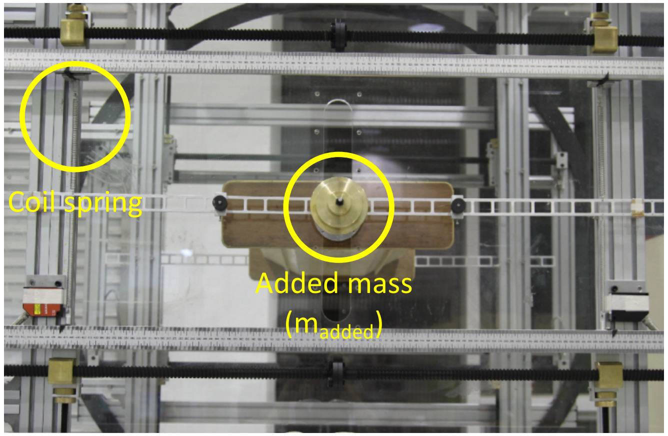
- Mass is tuned by added mass at the center of the model.
- Actually, total mass of structure is firstly measured by an electronic scale. However, sometimes we have to add the mass after the installation of model and in that case, mass can be calculated by the following equation.
  - **Stiffness of the structure does not change by addition of mass.**

$$f_{vert} = \frac{1}{2\pi}\sqrt{\frac{k_{vert}}{m}}$$

$$k_{vert} = (2\pi f)^2 m_{vert} = (\text{constant})$$

$$\therefore (2\pi f_{before})^2 m_{before} = (2\pi f_{after})^2 (m_{before} + m_{added})$$

$$m_{before} = \frac{f_{after}^2}{f_{before}^2 - f_{after}^2} m_{added}$$



Target bridge: Younggang-Haejae

3. Comparing with the target wind velocity, determine the scale of natural frequency (λ_f) and decide the vertical natural frequency.
   - Critical wind velocity = 72.8m/s
   - In general, the maximum investigated range of wind velocity is larger than the critical wind velocity around 10m/s ~ 20m/s for safety side.
   - Considered maximum wind velocity of wind tunnel = 15m/s
   - In general, the scale of the natural frequency is firstly determined by the measured natural frequency and check the appropriateness later.

$$\text{Measured vertical frequency} = 2.167Hz \rightarrow \lambda_f = 6.20$$

$$\left(\frac{U_m}{U_p}\right) = \lambda_f \lambda_L \rightarrow U_p \leq \frac{U_m \lambda_L}{\lambda_f} \rightarrow \lambda_f \leq \frac{U_m \lambda_L}{U_p}$$

$$\lambda_f \leq 15 \times 50 \div 100 = 7.5 \quad \rightarrow \quad \text{Satisfied!}$$

   - If it is not satisfied, tune again the natural frequency by change of spring.

Example

▶ **Target bridge: Younggang-Haejae**

4. Decide the mass moment of inertia

$$I_m = I_p \lambda_L^4 = 485.04(ton \cdot m^2/m) \times (1/50)^4 = 0.077606(kg \cdot m^2/m)$$

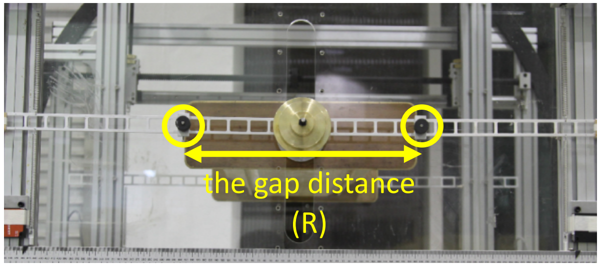
- Mass moment of inertia can be tuned by the change of the gap distance between added mass.
- According to the change of the gap distance, the mass moment of inertia for the model can be calculated by the following equation.
  - **Rotational stiffness does not change by the change of gap distance**

$$f_{tor} = \frac{1}{2\pi}\sqrt{\frac{k_{tor}}{I}}$$

$$k_{tor} = (2\pi f_{tor})^2 I = (\text{constant})$$

$$\therefore (2\pi f_{before})^2 I_{before} = (2\pi f_{after})^2 (I_{before} + I_{added})$$

$$I_{before} = \frac{f_{after}^2}{f_{before}^2 - f_{after}^2} m_{added} \times R^2$$



Example

Target bridge: Younggang-Haejae

5. Tune the frequency ratio
   - By the edit the gap distance between coil spring, we can adjust the torsional natural frequency without the change of other parameters.

Final result of tuning

- For final stage ($\lambda_L = 1/50$)

Parameters	Prototype	Target	Measured
Length	45.000m	0.900m	0.900m
Width	14.800m	0.296m	0.296m
Height	2.500m	0.050m	0.050m
Mass per length	26.650ton/m	10.660kg/m	10.774kg/m
Moment of inertia per length	485.040ton·m²/m	0.078kg·m²/m	0.07670kg·m²/m
Vertical natural frequency	0.349Hz	2.167Hz	2.1670Hz
Torsional natural frequency	1.422Hz	8.829Hz	8.9420Hz
Frequency ratio	4.074	4.074	4.126

Similitude for Atmospheric Flows

Target value

- The variation of the mean wind speed with height
- The variation of turbulence intensities and integral scales with height
- The spectra and cross-spectra of turbulence in the along /across/vertical

Similitude parameters for target model

	Essential	Recommendation
Girder	Along / Across wind turbulence intensity	Along wind turbulence scale Wind spectrum
Tower	Wind speed and Along wind turbulence intensity distribution according to height	Along wind turbulence scale Wind spectrum

- Because it is hard to fit an overall distribution of the turbulence intensity according to height, so tune **the turbulence intensity for the part of main vibration.**
- Consider that 1) a length scale of the wind tunnel is shorter than that of prototype / 2) a length scale is originally irregular

Grid

Methodology

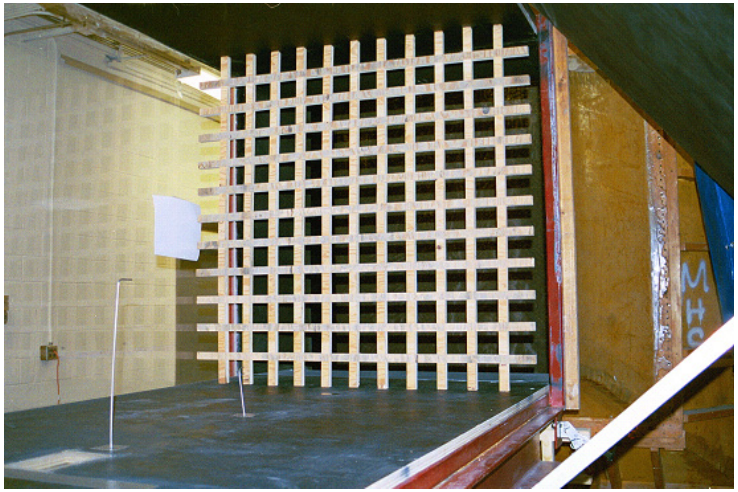
- Change the width of bar and interval between each bar.

Features

- Easy to make and install in wind tunnel, but hard to edit for the target value.
- It can satisfy the target turbulence intensity, however, cannot make a target turbulence length scale, height profile and spectra.

Application

- 2D section model test to check the vortex induced vibration or flutter instability under a turbulence condition.
- Hard to check the buffeting response because the spectrum is dominant for buffeting response.



Bricks, Spire, Blocks

Methodology

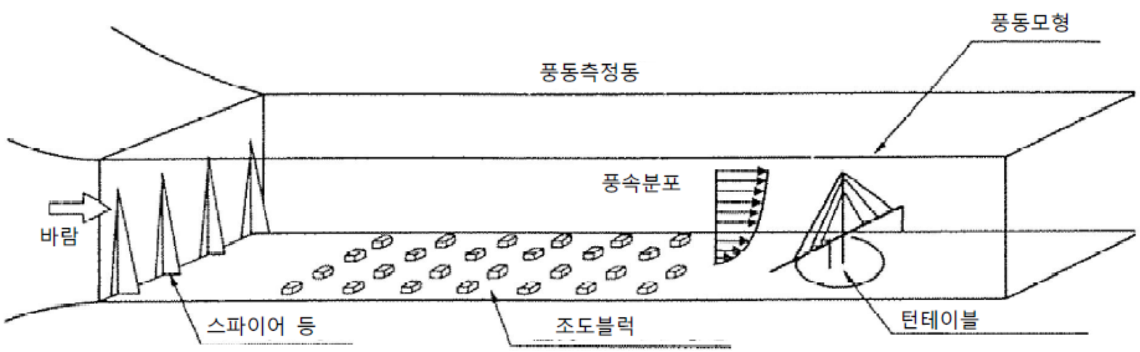
- Number of block: for boundary zone, homogeneity of flow
- Number of bricks: for thickness of boundary range
- Shape of spire: for higher range

Features

- Complex, but available of high quality of similarity for boundary layer flow
  - Turbulence length scale, spectrum, height profile..

Application

- 3D full scale model test



Example of Similitude for Atmospheric Flows

▶ Target bridge: Hwayang-Juckgeum

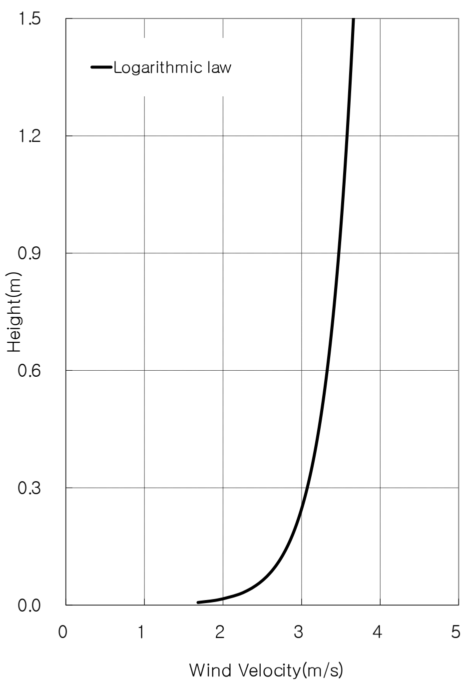
1. Set the target value
   - According to 10 mean wind velocity for certain terrain category
   - U_{10} = 30.9m/s, Terrain category = I

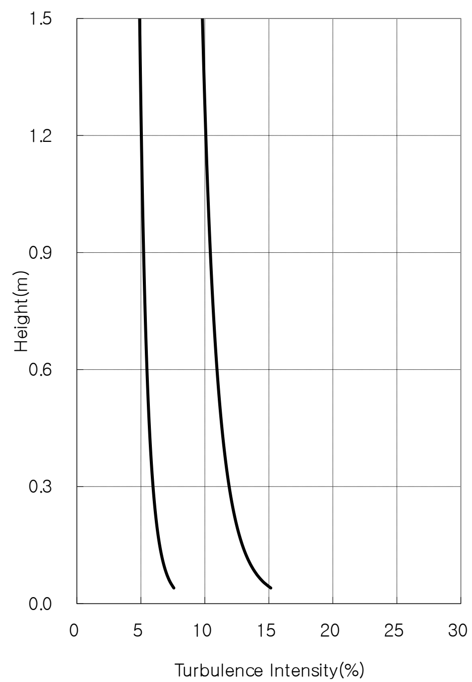
$$u^* = \frac{1}{2.5} U_{10} \frac{1}{\ln(10/z_0)} = \frac{1}{2.5} \cdot 30.9 \cdot \frac{1}{\ln(10/0.01)} = 1.789$$

$$U(z) = 2.5 \cdot u^* \cdot \ln\left(\frac{z}{z_0}\right)$$

$$I_u(z) = \sqrt{\frac{6u^{*2}}{U(z)}}$$

$$I_v(z) = I_u(z)/2$$

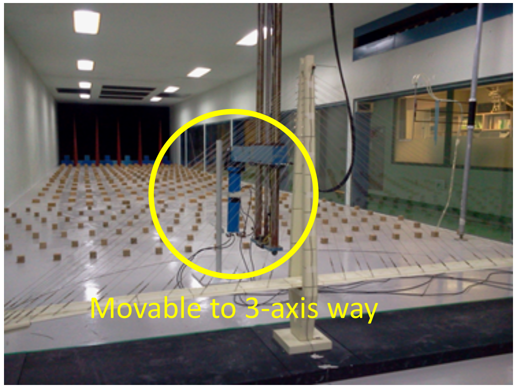




Example of Similitude for Atmospheric Flows

▶ **Target bridge: Hwayang-Juckgeum**

2. Set up the brick, spire and blocks and find the appropriate RPM of the wind tunnel for the equality of the mean wind velocity at the average height of the deck with the wind velocity profile.
3. Check the mean wind velocity and turbulence intensity according to the height

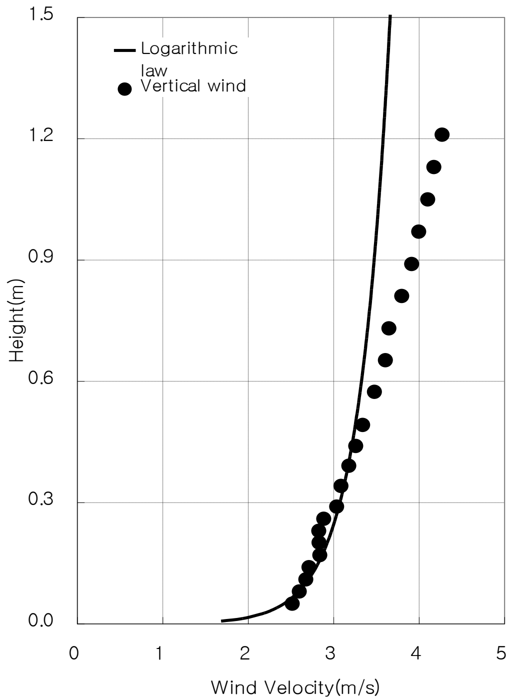


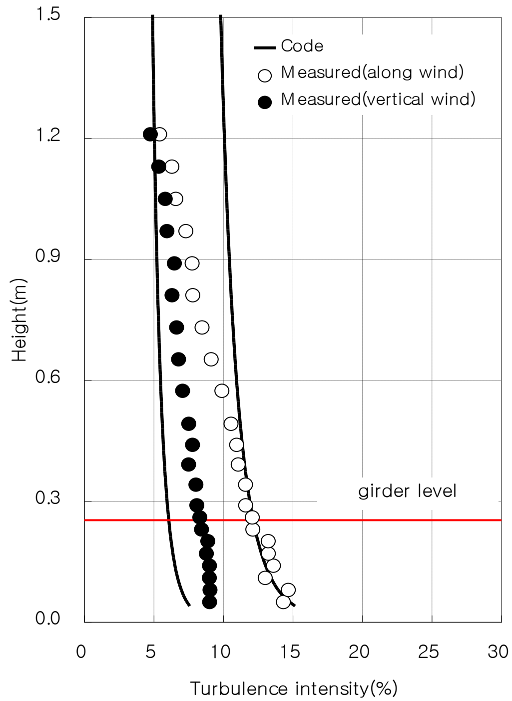
- If the result is bad, go to the step 2 and edit the brick, spire and blocks.

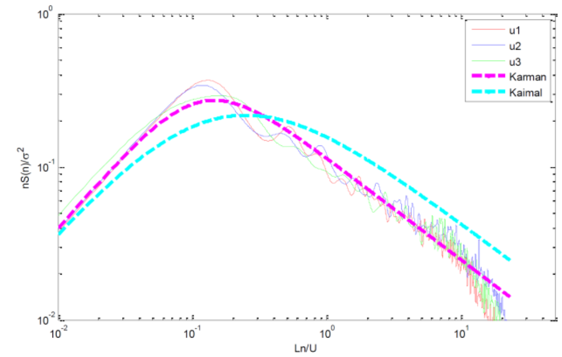
4. Check the spectra and turbulence length scale at the average height of the girder
5. Check the longitudinal distribution of wind velocity / turbulence intensity
   - For the decision of installation position for model

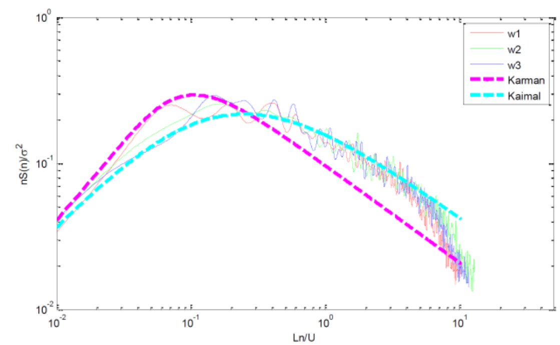
Example of Similitude for Atmospheric Flows

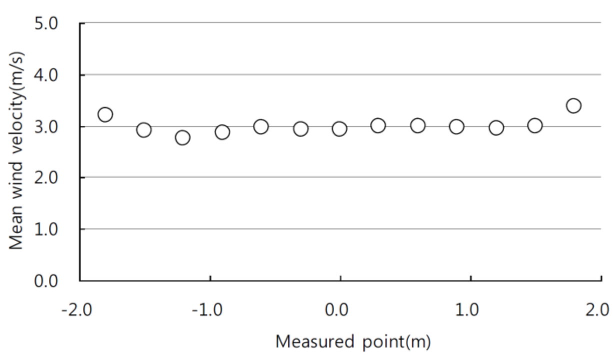
Result

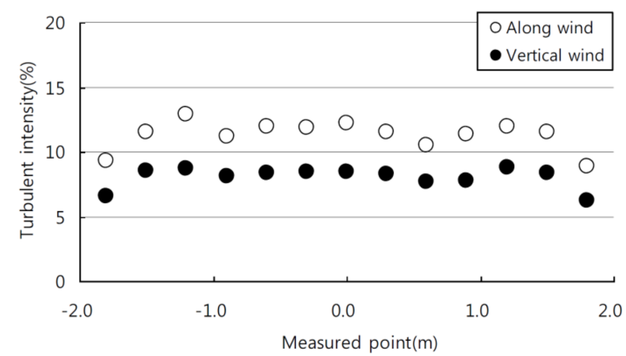












Unsteady Self-Excited Forces(2-D model)

Aerodynamic damping term

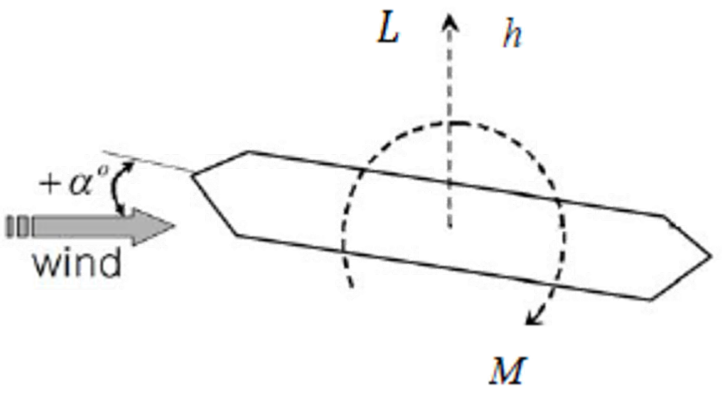
$$L_{se} = \frac{1}{2}\rho U^2 B\left[KH_1^*\frac{\dot{h}}{U} + KH_2^*\frac{B\dot{\alpha}}{U} + K^2H_3^*\alpha + K^2H_4^*\frac{h}{B}\right]$$

$$M_{se} = \frac{1}{2}\rho U^2 B^2\left[KA_1^*\frac{\dot{h}}{U} + KA_2^*\frac{B\dot{\alpha}}{U} + K^2A_3^*\alpha + K^2A_4^*\frac{h}{B}\right]$$

Aerodynamic stiffness term

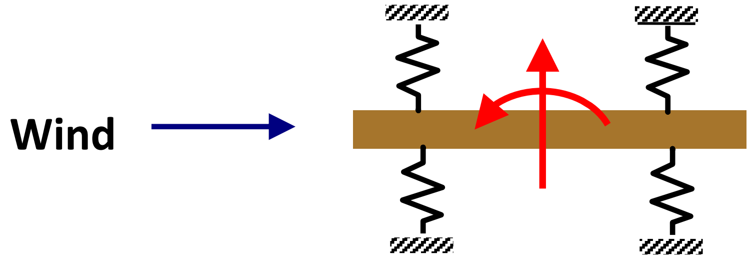
Flutter derivatives about vertical motion

Flutter derivatives about rotational motion



The methodology to get the flutter derivatives

Free-vibration test



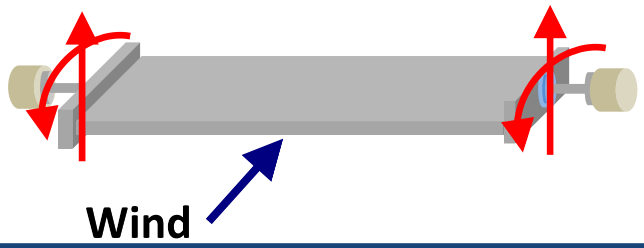
▶ **Advantage**

- Cost effective: No heavy experiment system.

▶ **Disadvantage**

- Accuracy: Results are different at each cases. So, we should do same experiment at least 10 times.
- Low upper limit: As wind speed higher, the vibration disappear faster.
- Assumption violation: No steady-state

Forced-vibration test



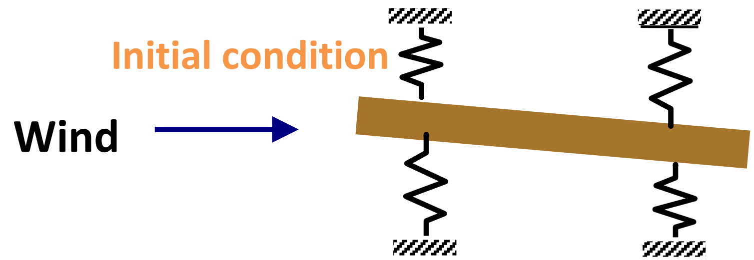
▶ **Advantage**

- Right assumption: The experiment system make the model move sinusoidal.
- No limit: Because we control the displacement, we can measure as we want.

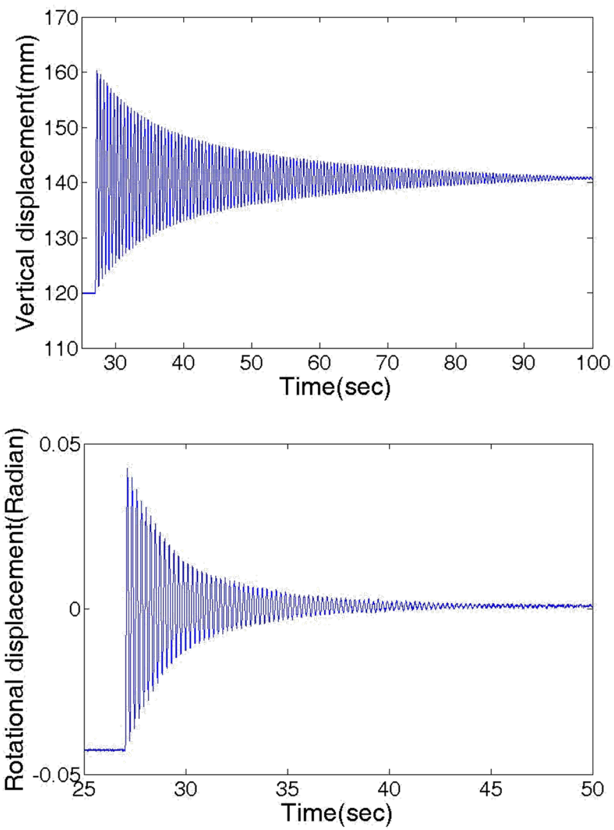
▶ **Disadvantage**

- Initial cost: Relatively very high cost. The system is very heavy and non-movable.
- No interaction: We cannot consider interaction between wind and model.

Extraction method in free-vibration test

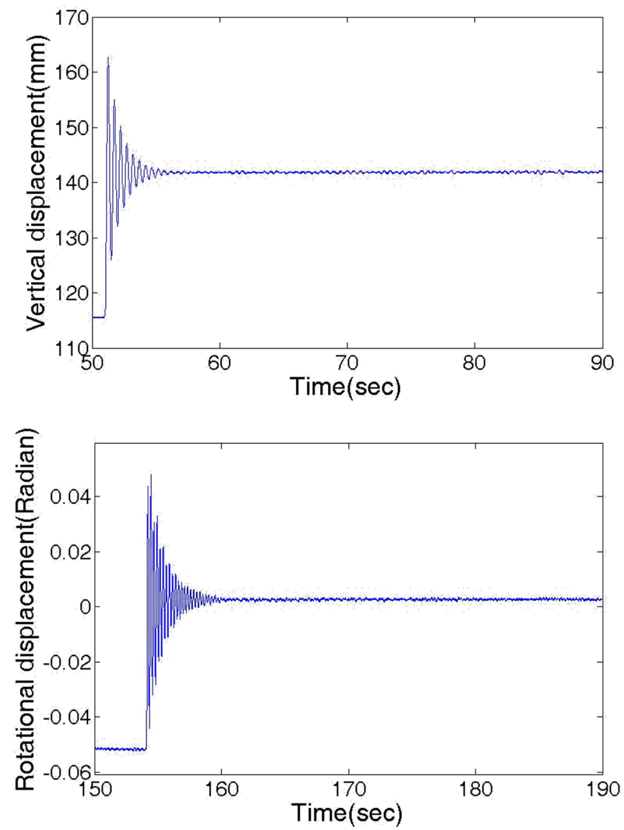


- **No wind**



Damping, Stiffness Change

- **With wind(6.6m/s)**



MITD (Modified Ibrahim time domain method) (1)

Space-state form of 2-DOF dynamic system

- A 2-DOF Dynamic equation(No external force)

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{0}$$

- Space-state form

$$\begin{bmatrix} \mathbf{\dot{U}} \\ \mathbf{\ddot{U}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{\dot{U}} \end{bmatrix}$$

If we set $\mathbf{X} = \begin{bmatrix} \mathbf{U} \\ \mathbf{\dot{U}} \end{bmatrix}$, $\mathbf{\dot{X}} = \mathbf{\Phi X} \quad \Leftrightarrow \quad \mathbf{X}(\mathrm{t}) = \mathrm{e}^{\mathbf{\Phi}\mathrm{t}}\mathbf{X}(0)$

$$\mathrm{e}^{\mathbf{\Phi}\mathrm{t}} = \sum_{\mathrm{n}=0}^{\infty} \frac{1}{\mathrm{n}!} \mathbf{\Phi}^{\mathrm{n}} \mathrm{t}^{\mathrm{n}} = \sum_{\mathrm{n}=0}^{\infty} \frac{1}{\mathrm{n}!} (\boldsymbol{\phi}\mathbf{\Lambda}\boldsymbol{\phi}^{-1})^{\mathrm{n}} \mathrm{t}^{\mathrm{n}} = \sum_{\mathrm{n}=0}^{\infty} \frac{1}{\mathrm{n}!} \boldsymbol{\phi}\mathbf{\Lambda}^{\mathrm{n}}\boldsymbol{\phi}^{-1} \mathrm{t}^{\mathrm{n}} = \boldsymbol{\phi}\left(\sum_{\mathrm{n}=0}^{\infty} \frac{1}{\mathrm{n}!} \mathbf{\Lambda}^{\mathrm{n}} \mathrm{t}^{\mathrm{n}}\right)\boldsymbol{\phi}^{-1}$$

$$= \boldsymbol{\phi}\mathrm{e}^{\mathbf{\Lambda}\mathrm{t}}\boldsymbol{\phi}^{-1}$$

Where,

$\boldsymbol{\phi}$ and $\mathbf{\Lambda}$ are the eigenvector matrix and diagonal matrix of eigenvalues of the system

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix} \quad \mathrm{e}^{\mathbf{\Lambda}\mathrm{t}} = \begin{bmatrix} \mathrm{e}^{\lambda_1 \mathrm{t}} & & & \\ & \mathrm{e}^{\lambda_2 \mathrm{t}} & & \\ & & \mathrm{e}^{\lambda_3 \mathrm{t}} & \\ & & & \mathrm{e}^{\lambda_4 \mathrm{t}} \end{bmatrix}$$

MITD (Modified Ibrahim time domain method) (2)

Space-state form of 2-DOF dynamic system

$$\mathbf{X}(\mathrm{t}) = \begin{bmatrix} \mathbf{U} \\ \dot{\mathbf{U}} \end{bmatrix} = \mathrm{e}^{\mathbf{\Phi}\mathrm{t}}\mathbf{X}(0) = \boldsymbol{\phi}\mathbf{e}^{\Lambda\mathrm{t}}\boldsymbol{\phi}^{-1}\mathbf{X}(0) = \boldsymbol{\phi}\mathbf{e}^{\Lambda\mathrm{t}}\mathbf{d} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix} \begin{bmatrix} \mathrm{d}_1 \mathrm{e}^{\lambda_1 \mathrm{t}} \\ \mathrm{d}_2 \mathrm{e}^{\lambda_2 \mathrm{t}} \\ \mathrm{d}_3 \mathrm{e}^{\lambda_3 \mathrm{t}} \\ \mathrm{d}_4 \mathrm{e}^{\lambda_4 \mathrm{t}} \end{bmatrix} = \begin{bmatrix} \sum_{\mathrm{j}=1}^{4} \phi_{1j} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} \\ \sum_{\mathrm{j}=1}^{4} \phi_{2j} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} \\ \sum_{\mathrm{j}=1}^{4} \phi_{3j} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} \\ \sum_{\mathrm{j}=1}^{4} \phi_{4j} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} \end{bmatrix}$$

- **Characteristics of displacement and velocity**

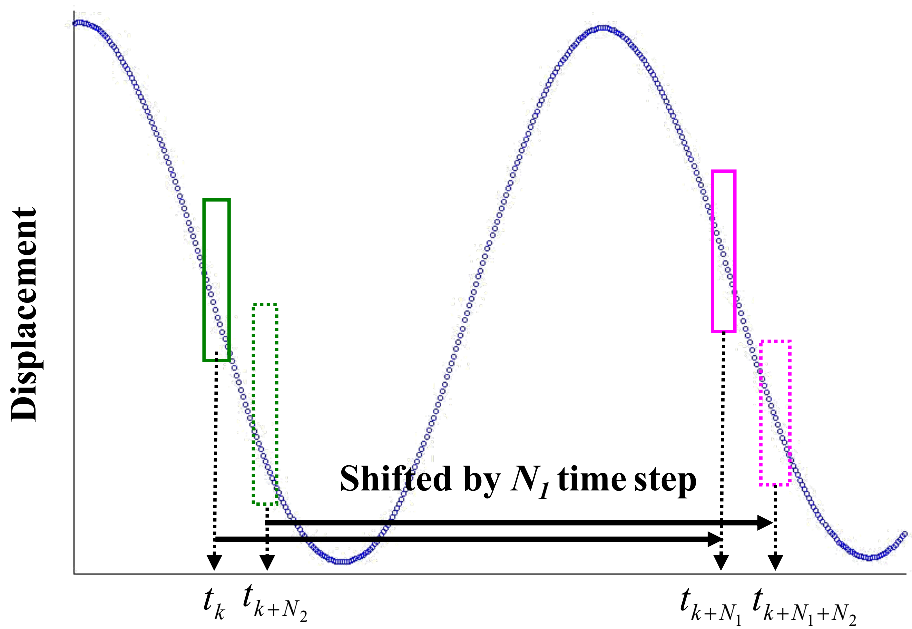
$$\sum_{\mathrm{j}=1}^{4} \phi_{3\mathrm{j}} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} = \frac{d}{dt}\left(\sum_{\mathrm{j}=1}^{4} \phi_{1\mathrm{j}} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} \right) = \sum_{\mathrm{j}=1}^{4} \lambda_j \phi_{1\mathrm{j}} \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} \rightarrow \sum_{\mathrm{j}=1}^{4} (\phi_{3\mathrm{j}} - \lambda_j \phi_{1\mathrm{j}}) \mathrm{d}_\mathrm{j} \mathrm{e}^{\lambda_\mathrm{j} \mathrm{t}} = 0$$

For all initial condition

$$\therefore \phi_{3\mathrm{j}} = \phi_{1\mathrm{j}} \lambda_j \,, \phi_{4\mathrm{j}} = \phi_{2\mathrm{j}} \lambda_j$$

What we need is only displacement data!

Shift operator



$$\left[\begin{pmatrix}\mathbf{U}\left(t_{N_1}\right)\\ \mathbf{U}\left(t_{N_1+N_2}\right)\end{pmatrix},\begin{pmatrix}\mathbf{U}\left(t_{N_1+1}\right)\\ \mathbf{U}\left(t_{N_1+N_2+1}\right)\end{pmatrix},\cdots\begin{pmatrix}\mathbf{U}\left(t_{N-N_2-1}\right)\\ \mathbf{U}\left(t_{N-1}\right)\end{pmatrix}\right]\approx\mathbf{S}\left[\begin{pmatrix}\mathbf{U}(t_0)\\ \mathbf{U}\left(t_{N_2}\right)\end{pmatrix},\begin{pmatrix}\mathbf{U}(t_1)\\ \mathbf{U}\left(t_{N_2+1}\right)\end{pmatrix},\cdots\begin{pmatrix}\mathbf{U}\left(t_{N-N_1-N_2-1}\right)\\ \mathbf{U}\left(t_{N-N_1-1}\right)\end{pmatrix}\right]$$

$$\mathbf{U}_{\mathrm{ob}}\left(t_{N_1}\right)\approx\mathbf{S}\mathbf{U}_{\mathrm{ob}}(t_0)$$

MITD (Modified Ibrahim time domain method) (4)

MITD

- **Minimization of the error**

$$\min_{\mathbf{S}} \boldsymbol{\Pi} = \left\| \left(\mathbf{U_{ob}}(\mathrm{t_{N_1}}) - \mathbf{S}\mathbf{U_{ob}}(\mathrm{t_0}) \right) \mathbf{W^T} \right\|_2^2$$

$$= \left(\left(\mathbf{U_{ob}}(\mathrm{t_{N_1}}) - \mathbf{S}\mathbf{U_{ob}}(\mathrm{t_0}) \right) \mathbf{W^T} \right)^{\mathbf{T}} \left(\mathbf{U_{ob}}(\mathrm{t_{N_1}}) - \mathbf{S}\mathbf{U_{ob}}(\mathrm{t_0}) \right) \mathbf{W^T}$$

$$= \mathbf{W}\mathbf{U_{ob}}^{\mathbf{T}}(\mathrm{t_{N_1}})\mathrm{U_{ob}}(\mathrm{t_{N_1}})\mathbf{W^T} - \mathbf{W}\mathbf{U_{ob}}^{\mathbf{T}}(\mathrm{t_0})\mathbf{S^T}\mathbf{U_{ob}}(\mathrm{t_{N_1}})\mathbf{W^T}$$

$$-\mathbf{W}\mathbf{U_{ob}}^{\mathbf{T}}(\mathrm{t_{N_1}})\mathbf{S}\mathbf{U_{ob}}(\mathrm{t_0})\mathbf{W^T} + \mathbf{W}\mathbf{U_{ob}}^{\mathbf{T}}(\mathrm{t_0})\mathbf{S^T}\mathbf{S}\mathbf{U_{ob}}(\mathrm{t_0})\mathbf{W^T}$$

$$\frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{s}} = -2\mathbf{W}\mathbf{U_{ob}}^{\mathbf{T}}(\mathrm{t_0})\mathbf{U_{ob}}(\mathrm{t_{N_1}})\mathbf{W^T} + 2\mathbf{W}\mathbf{U_{ob}}^{\mathbf{T}}(\mathrm{t_0})\mathbf{S}\mathbf{U_{ob}}(\mathrm{t_0})\mathbf{W^T} = \mathbf{0}$$

$$\mathbf{S} = \mathbf{U_{ob}}(\mathrm{t_{N_1}})\mathbf{W^T}\left(\mathbf{U_{ob}}(\mathrm{t_0})\mathbf{W^T}\right)^{-\mathbf{1}}$$

- **Weighting factor**

$\mathbf{W} = \mathbf{U_{ob}}(\mathrm{t_0})$ - ITD

$\mathbf{W}_{\boldsymbol{p}} = \mathbf{U}_{\boldsymbol{p-1}}(\mathrm{t_0})$ - MITD $\left(\mathbf{U}_{\mathrm{p-1}}(\mathrm{t}) = \mathrm{e}^{\boldsymbol{\Phi}_{\mathrm{p-1}}\mathrm{t}}\mathbf{U}(0) \right)$

→ Repeat these until S matrix converses

MITD (Modified Ibrahim time domain method) (5)

Identification of eigenvector and eigenvalue

$$
\mathbf{U_{ob}}(t_0)=\begin{bmatrix} h(t_0) & h(t_1) & & h(t_{N-N_1-N_2-1}) \\ \alpha(t_0) & \alpha(t_1) & \cdots & \alpha(t_{N-N_1-N_2-1}) \\ h(t_{N_2}) & h(t_{N_2+1}) & & h(t_{N-N_1-1}) \\ \alpha(t_{N_2}) & \alpha(t_{N_2+1}) & & \alpha(t_{N-N_1-1}) \end{bmatrix}=\begin{bmatrix} \sum_{j=1}^{4}\phi_{1j}d_j & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j\Delta t} & & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N-N_1-N_2-1)\Delta t} \\ \sum_{j=1}^{4}\phi_{2j}d_j & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j\Delta t} & \cdots & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N-N_1-N_2-1)\Delta t} \\ \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_jN_2\Delta t} & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N_2+1)\Delta t} & & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N-N_1-1)\Delta t} \\ \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_jN_2\Delta t} & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N_2+1)\Delta t} & & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N-N_1-1)\Delta t} \end{bmatrix}
$$

$$
=\begin{bmatrix} \phi_{11}d_1 & \phi_{12}d_2 & \phi_{13}d_3 & \phi_{14}d_4 \\ \phi_{21}d_1 & \phi_{22}d_2 & \phi_{23}d_3 & \phi_{24}d_4 \\ \phi_{11}d_1e^{\lambda_1N_2\Delta t} & \phi_{12}d_2e^{\lambda_2N_2\Delta t} & \phi_{13}d_3e^{\lambda_3N_2\Delta t} & \phi_{14}d_4e^{\lambda_4N_2\Delta t} \\ \phi_{21}d_1e^{\lambda_1N_2\Delta t} & \phi_{22}d_2e^{\lambda_2N_2\Delta t} & \phi_{23}d_3e^{\lambda_3N_2\Delta t} & \phi_{24}d_4e^{\lambda_4N_2\Delta t} \end{bmatrix}\boxed{\begin{bmatrix} 1 & e^{\lambda_1\Delta t} & & e^{\lambda_1(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_2\Delta t} & \cdots & e^{\lambda_2(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_3\Delta t} & & e^{\lambda_3(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_4\Delta t} & & e^{\lambda_4(N-N_1-N_2-1)\Delta t} \end{bmatrix}}
$$

$$
\mathbf{U_{ob}}(t_{N_1})=\begin{bmatrix} h(t_{N_1}) & h(t_{N_1+1}) & & h(t_{N-N_2-1}) \\ \alpha(t_{N_1}) & \alpha(t_{N_1+1}) & \cdots & \alpha(t_{N-N_2-1}) \\ h(t_{N_1+N_2}) & h(t_{N_1+N_2+1}) & & h(t_{N-1}) \\ \alpha(t_{N_1+N_2}) & \alpha(t_{N_1+N_2+1}) & & \alpha(t_{N-1}) \end{bmatrix}=\begin{bmatrix} \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_jN_1\Delta t} & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N_1+1)\Delta t} & & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N-N_2-1)\Delta t} \\ \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_jN_1\Delta t} & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N_1+1)\Delta t} & \cdots & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N-N_2-1)\Delta t} \\ \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N_1+N_2)\Delta t} & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N_1+N_2+1)\Delta t} & & \sum_{j=1}^{4}\phi_{1j}d_je^{\lambda_j(N-1)\Delta t} \\ \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N_1+N_2)\Delta t} & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N_1+N_2+1)\Delta t} & & \sum_{j=1}^{4}\phi_{2j}d_je^{\lambda_j(N-1)\Delta t} \end{bmatrix}
$$

$$
=\begin{bmatrix} \phi_{11}d_1e^{\lambda_1N_1\Delta t} & \phi_{12}d_2e^{\lambda_2N_1\Delta t} & \phi_{13}d_3e^{\lambda_3N_1\Delta t} & \phi_{14}d_4e^{\lambda_4N_1\Delta t} \\ \phi_{21}d_1e^{\lambda_1N_1\Delta t} & \phi_{22}d_2e^{\lambda_2N_1\Delta t} & \phi_{23}d_3e^{\lambda_3N_1\Delta t} & \phi_{24}d_4e^{\lambda_4N_1\Delta t} \\ \phi_{11}d_1e^{\lambda_1(N_1+N_2)\Delta t} & \phi_{12}d_2e^{\lambda_2(N_1+N_2)\Delta t} & \phi_{13}d_3e^{\lambda_3(N_1+N_2)\Delta t} & \phi_{14}d_4e^{\lambda_4(N_1+N_2)\Delta t} \\ \phi_{21}d_1e^{\lambda_1(N_1+N_2)\Delta t} & \phi_{22}d_2e^{\lambda_2(N_1+N_2)\Delta t} & \phi_{23}d_3e^{\lambda_3(N_1+N_2)\Delta t} & \phi_{24}d_4e^{\lambda_4(N_1+N_2)\Delta t} \end{bmatrix}\boxed{\begin{bmatrix} 1 & e^{\lambda_1\Delta t} & & e^{\lambda_1(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_2\Delta t} & \cdots & e^{\lambda_2(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_3\Delta t} & & e^{\lambda_3(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_4\Delta t} & & e^{\lambda_4(N-N_1-N_2-1)\Delta t} \end{bmatrix}}
$$

MITD (Modified Ibrahim time domain method) (6)

Identification of eigenvector and eigenvalue

$$\mathbf{U_{ob}}(t_{N_1}) = \mathbf{S}\mathbf{U_{ob}}(t_0)$$

$$\mathbf{S}\mathbf{U_{ob}}(t_0) - \mathbf{U_{ob}}(t_{N_1}) = 0$$

$$\mathbf{S}\begin{bmatrix} \phi_{11}d_1 & \phi_{12}d_2 & \phi_{13}d_3 & \phi_{14}d_4 \\ \phi_{21}d_1 & \phi_{22}d_2 & \phi_{23}d_3 & \phi_{24}d_4 \\ \phi_{11}d_1e^{\lambda_1 N_2\Delta t} & \phi_{12}d_2e^{\lambda_2 N_2\Delta t} & \phi_{13}d_3e^{\lambda_3 N_2\Delta t} & \phi_{14}d_4e^{\lambda_4 N_2\Delta t} \\ \phi_{21}d_1e^{\lambda_1 N_2\Delta t} & \phi_{22}d_2e^{\lambda_2 N_2\Delta t} & \phi_{23}d_3e^{\lambda_3 N_2\Delta t} & \phi_{24}d_4e^{\lambda_4 N_2\Delta t} \end{bmatrix}\begin{bmatrix} 1 & e^{\lambda_1\Delta t} & & e^{\lambda_1(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_2\Delta t} & \cdots & e^{\lambda_2(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_3\Delta t} & & e^{\lambda_3(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_4\Delta t} & & e^{\lambda_4(N-N_1-N_2-1)\Delta t} \end{bmatrix}$$
$$-\begin{bmatrix} \phi_{11}d_1e^{\lambda_1 N_1\Delta t} & \phi_{12}d_2e^{\lambda_2 N_1\Delta t} & \phi_{13}d_3e^{\lambda_3 N_1\Delta t} & \phi_{14}d_4e^{\lambda_4 N_1\Delta t} \\ \phi_{21}d_1e^{\lambda_1 N_1\Delta t} & \phi_{22}d_2e^{\lambda_2 N_1\Delta t} & \phi_{23}d_3e^{\lambda_3 N_1\Delta t} & \phi_{24}d_4e^{\lambda_4 N_1\Delta t} \\ \phi_{11}d_1e^{\lambda_1(N_1+N_2)\Delta t} & \phi_{12}d_2e^{\lambda_2(N_1+N_2)\Delta t} & \phi_{13}d_3e^{\lambda_3(N_1+N_2)\Delta t} & \phi_{14}d_4e^{\lambda_4(N_1+N_2)\Delta t} \\ \phi_{21}d_1e^{\lambda_1(N_1+N_2)\Delta t} & \phi_{22}d_2e^{\lambda_2(N_1+N_2)\Delta t} & \phi_{23}d_3e^{\lambda_3(N_1+N_2)\Delta t} & \phi_{24}d_4e^{\lambda_4(N_1+N_2)\Delta t} \end{bmatrix}\begin{bmatrix} 1 & e^{\lambda_1\Delta t} & & e^{\lambda_1(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_2\Delta t} & \cdots & e^{\lambda_2(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_3\Delta t} & & e^{\lambda_3(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_4\Delta t} & & e^{\lambda_4(N-N_1-N_2-1)\Delta t} \end{bmatrix} = 0$$

$$\mathbf{S}\begin{bmatrix} \phi_{11}d_1 & \phi_{12}d_2 & \phi_{13}d_3 & \phi_{14}d_4 \\ \phi_{21}d_1 & \phi_{22}d_2 & \phi_{23}d_3 & \phi_{24}d_4 \\ \phi_{11}d_1e^{\lambda_1 N_2\Delta t} & \phi_{12}d_2e^{\lambda_2 N_2\Delta t} & \phi_{13}d_3e^{\lambda_3 N_2\Delta t} & \phi_{14}d_4e^{\lambda_4 N_2\Delta t} \\ \phi_{21}d_1e^{\lambda_1 N_2\Delta t} & \phi_{22}d_2e^{\lambda_2 N_2\Delta t} & \phi_{23}d_3e^{\lambda_3 N_2\Delta t} & \phi_{24}d_4e^{\lambda_4 N_2\Delta t} \end{bmatrix}\begin{bmatrix} 1 & e^{\lambda_1\Delta t} & & e^{\lambda_1(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_2\Delta t} & \cdots & e^{\lambda_2(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_3\Delta t} & & e^{\lambda_3(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_4\Delta t} & & e^{\lambda_4(N-N_1-N_2-1)\Delta t} \end{bmatrix}$$
$$-\begin{bmatrix} \phi_{11}d_1 & \phi_{12}d_2 & \phi_{13}d_3 & \phi_{14}d_4 \\ \phi_{21}d_1 & \phi_{22}d_2 & \phi_{23}d_3 & \phi_{24}d_4 \\ \phi_{11}d_1e^{\lambda_1 N_2\Delta t} & \phi_{12}d_2e^{\lambda_2 N_2\Delta t} & \phi_{13}d_3e^{\lambda_3 N_2\Delta t} & \phi_{14}d_4e^{\lambda_4 N_2\Delta t} \\ \phi_{21}d_1e^{\lambda_1 N_2\Delta t} & \phi_{22}d_2e^{\lambda_2 N_2\Delta t} & \phi_{23}d_3e^{\lambda_3 N_2\Delta t} & \phi_{24}d_4e^{\lambda_4 N_2\Delta t} \end{bmatrix}\begin{bmatrix} e^{\lambda_1 N_1\Delta t} & & & \\ & e^{\lambda_2 N_1\Delta t} & & \\ & & e^{\lambda_3 N_1\Delta t} & \\ & & & e^{\lambda_4 N_1\Delta t} \end{bmatrix}\begin{bmatrix} 1 & e^{\lambda_1\Delta t} & & e^{\lambda_1(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_2\Delta t} & \cdots & e^{\lambda_2(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_3\Delta t} & & e^{\lambda_3(N-N_1-N_2-1)\Delta t} \\ 1 & e^{\lambda_4\Delta t} & & e^{\lambda_4(N-N_1-N_2-1)\Delta t} \end{bmatrix}$$
$$= 0$$

Identification of eigenvector and eigenvalue

$$
\left\{ S \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{11} e^{\lambda_1 N_2 \Delta t} & \phi_{12} e^{\lambda_2 N_2 \Delta t} & \phi_{13} e^{\lambda_3 N_2 \Delta t} & \phi_{14} e^{\lambda_4 N_2 \Delta t} \\ \phi_{21} e^{\lambda_1 N_2 \Delta t} & \phi_{22} e^{\lambda_2 N_2 \Delta t} & \phi_{23} e^{\lambda_3 N_2 \Delta t} & \phi_{24} e^{\lambda_4 N_2 \Delta t} \end{bmatrix} - \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{11} e^{\lambda_1 N_2 \Delta t} & \phi_{12} e^{\lambda_2 N_2 \Delta t} & \phi_{13} e^{\lambda_3 N_2 \Delta t} & \phi_{14} e^{\lambda_4 N_2 \Delta t} \\ \phi_{21} e^{\lambda_1 N_2 \Delta t} & \phi_{22} e^{\lambda_2 N_2 \Delta t} & \phi_{23} e^{\lambda_3 N_2 \Delta t} & \phi_{24} e^{\lambda_4 N_2 \Delta t} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 N_1 \Delta t} & & & \\ & e^{\lambda_2 N_1 \Delta t} & & \\ & & e^{\lambda_3 N_1 \Delta t} & \\ & & & e^{\lambda_4 N_1 \Delta t} \end{bmatrix} \right\} \begin{bmatrix} \mathbf{1} & \boldsymbol{e^{\lambda_1 \Delta t}} & \cdots & \boldsymbol{e^{\lambda_1 (N-N_1-N_2-1)\Delta t}} \\ \mathbf{1} & \boldsymbol{e^{\lambda_2 \Delta t}} & & \boldsymbol{e^{\lambda_2 (N-N_1-N_2-1)\Delta t}} \\ \mathbf{1} & \boldsymbol{e^{\lambda_3 \Delta t}} & & \boldsymbol{e^{\lambda_3 (N-N_1-N_2-1)\Delta t}} \\ \mathbf{1} & \boldsymbol{e^{\lambda_4 \Delta t}} & & \boldsymbol{e^{\lambda_4 (N-N_1-N_2-1)\Delta t}} \end{bmatrix} = \mathbf{0}
$$

$$
S = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{11} e^{\lambda_1 N_2 \Delta t} & \phi_{12} e^{\lambda_2 N_2 \Delta t} & \phi_{13} e^{\lambda_3 N_2 \Delta t} & \phi_{14} e^{\lambda_4 N_2 \Delta t} \\ \phi_{21} e^{\lambda_1 N_2 \Delta t} & \phi_{22} e^{\lambda_2 N_2 \Delta t} & \phi_{23} e^{\lambda_3 N_2 \Delta t} & \phi_{24} e^{\lambda_4 N_2 \Delta t} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 N_1 \Delta t} & & & \\ & e^{\lambda_2 N_1 \Delta t} & & \\ & & e^{\lambda_3 N_1 \Delta t} & \\ & & & e^{\lambda_4 N_1 \Delta t} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{11} e^{\lambda_1 N_2 \Delta t} & \phi_{12} e^{\lambda_2 N_2 \Delta t} & \phi_{13} e^{\lambda_3 N_2 \Delta t} & \phi_{14} e^{\lambda_4 N_2 \Delta t} \\ \phi_{21} e^{\lambda_1 N_2 \Delta t} & \phi_{22} e^{\lambda_2 N_2 \Delta t} & \phi_{23} e^{\lambda_3 N_2 \Delta t} & \phi_{24} e^{\lambda_4 N_2 \Delta t} \end{bmatrix}^{-1}
$$

$$
= \boldsymbol{\phi_S \Lambda_S \phi_S^{-1}}
$$

Identification of eigenvector and eigenvalue

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix} = \frac{1}{\mathrm{N_1 \Delta t}} \ln \mathbf{\Lambda_S}$$

Where $\mathbf{\Lambda_S}$ is diagonal matrix of eigenvalues of $\mathbf{S}$

$$\boldsymbol{\phi}_s = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{11} e^{\lambda_1 N_2 \Delta t} & \phi_{12} e^{\lambda_2 N_2 \Delta t} & \phi_{13} e^{\lambda_3 N_2 \Delta t} & \phi_{14} e^{\lambda_4 N_2 \Delta t} \\ \phi_{21} e^{\lambda_1 N_2 \Delta t} & \phi_{22} e^{\lambda_2 N_2 \Delta t} & \phi_{23} e^{\lambda_3 N_2 \Delta t} & \phi_{24} e^{\lambda_4 N_2 \Delta t} \end{bmatrix} = \begin{bmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{1j} e^{\lambda_j N_2 \Delta t} \\ \phi_{2j} e^{\lambda_j N_2 \Delta t} \end{bmatrix}$$

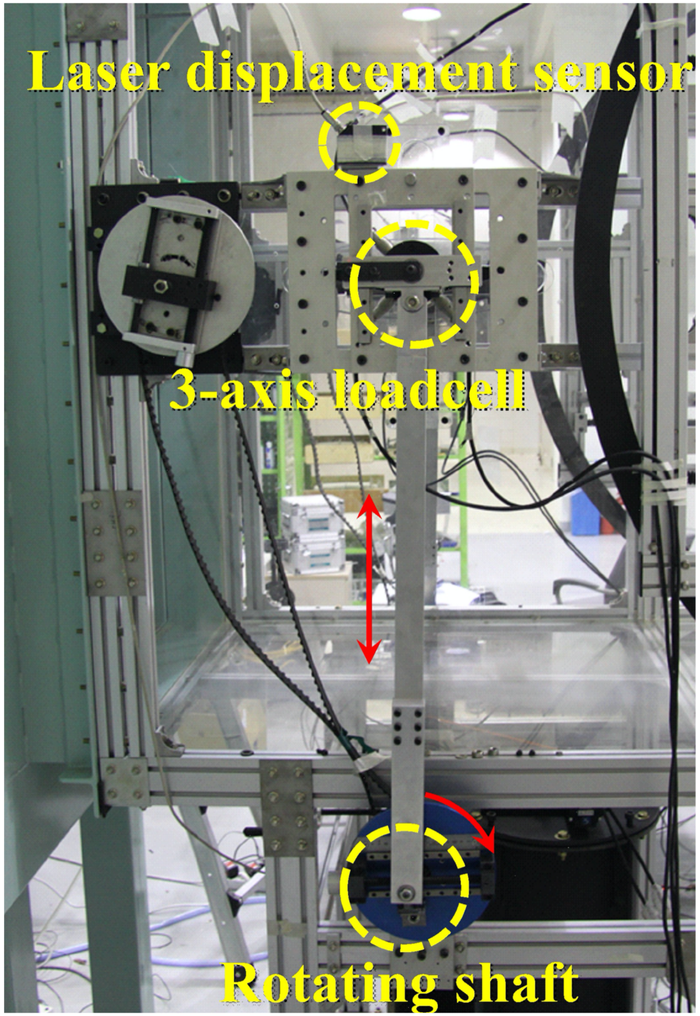
$$\begin{bmatrix} \boldsymbol{\phi}_{1j} \\ \boldsymbol{\phi}_{2j} \end{bmatrix} = [\mathbf{I}_{2\times 2} \quad \mathbf{0}_{2\times 2}] \boldsymbol{\phi}_s = \mathbf{G} \boldsymbol{\phi}_s$$

$$\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_{1j} \\ \boldsymbol{\phi}_{2j} \\ \boldsymbol{\phi}_{3j} \\ \boldsymbol{\phi}_{4j} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_{1j} \\ \boldsymbol{\phi}_{2j} \\ \lambda_j \boldsymbol{\phi}_{1j} \\ \lambda_j \boldsymbol{\phi}_{2j} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G\phi}_s \\ \boldsymbol{G\Lambda\phi}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{G} \frac{1}{N_1 \Delta t} \ln \boldsymbol{\Lambda_S} \end{bmatrix} \boldsymbol{\phi}_s$$

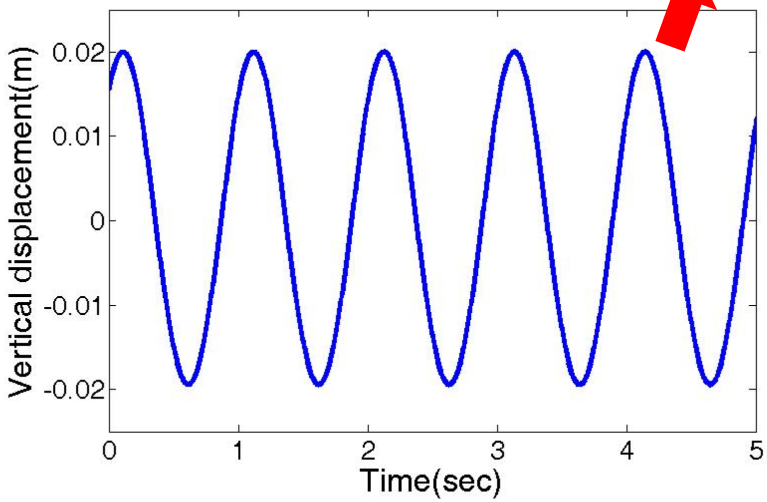
$$\therefore \boldsymbol{\Phi} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} = \boldsymbol{\phi\Lambda\phi}^{-1} = \frac{1}{N_1 \Delta t} \begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{G} \frac{1}{N_1 \Delta t} \ln \boldsymbol{\Lambda_S} \end{bmatrix} \boldsymbol{\phi}_s \ln \boldsymbol{\Lambda_S} \left[\begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{G} \frac{1}{N_1 \Delta t} \ln \boldsymbol{\Lambda_S} \end{bmatrix} \boldsymbol{\phi}_s \right]^{-1}$$

Extraction method in forced-vibration test

Basic concept

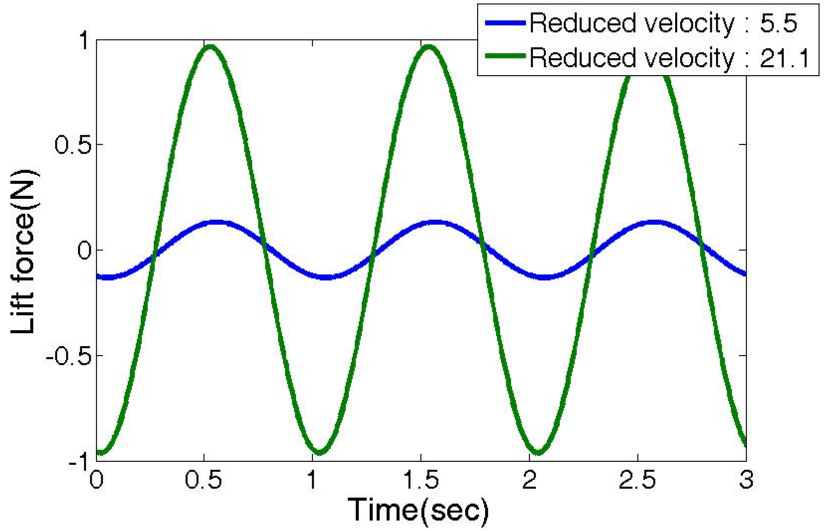


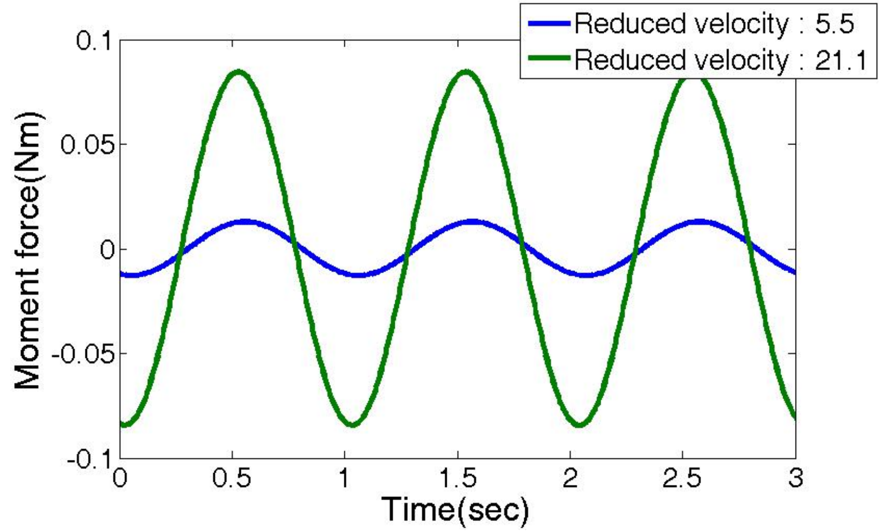
- Control the amplitude of the motion.
- Oscillate the system at the frequency which we want.
- We can extract the aero-derivatives at the high reduced velocity.



Sin fitting method (1)

Concept of sin fitting





$$L_{ae}(t) = L_h sin(2\pi\omega_h t + \theta_{L_h}) \qquad M_{ae}(t) = M_h sin(2\pi\omega_h t + \theta_{M_h})$$

- First, we fit our data as sin function.
- Using amplitude difference and phase difference, we can calculate flutter derivatives.

Sin fitting method (2)

Concept of sin fitting

$$L_{ae} = \frac{1}{2}\rho U^2 B\left(K_h H_1^* \frac{\dot{h}}{U} + K_\alpha H_2^* \frac{B\dot{\alpha}}{U} + K_\alpha^2 H_3^* \alpha + K_h^2 H_4^* \frac{h}{B} \right)$$

$$M_{ae} = \frac{1}{2}\rho U^2 B^2\left(K_h A_1^* \frac{\dot{h}}{U} + K_\alpha A_2^* \frac{B\dot{\alpha}}{U} + K_\alpha^2 A_3^* \alpha + K_h^2 A_4^* \frac{\alpha}{B} \right)$$

Vertical motion

$$h(t) = h_0 \sin(\omega_h t)$$

$$L_{ae} = \frac{1}{2}\rho U^2 B\left(K_h H_1^* \frac{\dot{h}}{U} + K_h^2 H_4^* \frac{h}{B} \right)$$

$$M_{ae} = \frac{1}{2}\rho U^2 B^2\left(K_h A_1^* \frac{\dot{h}}{U} + K_h^2 A_4^* \frac{\alpha}{B} \right)$$

Rotational motion

$$\alpha(t) = \alpha_0 \sin(\omega_h t)$$

$$L_{ae} = \frac{1}{2}\rho U^2 B\left(K_\alpha H_2^* \frac{B\dot{\alpha}}{U} + K_\alpha^2 H_3^* \alpha \right)$$

$$M_{ae} = \frac{1}{2}\rho U^2 B^2\left(K_\alpha A_2^* \frac{B\dot{\alpha}}{U} + K_\alpha^2 A_3^* \alpha \right)$$

Formulation of sin fitting method

$$L_{ae} = \frac{1}{2}\rho U^2 B(K_h H_1^* \frac{\dot{h}}{U} + K_h^2 H_4^* \frac{h}{B})$$

Vertical motion, Lift force

$$L_h \sin(\omega_h t + \theta_{Lh}) = \frac{1}{2}\rho U^2 B(K_h H_1^* \frac{h_0 \frac{d\sin(\omega_h t)}{dt}}{U} + K_h^2 H_4^* \frac{h_0 \sin(\omega_h t)}{B})$$

Assume $h(t) = h_0 \sin(\omega_h t)$

$$L_h \int_{-\infty}^{\infty} \sin(\omega_h t + \theta_{Lh}) e^{-i\omega t} dt$$

$$= \frac{1}{2}\rho U^2 B h_0 (\frac{K_h H_1^*}{U} \int_{-\infty}^{\infty} \frac{d\sin(\omega_h t)}{dt} e^{-i\omega t} dt + \frac{K_h^2 H_4^*}{B} \int_{-\infty}^{\infty} \sin(\omega_h t) e^{-i\omega t} dt)$$

$$= \frac{1}{2}\rho U^2 B h_0 (\frac{K_h H_1^*}{U} i\omega_h \int_{-\infty}^{\infty} \sin(\omega_h t) e^{-i\omega t} dt + \frac{K_h^2 H_4^*}{B} \int_{-\infty}^{\infty} \sin(\omega_h t) e^{-i\omega t} dt)$$

$$= \frac{1}{2}\rho U^2 B h_0 (\frac{K_h H_1^*}{U} i\omega_h + \frac{K_h^2 H_4^*}{B}) \int_{-\infty}^{\infty} \sin(\omega_h t) e^{-i\omega t} dt$$

$$\because \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{i\omega t} dt = i\omega F(\omega)$$

Formulation of sin fitting method

$$L_{ae}=\frac{1}{2}\rho U^2 B(K_h H_1^* \frac{\dot{h}}{U}+K_h^2 H_4^* \frac{h}{B})$$

Vertical motion, Lift force

$$L_h \int_{-\infty}^{\infty}\left[\sin(\omega_h t)\cos(\theta_{Lh})+\cos(\omega_h t)\sin(\theta_{Lh})\right]e^{-i\omega t}dt$$

$$=L_h\cos(\theta_{Lh})\int_{-\infty}^{\infty}\sin(\omega_h t)e^{-i\omega t}dt+L_h\sin(\theta_{Lh})\frac{1}{\omega_h}\int_{-\infty}^{\infty}\omega_h\cos(\omega_h t)e^{-i\omega t}dt$$

$$=L_h\cos(\theta_{Lh})\int_{-\infty}^{\infty}\sin(\omega_h t)e^{-i\omega t}dt+L_h\sin(\theta_{Lh})\frac{1}{\omega_h}\int_{-\infty}^{\infty}\frac{d\sin(\omega_h t)}{dt}e^{-i\omega t}dt$$

$$=L_h\cos(\theta_{Lh})\int_{-\infty}^{\infty}\sin(\omega_h t)e^{-i\omega t}dt+L_h\sin(\theta_{Lh})\frac{i\omega_h}{\omega_h}\int_{-\infty}^{\infty}\sin(\omega_h t)e^{-i\omega t}dt$$

$$=L_h\left\{\cos(\theta_{Lh})+i\sin(\theta_{Lh})\right\}\int_{-\infty}^{\infty}\sin(\omega_h t)e^{-i\omega t}dt$$

$$\therefore \frac{1}{2}\rho U^2 B h_0\left(\frac{K_h^2 H_4^*}{B}+\frac{K_h H_1^*}{U}i\omega_h\right)=L_h\left\{\cos(\theta_{Lh})+i\sin(\theta_{Lh})\right\}$$

$$\boxed{H_1^*=\frac{L_h\sin(\theta_{Lh})}{\frac{1}{2}\rho B^2\omega_h^2 h_0} \quad H_4^*=\frac{L_h\cos(\theta_{Lh})}{\frac{1}{2}\rho B^2\omega_h^2 h_0}}$$

Concept of EEE

$$\boxed{\operatorname*{Min}_{\mathbf{X}} \Pi(\mathbf{X}) = \frac{1}{2}\sum_{i=1}^{nt} \left\| \mathbf{F_{kn}}(t_i) - \mathbf{F_{un}}(\mathbf{X}, t_i) \right\|_2^2}$$

Minimization problem !!

$$= \frac{1}{2}\sum_{i=1}^{nt} \left(\mathbf{F}_{kn}(t_i) - \mathbf{F}_{un}(\mathbf{X}, t_i)\right)^T \left(\mathbf{F}_{kn}(t_i) - \mathbf{F}_{un}(\mathbf{X}, t_i)\right)$$

$$= \frac{1}{2}\sum_{i=1}^{nt} \left(\mathbf{F}_{kn}^T(t_i)\mathbf{F}_{kn}(t_i) - 2\mathbf{F}_{un}^T(\mathbf{X}, t_i)\mathbf{F}_{kn}(t_i) + \mathbf{F}_{un}^T(\mathbf{X}, t_i)\mathbf{F}_{un}(\mathbf{X}, t_i)\right)$$

$$= \frac{1}{2}\sum_{i=1}^{nt} \left(\mathbf{F}_{kn}^T(t_i)\mathbf{F}_{kn}(t_i) - 2\mathbf{X}^T\mathbf{s}^T(t_i)\mathbf{F}_{kn}(t_i) + \mathbf{X}^T\mathbf{s}^T(t_i)\mathbf{s}(t_i)\mathbf{X}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{nt} \mathbf{F}_{kn}^T(t_i)\mathbf{F}_{kn}(t_i) - \mathbf{X}^T\mathbf{G} + \frac{1}{2}\mathbf{X}^T\mathbf{S}\mathbf{X}$$

$$\textit{Where,}\quad \mathbf{S} = \sum_{i=1}^{nt} \mathbf{s}(t_i)^T\mathbf{s}(t_i)\,,\ \ \mathbf{G} = \sum_{i=1}^{nt} \mathbf{s}(t_i)^T\mathbf{F_{kn}}(t_i)\ \ ,\ \ \mathbf{F_{un}}(t_i) = \mathbf{s}(t_i)\mathbf{X}$$

Concept of EEE

$$\underset{\mathbf{X}}{\mathrm{Min}}\,\Pi(\mathbf{X}) = \frac{1}{2}\sum_{i=1}^{nt}\mathbf{F}_{kn}^{T}(t_i)\mathbf{F}_{kn}(t_i) - \mathbf{X}^{T}\mathbf{G} + \frac{1}{2}\mathbf{X}^{T}\mathbf{S}\mathbf{X}$$

$$\frac{\partial\Pi(\mathbf{X})}{\partial\mathbf{X}} = \mathbf{S}\mathbf{X} - \mathbf{G} = 0$$

$$\mathbf{X} = \mathbf{S}^{-1}\mathbf{G}$$

A unique solution is always determined by above eq. as long as a sufficient amount of measured dynamic responses of a section model are provided.

- **Apply EEE method for the no-wind state**

$$\underline{\mathbf{Ma}}(t_i)+\underline{\mathbf{Cv}}(t_i)+\underline{\mathbf{Ku}}(t_i)=\underline{\mathbf{R}(t_i)}=-\underline{\mathbf{P}(t_i)}$$

Known value

Unknown value

$$\mathbf{Ma}(t_i)+\mathbf{Cv}(t_i)+\mathbf{Ku}(t_i)=\mathbf{F}_{\mathbf{un}}(t_i)$$

$$\mathbf{R}(t_i)=\mathbf{F}_{\mathbf{kn}}(t_i)$$

$$\mathbf{F}_{\mathbf{un}}(t_i)=\mathbf{s}(t_i)\mathbf{X}$$

$$\mathbf{s}(t_i)=\begin{bmatrix} a_h(t_i) & a_\alpha(t_i) & v_h(t_i) & v_\alpha(t_i) & u_h(t_i) & u_\alpha(t_i) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_h(t_i) & a_\alpha(t_i) & v_h(t_i) & v_\alpha(t_i) & u_h(t_i) & u_\alpha(t_i) \end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix} m_{11} & m_{12} & c_{12} & c_{12} & k_{12} & k_{12} & m_{21} & m_{22} & c_{21} & c_{22} & k_{21} & k_{22} \end{bmatrix}^T$$

$$\frac{\partial\Pi(\mathbf{X})}{\partial\mathbf{X}}=-\mathbf{G}+\mathbf{SX}=\mathbf{0}$$

$$\mathbf{X}=\mathbf{S}^{-1}\mathbf{G}$$

Apply EEE method for the wind state

Known value
Unknown value

$$\underline{\mathbf{Ma}(t_i)}+\underline{\mathbf{Cv}(t_i)}+\underline{\mathbf{Ku}(t_i)}=\underline{\mathbf{R}(t_i)}+\underline{\mathbf{C}_{\mathbf{ae}}}\underline{\mathbf{v}(t_i)}+\underline{\mathbf{K}_{ae}}\underline{\mathbf{u}(t_i)}$$

$$Where, \mathbf{C}_{\mathbf{ae}}=\frac{1}{2}\rho\omega_{ex}B^2\begin{bmatrix} H_1 & BH_2 \\ BA_1 & B^2A_2 \end{bmatrix}, \mathbf{K}_{\mathbf{ae}}=\frac{1}{2}\rho\omega_{ex}^2B^2\begin{bmatrix} H_4 & BH_3 \\ BA_4 & B^2A_3 \end{bmatrix}$$

$$\begin{aligned}\mathbf{F}_{\mathbf{kn}}(t_i) &= \mathbf{Ma}(t_i)+\mathbf{Cv}(t_i)+\mathbf{Ku}(t_i)-\mathbf{R}(t_i)\\ &= \mathbf{Ma}(t_i)+\mathbf{Cv}(t_i)+\mathbf{Ku}(t_i)+\mathbf{P}(t_i)\end{aligned}$$

$$\mathbf{F}_{\mathbf{un}}(t_i)=\mathbf{C}_{\mathbf{ae}}\dot{\mathbf{u}}(t_i)+\mathbf{K}_{\mathbf{ae}}\mathbf{u}(t_i)=\mathbf{s}(t_i)\mathbf{X}$$

$$\mathbf{s}(t_i)=\begin{bmatrix} v_h(t_i) & v_\alpha(t_i) & u_h(t_i) & u_\alpha(t_i) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_h(t_i) & v_\alpha(t_i) & u_h(t_i) & u_\alpha(t_i) \end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix} H_1 & H_2 & H_4 & H_3 & A_1 & A_2 & A_4 & A_3 \end{bmatrix}^T$$

$$\mathbf{X}=\mathbf{S}^{-1}\mathbf{G}$$

THANK YOU

for your attention!