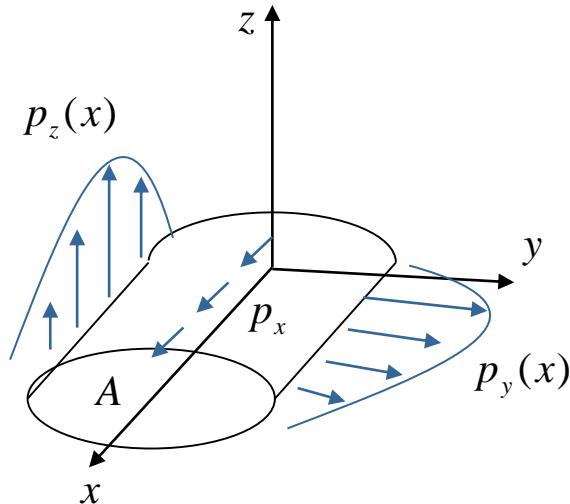


Beams

❖ Beams

- 2 dimensions $\ll 1$
 - Could have
 - out of plane bending
 - extensions
 - twisting
 - warping
 - transverse shear
- all kinematic assumptions

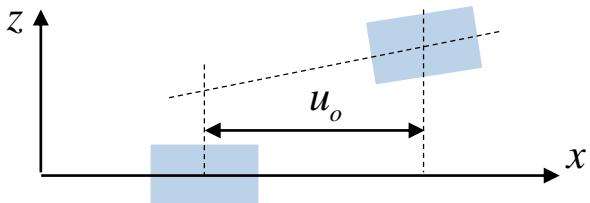


Beams

kinematics

→ Bernoulli-Euler Beam

- plane section remain plane \perp to the centerline



$$u(x, y, z) = u_o - y \frac{du}{dx} - z \frac{dw}{dx}$$

$$v(x, y, z) = v_o(x)$$

$$w(x, y, z) = w_o(x)$$

- Strain-Displacement Relation

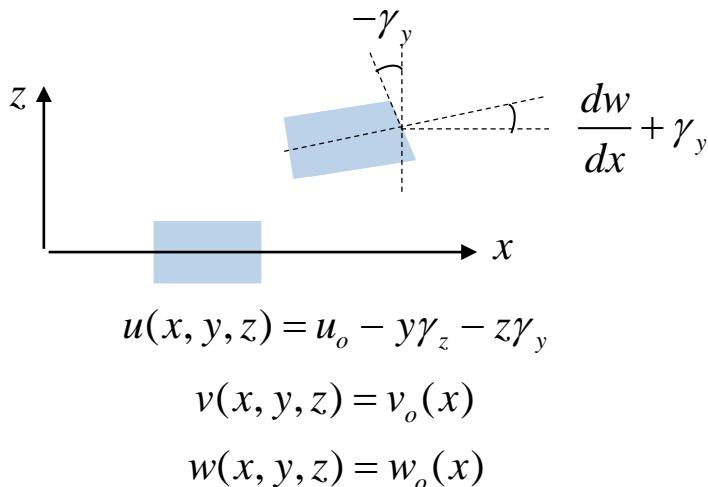
$$\begin{aligned} S_1 &= \frac{du}{dx} = \frac{du_o}{dx} + y \left(-\frac{d^2 v_o}{dx^2} \right) - z \left(-\frac{d^2 w_o}{dx^2} \right) \\ &= \varepsilon_o + y \kappa_z - z \kappa_y \end{aligned} \quad S_2 = \frac{dv}{dy} = 0 \quad S_3 = \frac{dw}{dz} = 0$$

$$S_4 = S_5 = 0 \quad S_6 = 2S_{12} = \frac{du}{dx} + \frac{dv}{dx}$$

unknowns $\varepsilon_o, \kappa_z, \kappa_y$
at each section

Beams

- Transverse shear kinematics – Timoshenko beam theory
Allow shear in beam



- Strains

$$S_1 = \varepsilon_o + y \frac{d\gamma_z}{dx} + z \frac{d\gamma_y}{dx}$$

5 parameters

$$S_5 = \frac{dw_o}{dx} + \gamma_y$$

$$\varepsilon_o$$

$$\frac{d\gamma_z}{dx} \quad \frac{d\gamma_y}{dx}$$

$$\frac{dw_o}{dx} + \gamma_y \quad \frac{dv_o}{dx} + \gamma_z$$

$$S_6 = \frac{dv_o}{dx} + \gamma_z$$

extension

2 curvatures

2 shear

Beams

- Torsion Kinematics
 - simplest is St. Venant's torsion
 - inherently a 2-D problem
 - shear distribution on the cross section

* Prandtl → stress distribution
any cross section

$$\rightarrow \varphi' = \frac{d\varphi}{dx} \text{ is constant}$$

→ each section rotates as a rigid body

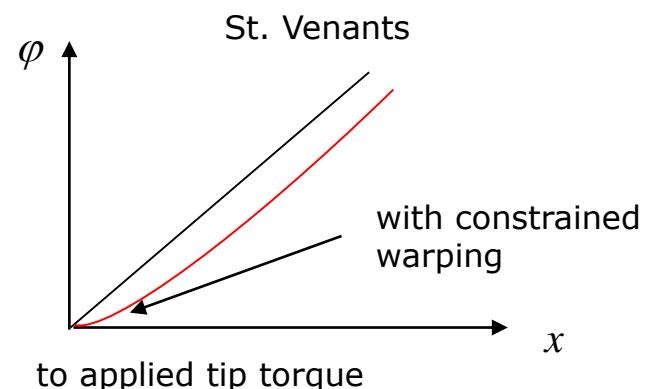
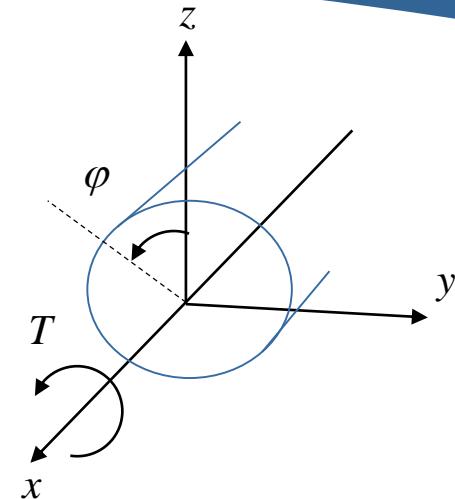
$$u = f(y, z)\varphi' \quad : \text{warping} \quad v = -z\varphi \quad w = y\varphi$$

→ cross section is free to warp

$$S_1 = \varphi'' f(y, z) = 0 \quad S_5 = \varphi' \left(\frac{\partial f}{\partial z} + y \right)$$

$$S_2 = S_3 = S_4 = 0 \quad S_6 = \varphi' \left(\frac{\partial f}{\partial y} - z \right)$$

can allow φ' to vary as a function of x
→ constrained warping



Beams

➤ General Beam Model ("Rehfield")

$$u = u_o + y\gamma_z + z\gamma_y + \varphi''f(y, z)$$

$$v = v_o - z\varphi$$

$$w = w_o + y\varphi$$

- 7 descriptive variables per cross section

$$\varepsilon_o, \quad \frac{d\gamma_z}{dx}, \quad \frac{d\gamma_y}{dx}, \quad \frac{dw_o}{dx} + \gamma_y, \quad \frac{dv_o}{dx} + \gamma_z, \quad \varphi', \quad \varphi''$$

➤ Bending of a Beam

- E-B (Euler-Bernoulli) kinematics

$$u = u_o - y \frac{dv_o}{dx} - z \frac{dw_o}{dx}$$

$$v = v_o(x)$$

$$w = w_o(x)$$

$$S_1 = \frac{du_o}{dx} + y \left(-\frac{d^2 v_o}{dx^2} \right) + z \left(-\frac{d^2 w_o}{dx^2} \right) = \varepsilon_o + y\kappa_z + z\kappa_y$$
$$S_2 \rightarrow S_6 = 0$$

Beams

➤ Constitutive Properties

Option i) reduce from $\begin{bmatrix} T \\ D \end{bmatrix} = \begin{bmatrix} c^E & -e_t \\ e & \varepsilon^S \end{bmatrix} \begin{Bmatrix} S \\ E \end{Bmatrix}$ (1)

to $\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} c_{11}^E & -e_t \\ e & \varepsilon^S \end{bmatrix} \begin{Bmatrix} S_1 \\ E \end{Bmatrix}$ (2)

4x4

Option ii) reduce (1) to include only normal strains

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ D \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ E \end{Bmatrix}$$

Then say $S_2 = S_3 = -\nu S_1$

reduce to

$$\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} c_{11}^E & -e_t \\ e & \varepsilon^S \end{bmatrix} \begin{Bmatrix} S_1 \\ E \end{Bmatrix}$$

Beams

Option iii) stress form (2)

$$\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} s^E & d_t \\ d & \varepsilon^T \end{bmatrix} \begin{Bmatrix} T \\ E \end{Bmatrix}$$

reduce this assuming only T_1

$$\begin{bmatrix} S_1 \\ D \end{bmatrix} = \begin{bmatrix} s_{11}^E & d_t \\ d & \varepsilon^T \end{bmatrix} \begin{Bmatrix} T_1 \\ E \end{Bmatrix} \quad S_1 = s_{11} T_1 + dE$$

$$T_1 = c_{11}^{-1} S_1 - s_{11}^{-1} dE$$

invert this to obtain

$$\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} c_{11}^E & -e_t \\ e & \varepsilon^S \end{bmatrix} \begin{Bmatrix} S_1 \\ E \end{Bmatrix}$$

 effective

no piezoelectric coupling in structure

$$\begin{bmatrix} T_1 \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} c_{11}^E & -e_{11} & -e_{12} & -e_{13} \\ e_{11} & \varepsilon_1^S & 0 & 0 \\ e_{12} & 0 & \varepsilon_2^S & 0 \\ e_{13} & 0 & 0 & \varepsilon_3^S \end{bmatrix} \begin{Bmatrix} S_1 \\ E_1 \\ E_2 \\ E_3 \end{Bmatrix}$$

effective properties

Beams

➤ Variational Principle

$$\int_{t_1}^{t_2} [\delta T - \delta U_1^m + \delta W_1^m + \delta W_1^E] dt = 0$$

- First Kinetic Energy

$$\delta T = \int_V \rho \vec{V} \bullet \delta \vec{V} dV = \int_V \bar{m} [\dot{u}_o \quad \dot{v}_o \quad \dot{w}_o] \begin{bmatrix} \delta \dot{u}_o \\ \delta \dot{v}_o \\ \delta \dot{w}_o \end{bmatrix} dx$$

$$\bar{m} = \int_A \rho dA \quad \text{Effective mass per unit length}$$

$\dot{u} \cong \dot{u}_o$ Ignore rotary inertia

➤ Mechanical energies

$$\delta U_1^m = \int_V T \delta S dV = \int_x \left\{ \int_{A_1} \left\{ c_{11}^E (\varepsilon_o + y\kappa_z + z\kappa_y) - \overline{\vec{e}_t \bullet \vec{E}} \right\} (\delta \varepsilon_o + y\delta\kappa_z + z\delta\kappa_y) dA \right\} dx$$

$$+ \int_{A_1} c_{11} (\varepsilon_o + y\kappa_z + z\kappa_y) (\delta \varepsilon_o + y\delta\kappa_z + z\delta\kappa_y) dA$$

$$\delta U_1^m = \int_x \left\{ [\varepsilon_o \quad \kappa_z \quad \kappa_y] \bar{K} \begin{bmatrix} \delta \varepsilon_o \\ \delta \kappa_z \\ \delta \kappa_y \end{bmatrix} - \bar{\Theta} \begin{bmatrix} \delta \varepsilon_o \\ \delta \kappa_z \\ \delta \kappa_y \end{bmatrix} \right\} dx$$

3x3 1x3

Beams

where, $\bar{K} = \int_A \begin{bmatrix} c_{11} & yc_{11} & zc_{11} \\ yc_{11} & y^2c_{11} & yzc_{11} \\ zc_{11} & yzc_{11} & z^2c_{11} \end{bmatrix} dA$

$$\bar{\Theta} = \begin{bmatrix} P^E & -M_z^E & -M_y^E \end{bmatrix}$$

where, $P^E = \int_A \vec{e}_t \cdot \vec{E} dA = \int_A T_1^E dA$ $-M_z^E = \int_A y T_1^E dA$ $-M_y^E = \int_A z T_1^E dA$

➤ Work terms

$$\delta W_1^m = \int_x \begin{bmatrix} P_x \delta \varepsilon_o & -M_z \delta \kappa_z & -M_y \delta \kappa_y \end{bmatrix} dx$$

$$P_x = \int_A T_1 dA$$

$$-M_y = \int_A z T_1 dA$$

$$-M_z = \int_A y T_1 dA$$

$$\delta W_1^m = \int_x \left\{ P_x \delta u_o + P_y \delta v_o + P_z \delta w_o + m_x 0 + m_y \left(-\frac{dw_o}{dx} \right) + m_z \left(\frac{dv_o}{dx} \right) \right\} dx$$

Beams

- Integrating by parts,

$$\begin{aligned}
 &= \left(\int P_x \right) \delta u_o \Big|_0^l - \int_x \left(\int P_x \right) \left(\frac{d \delta u_o}{dx} \right) dx + \left(\int P_y \right) \delta v_o \Big|_0^l + \int_x \left[\left(- \int P_y \right) + m_z \right] \left(\frac{d \delta v_o}{dx} \right) dx \\
 &\quad + \left(\int P_z \right) \delta w_o \Big|_0^l + \int_x \left[\left(- \int P_z \right) - m_y \right] \left(\frac{d \delta w_o}{dx} \right) dx \\
 &= \left(\int P_x \right) \delta u_o \Big|_0^l - \int_x \left[\int P_x \right] \left(\frac{d \delta u_o}{dx} \right) dx + \underline{\left(\int P_y \right) \delta v_o \Big|_0^l + \left[\int_x \left\{ \left(- \int P_y \right) + m_z \right\} \right] \left(\frac{d \delta v_o}{dx} \right) \Big|_0^l} \\
 &\quad + \int_x \left[\int_x \left\{ \left(- \int P_y \right) + m_z \right\} \left(- \frac{d^2 \delta v_o}{dx^2} \right) \right] dx + \underline{\left(\int P_z \right) \delta w_o \Big|_0^l + \left[\int_x \left\{ \left(- \int P_z \right) - m_y \right\} \right] \left(\frac{d \delta w_o}{dx} \right) \Big|_0^l} \\
 &- M_z \xleftarrow{-M_z} \underline{\left[\int_x \left\{ \left(- \int P_z \right) + m_y \right\} \left(- \frac{d^2 \delta w_o}{dx^2} \right) \right]} \xrightarrow{\delta K_z} -M_y \\
 &\quad + \int_x \left[\int_x \left\{ \left(- \int P_z \right) + m_y \right\} \left(- \frac{d^2 \delta w_o}{dx^2} \right) \right] dx
 \end{aligned}$$

$$M_z'' + m_z' = P_y$$

$$M_y' + m_y'' = P_z$$

$$\delta W_1^m = \int_x \begin{bmatrix} P_x \delta \epsilon_o & -M_z \delta \kappa_z & -M_y \delta \kappa_y \end{bmatrix} dx \quad \text{Stress resultants}$$

Beams

Few options

i) add electrical terms $\delta W_1^E, \delta U_1^E$

ii) just look at actuation

→ Assume Prescribed \vec{E}

→ $\delta E = 0$ ignore electrical terms

→ Actuation Equations only

Assume for now quasi-static $\delta T = 0$

simplified

$$\int_{t_1}^{t_2} [\delta U_1^m - \delta W_1^m] dt = 0$$

substituting

$$\int_{t_1}^{t_2} \int_x \left\{ \begin{bmatrix} \varepsilon_o & \kappa_z & \kappa_y \end{bmatrix} \bar{C} \begin{bmatrix} \delta \varepsilon_o \\ \delta \kappa_z \\ \delta \kappa_y \end{bmatrix} - \begin{bmatrix} P^E & -M_z^E & -M_y^E \end{bmatrix} \begin{bmatrix} \delta \varepsilon_o \\ \delta \kappa_z \\ \delta \kappa_y \end{bmatrix} - \begin{bmatrix} P^m & -M_z^m & -M_y^m \end{bmatrix} \begin{bmatrix} \delta \varepsilon_o \\ \delta \kappa_z \\ \delta \kappa_y \end{bmatrix} \right\} dx dt = 0$$

$$\int_{t_1}^{t_2} \int_x \left\{ \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} \delta \varepsilon_o \\ \delta \kappa_z \\ \delta \kappa_y \end{bmatrix} \right\} dx dt = 0$$