Governing Equations of Fluid Flow and Heat Transfer

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0$$

$$\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}$$

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My}$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_{Mz}$$

τ_{ij} : stress component acts in the *j*-direction on a surface normal to *i*-direction

$$\rho \frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \right] + \mathrm{div}(k \ \mathrm{grad}\ T) + S_E$$

$$\rho \frac{Di}{Dt} = -p \ \mathrm{div}\ \mathbf{u} + \mathrm{div}(k \ \mathrm{grad}\ T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i$$

2.2 Equations of State

- Thermodynamic variables

$$\rho, p, i \text{ and } T$$

  - Assumption of thermodynamic equilibrium

- Equations of the state
  - Relate two state variables to the other variables

$$p = p(\rho, T) \qquad i = i(\rho, T)$$

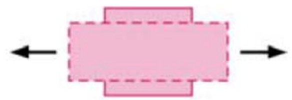
- Compressible fluids
  - EOS provides the linkage between the energy equation and other governing equations.

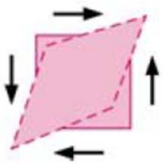
- Incompressible fluids
  - No linkage between the energy equation and the others.
  - The flow field can be solved by considering mass and momentum equations.

2.3 Navier-Stokes Equations for a Newtonian Fluid

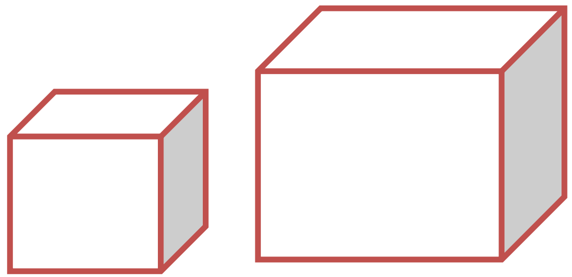
- Viscous stresses τ_{ij} in momentum and energy equations
  - Viscous stresses can be expressed as functions of the local deformation rate (or strain rate).
  - In 3D flows the local rate of deformation is composed of
    - the linear deformation rate
    - the volumetric deformation rate.
  - All gases and many liquids are isotropic.

- The rate of linear deformation of a fluid element
  - Nine components in 3D
  - Linear elongating deformation
  - Shearing linear deformation components





- The rate of volume deformation of a fluid element



2.3 Navier-Stokes Equations for a Newtonian Fluid

❖ Viscous stresses τ_{ij} in momentum and energy equations

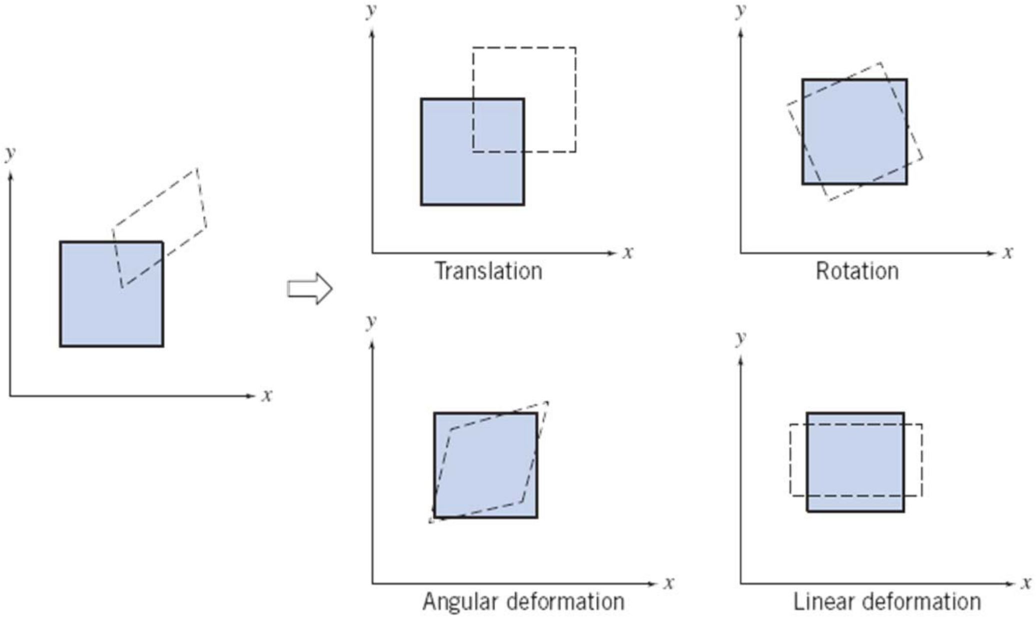


Fig. 5.5 Pictorial representation of the components of fluid motion.

All gases and many liquids are isotropic.

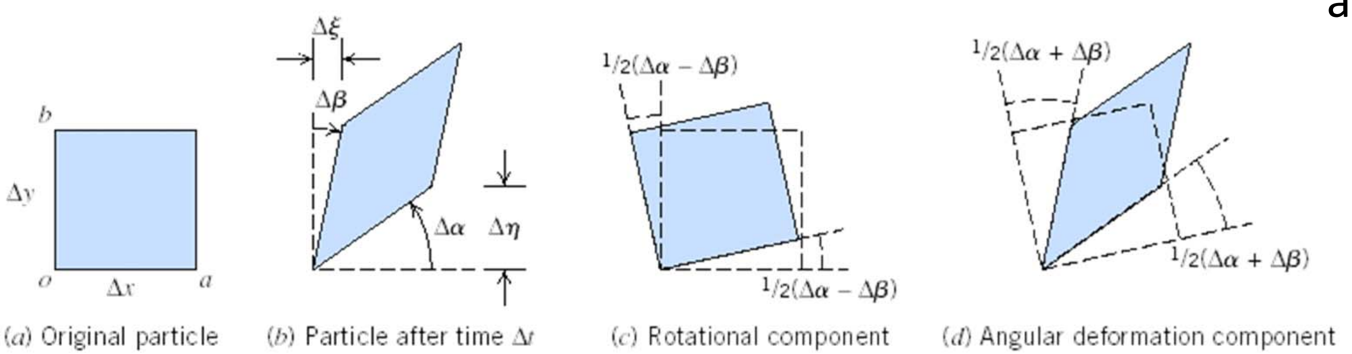
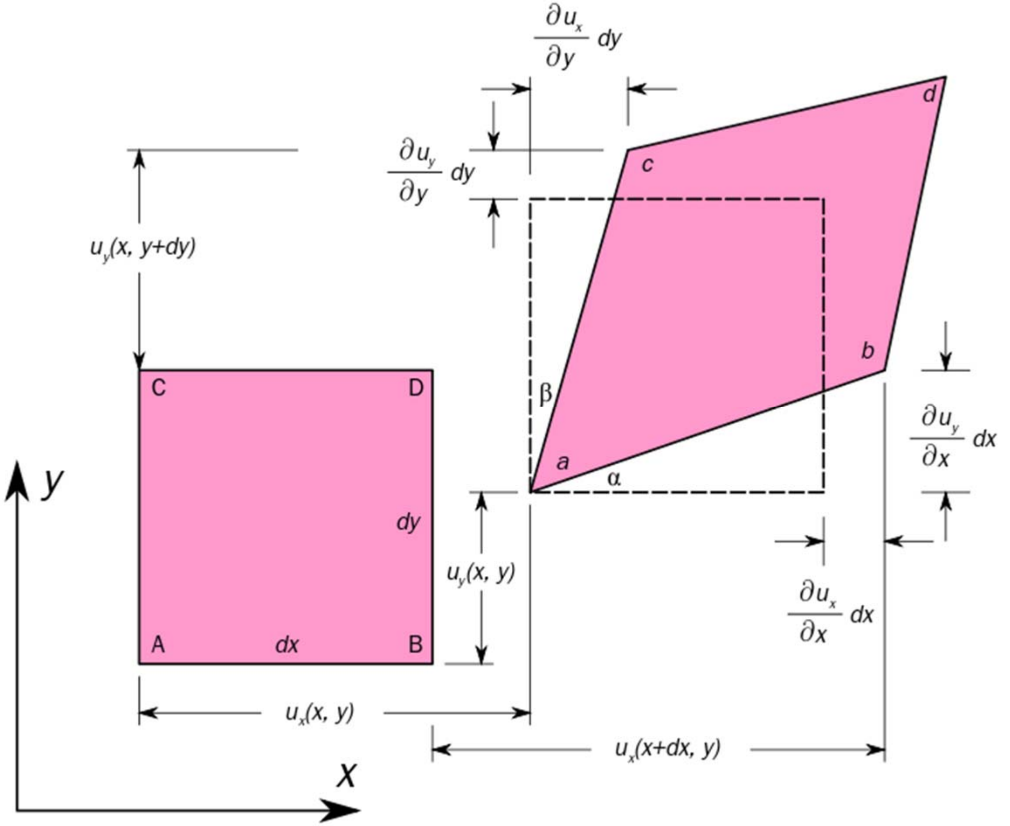


Fig. 5.7 Rotation and angular deformation of perpendicular line segments in a two-dimensional flow.

- Viscous stresses τ_{ij} in momentum and energy equations

$$\tan\alpha = \frac{\dfrac{\partial u_y}{\partial x}dx}{dx + \dfrac{\partial u_x}{\partial x}dx} = \frac{\dfrac{\partial u_y}{\partial x}}{1 + \dfrac{\partial u_x}{\partial x}}$$

$$\tan\beta = \frac{\dfrac{\partial u_x}{\partial y}dy}{dy + \dfrac{\partial u_y}{\partial y}dy} = \frac{\dfrac{\partial u_x}{\partial y}}{1 + \dfrac{\partial u_y}{\partial y}}$$



$$\tan\alpha \approx \frac{\partial u_y}{\partial x} \approx \alpha \qquad \tan\beta \approx \frac{\partial u_x}{\partial y} \approx \beta$$

$$\gamma_{xy} = \alpha + \beta$$

The rate at which
two sides close toward each other

$$s_{xy} = \frac{1}{2}\gamma_{xy}$$

2.3 Navier-Stokes Equations for a Newtonian Fluid

- Newtonian fluid
  - Viscous stresses are proportional to the rates of deformation.
  - Two constants of proportionality
    - Dynamic viscosity (μ): to relate stresses to linear deformations
    - Second viscosity (λ): to relate stresses to volumetric deformation

$$e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div}\,\mathbf{u}$$

- Viscous stress components

$$s_{xx} = \frac{\partial u}{\partial x} \quad s_{yy} = \frac{\partial v}{\partial y} \quad s_{zz} = \frac{\partial w}{\partial z} \qquad s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\,\text{div}\,\mathbf{u} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\,\text{div}\,\mathbf{u} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\,\text{div}\,\mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

  - Second viscosity

2.3 Navier-Stokes Equations for a Newtonian Fluid

- Momentum equations

$$\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}$$

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + S_{My}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu \frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}\right] + S_{Mz}$$

- Rearrangement

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda \operatorname{div} \mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}(\lambda \operatorname{div} \mathbf{u})\right]$$

- N.-S. equations can be written as follows with modified source terms; $S_M = S_M + [s_M]$

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\operatorname{div}(\mu \operatorname{grad} u)+S_{Mx}$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\operatorname{div}(\mu \operatorname{grad} v)+S_{My}$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\operatorname{div}(\mu \operatorname{grad} w)+S_{Mz}$$

- For incompressible fluids with constant μ

$$[s_{Mx}] = \left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}(\lambda \text{ div } \mathbf{u})\right]$$

$$= \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right] = \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)\right]$$

$$= \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = 0$$

Internal energy equation

$$\rho\frac{Di}{Dt} = -p \text{ div } \mathbf{u} + \text{div}(k \text{ grad } T) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$
$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z}$$
$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z} + S_i$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda \text{ div } \mathbf{u} \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda \text{ div } \mathbf{u} \quad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda \text{ div } \mathbf{u}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

- Dissipation function Φ
  - Always positive
  - Source of internal energy due to deformation work on the fluid particle.
  - Mechanical energy is converted into internal energy or heat.

$$\Phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} + \lambda(\text{div } \mathbf{u})^2$$

2.4 Conservative form of the governing equations of fluid flow

Mass	$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{u}) = 0$
x-momentum	$\frac{\partial(\rho u)}{\partial t} + div(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + div(\mu\, grad\, u) + S_{Mx}$
y-momentum	$\frac{\partial(\rho v)}{\partial t} + div(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + div(\mu\, grad\, v) + S_{My}$
z-momentum	$\frac{\partial(\rho w)}{\partial t} + div(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + div(\mu\, grad\, w) + S_{Mz}$
Internal energy	$\frac{\partial(\rho i)}{\partial t} + div(\rho i \mathbf{u}) = -p\, div\, \mathbf{u} + div(k\, grad\, T) + \Phi + S_i$
+ EOS	u, v, w, p, i, ρ, T

This system is mathematically closed!

2.5 Differential and integral forms of the general transport equations

- General form of fluid flow equations

$$\frac{\partial(\rho\phi)}{\partial t} + \text{div}(\rho\phi\mathbf{u}) = \text{div}(\Gamma \text{ grad } \phi) + S_\phi$$

Rate of increase of ϕ of fluid element	+ Net rate of flow of ϕ out of fluid element	= Rate of increase of ϕ due to diffusion	+ Rate of increase of ϕ due to sources
Temporal term	Convective term	Diffusive term	Source term

- By setting ϕ, Γ, S_ϕ,

$$\phi = 1, u, v, w, i$$
$$\Gamma = 0, \mu, k$$
$$S_\phi = 0, (S_{Mx} - \partial p / \partial x), \ldots,$$

$$\frac{\partial \rho}{\partial t} + div(\rho\mathbf{u}) = 0$$

Differential form

2.5 Differential and integral forms of the general transport equations

- Starting point for computational procedures in FVM
  - Integration of the general form over a 3D control volume (CV)

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t} dV + \int_{CV} \mathrm{div}(\rho\phi\mathbf{u}) dV = \int_{CV} \mathrm{div}(\Gamma \ \mathrm{grad} \ \phi) dV + \int_{CV} S_\phi dV$$

  - Gauss's divergence theorem
    - Volume integral ⇔ surface integral

$$\int_{CV} \mathrm{div}(\mathbf{a}) dV = \int_{A} \mathbf{n} \,.\, \mathbf{a} dA$$

    - $\mathbf{n}\cdot\mathbf{a}$**:** component of vector $\mathbf{a}$ in the direction of the vector $\mathbf{n}$ normal to surface element dA

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_{A} \mathbf{n} \,.\, (\rho\phi\mathbf{u}) dA = \int_{A} \mathbf{n} \,.\, (\Gamma \ \mathrm{grad} \ \phi) dA + \int_{CV} S_\phi dV$$

A special case of the Reynold' transport theorem

2.5 Differential and integral forms of the general transport equations

- Starting point for computational procedures in FVM

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) + \int_A \mathbf{n} \cdot (\rho\phi\mathbf{u}) dA = \int_A \mathbf{n} \cdot (\Gamma \operatorname{grad} \phi) dA + \int_{CV} S_\phi dV$$

- In time-dependent problems
  - Integrate with respect to time t over a small interval Δt

$$\int_{\Delta t} \frac{\partial}{\partial t}\left(\int_{CV} \rho\phi dV\right) dt + \int_{\Delta t} \int_A \mathbf{n} \cdot (\rho\phi\mathbf{u}) dA dt = \int_{\Delta t} \int_A \mathbf{n} \cdot (\Gamma \operatorname{grad} \phi) dA dt + \int_{\Delta t} \int_{CV} S_\phi dV dt$$

2.10 Auxiliary conditions for viscous fluid flow equations

- Initial and boundary conditions for compressible viscous flow
  - Initial conditions for unsteady flows
    - Everywhere in the solution region, ρ, $\mathbf{u}$ and T must be given at time t=0.
  - Boundary conditions
    - On solid Walls
      - No-slip condition: $\mathbf{u} = \mathbf{u}_w$
      - Fixed temperature $T = T_w$
      - Fixed heat flux $k\dfrac{\partial T}{\partial n} = -q_w$
    - On fluid boundaries
      - Inlet $\rho, \mathbf{u}$ and T
      - Outlet

$$-p + \mu\frac{\partial u_n}{\partial n} = F_n \qquad \mu\frac{\partial u_t}{\partial n} = F_t$$

    - Outflow boundaries
      - Far from solid objects in an external flow
      - Commonly, no change in any of the velocity components in the direction across the boundary
      - Open boundary

$$-p = F_n \qquad 0 = F_t$$

2.10 Auxiliary conditions for viscous fluid flow equations

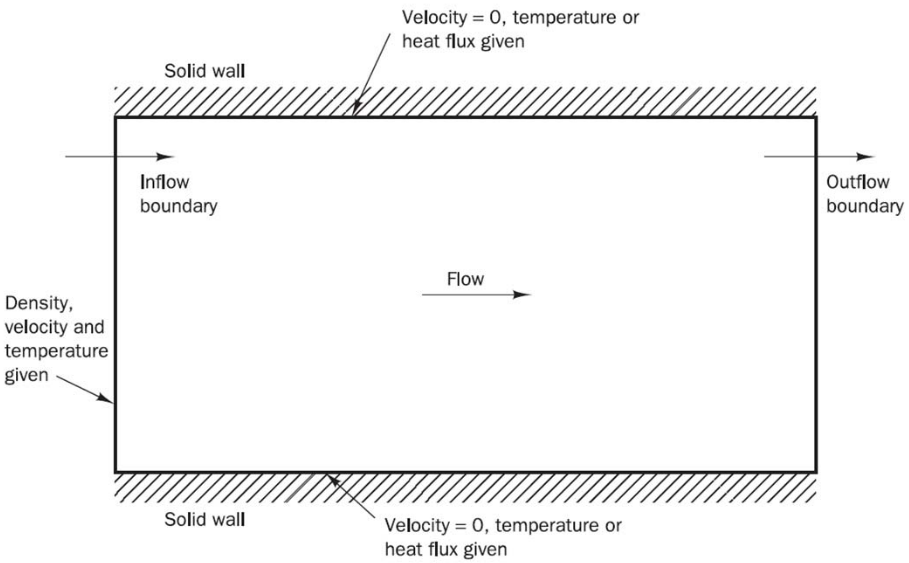
❖ Initial and boundary conditions for compressible viscous flow

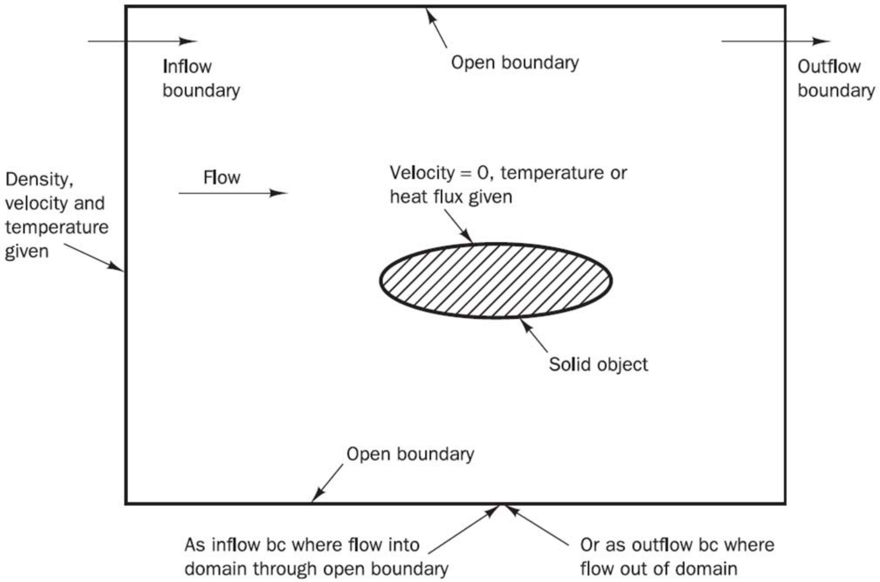
- Symmetry boundary condition

$$\partial\phi/\partial n = 0$$

- Cyclic (periodic boundary condition)

$$\phi_1 = \phi_2$$





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