Seoul National University
Dept. of Civil and Environmental Engineering

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457.212 Statistics for Civil & Environmental Engineers

In-Class Material: Class 11

Useful Distribution Models – Part I (A&T: 3.2)

"Distribution models" are useful because ...

- The probability function is the result of an underlying physical process and can be derived on the basis of certain physically reasonable assumptions.
- The function is the result of some limiting process (e.g.).
- It is widely known, and the necessary probability and statistical information (including probability tables) are widely available (e.g. probability table of)

1. **Normal** distribution

- Best known and most widely used. Also known as ________________ distribution.
- According to __________________________, the sum of random variables converges to a normal random variable as the number of the variables increases, no matter what distributions the variables are subjected to.
- Completely defined by the _______________ and the _____________________ of the random variable.

(a) PDF: $X \sim N(\mu, \sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right], \ -\infty < x < \infty$$

(b) CDF: no closed-form expression available

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx, \ -\infty < x < \infty$$

(c) Parameters: μ, σ

- μ: __________ of the random variable, i.e. $\mu = \mu_X \equiv \mathrm{E}[X]$
- σ: ______________________ of the random variable, i.e. $\sigma = \sigma_X \equiv \left\{ \mathrm{E}[(X-\mu_X)^2] \right\}^{0.5}$

(d) Shape of the PDF plots

- Symmetric around $x =$
- A change in μ_X ________________ the PDF horizontally by the same amount.
- The larger the value of σ_X gets, the more _______________ the PDF becomes around the central axis.

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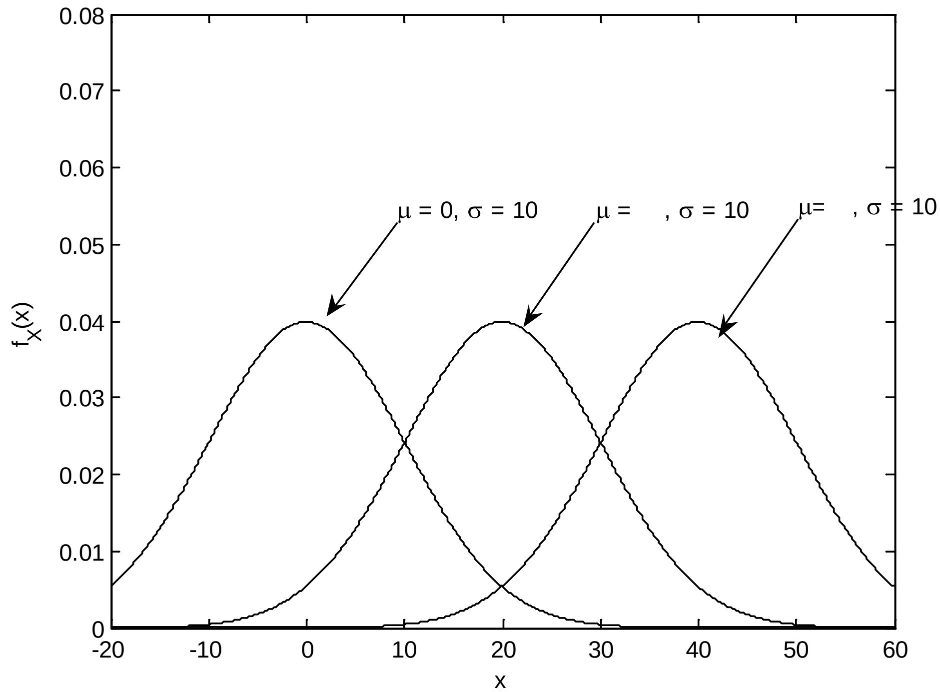

Figure 1. PDF's of normal random variables with different values of μ

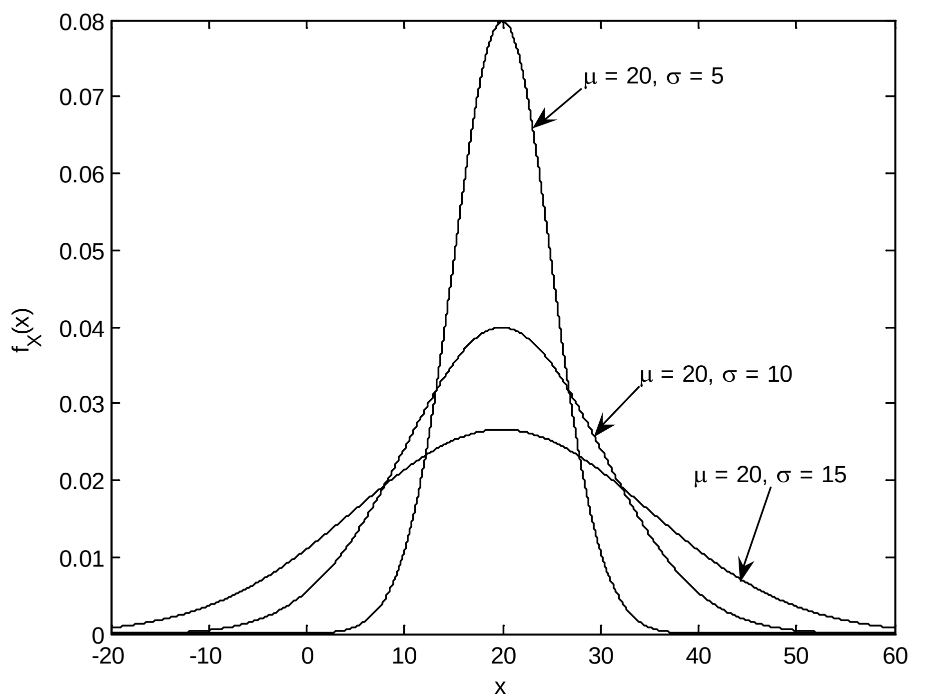

Figure 2. PDF's of normal random variables with different values of σ

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1a. **Standard** normal distribution

- A special case of the normal distribution: $\mu_X =$ ________, $\sigma_X =$ ________.
- The CDF of the standard normal distribution can be used for computing the CDF of any general normal random variable.

(a) PDF: $U \sim N(\quad,\quad)$

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right), \ -\infty < u < \infty$$

(b) CDF:

$$\Phi(u) = \int_{-\infty}^{u} \varphi(u)du, \quad -\infty < u < \infty$$

→ no closed-form expression available, but the table of the standard normal CDF $\Phi(\cdot)$ can be found in books or computer software (e.g. See Appendix A of A&T)

(c) Inverse CDF of standard normal distribution: $\Phi^{-1}(\cdot)$

$$\Phi(u_p) = p \quad \Leftrightarrow \quad u_p = \Phi^{-1}(p)$$

(d) Symmetry around $u =$ ____ :

$$\Phi(-u) = 1 - \Phi(u)$$
$$u_{1-p} = -u_p$$

→ The table of the standard normal CDF is often provided for positive *u* values only, but using the symmetry one can find the CDF for negative values as well.

(e) One can compute the CDF of a general normal random variable $X \sim N(\mu, \sigma)$ by use of the CDF of the standard normal random variable $U \sim N(0,1)$ as follows.

$$\begin{aligned} F_X(a) &= P(X \le a) \\ &= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \\ &= \int_{-\infty}^{\left(\frac{a-\mu}{\sigma}\right)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}u^2\right) \sigma du \\ &= \Phi\left(\frac{\quad}{\quad}\right) \end{aligned}$$

Hence, $P(a < X \le b) = F_X(\quad) - F_X(\quad) = \Phi\left(\frac{\quad}{\quad}\right) - \Phi\left(\frac{\quad}{\quad}\right)$

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Example 1: Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $u = 1.84$

(b) between $u = -1.97$ and $u = 0.86$

Example 2: The drainage demand during a storm (in mgd: million gallons/day): $X \sim N(1.2, 0.4)$. The maximum drain capacity is 1.5 mgd.

(a) Probability of flooding?

(b) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(c) The 90-percentile drainage demand?

2. **Lognormal** distribution

- Closely related to the ________ distribution.
- Defined for __________ values only.

(a) PDF: $X \sim LN(\lambda, \zeta)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right], \quad 0 < x < \infty$$

(b) CDF:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx\,, \quad 0 < x < \infty$$

→ no closed-form expression available, but can be computed by use of the table of the standard normal CDF $\Phi(\cdot)$ (as shown below)

(c) Parameters: λ, ζ

- λ: mean of ______________, i.e. $\lambda = \lambda_X \equiv \mathrm{E}[\ln X]$
- ζ: standard deviation of______________, i.e. $\zeta^2 = \zeta_X^2 = \sigma_{\ln X}^2$

(d) Shape of the PDF plots

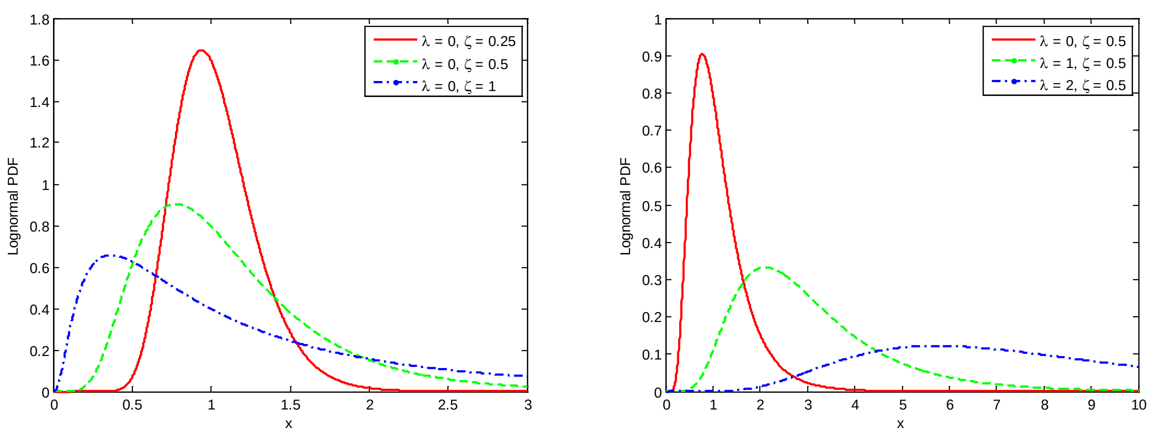


Figure 3. PDF's of lognormal random variables.

(e) Relationship between normal and lognormal distribution:

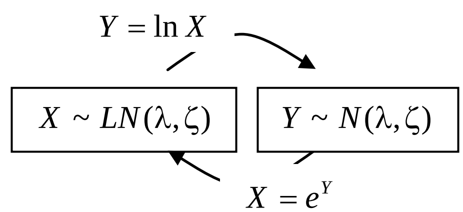
"The logarithm of a ___________ random variable is a __________ random variable."

$$X \sim LN(\lambda, \zeta) \Rightarrow \ln X \sim N(\lambda, \zeta)$$

(f) Can obtain the CDF of lognormal $X \sim LN(\lambda, \zeta)$ from the CDF of standard normal:

$$\begin{aligned} F_X(a) &= P(X \le a) \\ &= P(\ln X \le \ln a) \qquad \text{Since } \ln X \sim N(\lambda, \zeta), \\ &= \Phi\left(\frac{\ln a - \lambda}{\zeta}\right) \end{aligned}$$

(g) "The exponential function of a _____________ random variable is a ______________ random variable."



(h) $(\lambda, \zeta) \rightarrow (\mu, \delta)$: Find the mean and c.o.v. from the distribution parameters

$$\mu = \mathrm{E}[X] = \exp(\lambda + 0.5\zeta^2)$$
$$\delta = \sigma/\mu = \sqrt{\exp(\zeta^2) - 1} \quad (\cong \zeta \text{ for } \zeta << 1)$$

(i) $(\mu, \delta) \rightarrow (\lambda, \zeta)$: Find the distribution parameters from the mean and c.o.v.

$$\zeta = \sqrt{\ln(1 + \delta^2)} \quad (\cong \delta \text{ for } \delta << 1)$$
$$\lambda = \ln\mu - 0.5\ln(1 + \delta^2)$$

(j) $(x_{0.5}) \leftrightarrow (\lambda)$: Relationship between the median and λ

$$\lambda = \ln x_{0.5}, \quad x_{0.5} = e^{\lambda}$$

(k) $(\mu, \delta) \rightarrow (x_{0.5})$: Find the median from the mean and c.o.v.

$$x_{0.5} = \frac{\mu}{\sqrt{1 + \delta^2}}$$

Note: $x_{0.5} < \mu$ for the lognormal distribution.

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Example 3: The drainage demand during a storm (in mgd: million gallons/day) is assumed to follow the lognormal distribution with the same mean and standard deviation as Example 1 (mean 1.2, standard deviation 0.4). The maximum drain capacity is 1.5 mgd.

(a) Distribution parameters, i.e. λ and ζ?

(b) Probability of the flooding?

(c) Probability that the drainage demand during a storm will be between 1.0 and 1.6 mgd?

(d) The 90-percentile drainage demand?

Example 4: Consider a bridge whose uncertain capacity against "complete damage" limit-state caused by earthquake events is defined in terms of peak ground acceleration (PGA; unit: g) that the bridge can sustain. Suppose the median of the capacity is 1.03g and the coefficient of variation is 0.50. It is assumed that the capacity follows a lognormal distribution.

(a) Distribution parameters of the lognormal distribution, i.e. λ and ζ?

(b) The mean and standard deviation of the uncertain capacity, i.e. μ and σ?

(c) Suppose the peak ground acceleration from an earthquake event is 0.5g. What is the probability that the structure will exceed "complete damage" limit state?

x	PHI(x)	x	PHI(x)	x	PHI(x)	x	PHI(x)	x	PHI(x)
0.00	0.5	0.90	0.81593987	1.80	0.96406968	2.70	0.99653303	3.60	0.999840891
0.01	0.50398936	0.91	0.81858875	1.81	0.96485211	2.71	0.99663584	3.61	0.999846901
0.02	0.50797831	0.92	0.82121362	1.82	0.9656205	2.72	0.9967359	3.62	0.999852698
0.03	0.51196647	0.93	0.82381446	1.83	0.96637503	2.73	0.99683328	3.63	0.999858289
0.04	0.51595344	0.94	0.82639122	1.84	0.96711588	2.74	0.99692804	3.64	0.999863681
0.05	0.51993881	0.95	0.82894387	1.85	0.96784323	2.75	0.99702024	3.65	0.99986888
0.06	0.52392218	0.96	0.83147239	1.86	0.96855724	2.76	0.99710993	3.66	0.999873892
0.07	0.52790317	0.97	0.83397675	1.87	0.96925809	2.77	0.99719719	3.67	0.999878725
0.08	0.53188137	0.98	0.83645694	1.88	0.96994596	2.78	0.99728206	3.68	0.999883383
0.09	0.53585639	0.99	0.83891294	1.89	0.97062102	2.79	0.9973646	3.69	0.999887873
0.10	0.53982784	1.00	0.84134475	1.90	0.97128344	2.80	0.99744487	3.70	0.9998922
0.11	0.54379531	1.01	0.84375235	1.91	0.97193339	2.81	0.99752293	3.71	0.99989637
0.12	0.54775843	1.02	0.84613577	1.92	0.97257105	2.82	0.99759882	3.72	0.999900389
0.13	0.55171679	1.03	0.848495	1.93	0.97319658	2.83	0.9976726	3.73	0.99990426
0.14	0.55567	1.04	0.85083005	1.94	0.97381016	2.84	0.99774432	3.74	0.99990799
0.15	0.55961769	1.05	0.85314094	1.95	0.97441194	2.85	0.99781404	3.75	0.999911583
0.16	0.56355946	1.06	0.8554277	1.96	0.9750021	2.86	0.99788179	3.76	0.999915043
0.17	0.56749493	1.07	0.85769035	1.97	0.97558081	2.87	0.99794764	3.77	0.999918376
0.18	0.57142372	1.08	0.85992891	1.98	0.97614824	2.88	0.99801162	3.78	0.999921586
0.19	0.57534543	1.09	0.86214343	1.99	0.97670453	2.89	0.99807379	3.79	0.999924676
0.20	0.57925971	1.10	0.86433394	2.00	0.97724987	2.90	0.99813419	3.80	0.999927652
0.21	0.58316616	1.11	0.86650049	2.01	0.97778441	2.91	0.99819286	3.81	0.999930517
0.22	0.58706442	1.12	0.86864312	2.02	0.97830831	2.92	0.99824984	3.82	0.999933274
0.23	0.59095412	1.13	0.87076189	2.03	0.97882173	2.93	0.99830519	3.83	0.999935928
0.24	0.59483487	1.14	0.87285685	2.04	0.97932484	2.94	0.99835894	3.84	0.999938483
0.25	0.59870633	1.15	0.87492806	2.05	0.97981778	2.95	0.99841113	3.85	0.999940941
0.26	0.60256811	1.16	0.8769756	2.06	0.98030073	2.96	0.9984618	3.86	0.999943306
0.27	0.60641987	1.17	0.87899952	2.07	0.98077383	2.97	0.998511	3.87	0.999945582
0.28	0.61026125	1.18	0.88099989	2.08	0.98123723	2.98	0.99855876	3.88	0.999947772
0.29	0.61409188	1.19	0.8829768	2.09	0.9816911	2.99	0.99860511	3.89	0.999949878
0.30	0.61791142	1.20	0.88493033	2.10	0.98213558	3.00	0.9986501	3.90	0.999951904
0.31	0.62171952	1.21	0.88686055	2.11	0.98257082	3.01	0.99869376	3.91	0.999953852
0.32	0.62551583	1.22	0.88876756	2.12	0.98299698	3.02	0.99873613	3.92	0.999955726
0.33	0.62930002	1.23	0.89065145	2.13	0.98341419	3.03	0.99877723	3.93	0.999957527
0.34	0.63307174	1.24	0.8925123	2.14	0.98382262	3.04	0.99881711	3.94	0.999959259
0.35	0.63683065	1.25	0.89435023	2.15	0.98422239	3.05	0.99885579	3.95	0.999960924
0.36	0.64057643	1.26	0.89616532	2.16	0.98461367	3.06	0.99889332	3.96	0.999962525
0.37	0.64430875	1.27	0.89795768	2.17	0.98499658	3.07	0.99892971	3.97	0.999964064
0.38	0.64802729	1.28	0.89972743	2.18	0.98537127	3.08	0.998965	3.98	0.999965542
0.39	0.65173173	1.29	0.90147467	2.19	0.98573788	3.09	0.99899922	3.99	0.999966963
0.40	0.65542174	1.30	0.90319952	2.20	0.98609655	3.10	0.9990324	**4.00***	3.16712E-05
0.41	0.65909703	1.31	0.90490208	2.21	0.98644742	3.11	0.99906456	**4.05**	2.56088E-05
0.42	0.66275727	1.32	0.90658249	2.22	0.98679062	3.12	0.99909574	**4.10**	2.06575E-05
0.43	0.66640218	1.33	0.90824086	2.23	0.98712628	3.13	0.99912597	**4.15**	1.66238E-05
0.44	0.67003145	1.34	0.90987733	2.24	0.98745454	3.14	0.99915526	**4.20**	1.33457E-05
0.45	0.67364478	1.35	0.91149201	2.25	0.98777553	3.15	0.99918365	**4.25**	1.06885E-05
0.46	0.67724189	1.36	0.91308504	2.26	0.98808937	3.16	0.99921115	**4.30**	8.53991E-06
0.47	0.68082249	1.37	0.91465655	2.27	0.98839621	3.17	0.99923781	**4.35**	6.80688E-06
0.48	0.6843863	1.38	0.91620668	2.28	0.98869616	3.18	0.99926362	**4.40**	5.41254E-06
0.49	0.68793305	1.39	0.91773556	2.29	0.98898934	3.19	0.99928864	**4.45**	4.29351E-06
0.50	0.69146246	1.40	0.91924334	2.30	0.98927589	3.20	0.99931286	**4.50**	3.39767E-06
0.51	0.69497427	1.41	0.92073016	2.31	0.98955592	3.21	0.99933633	**4.55**	2.68230E-06
0.52	0.69846821	1.42	0.92219616	2.32	0.98982956	3.22	0.99935905	**4.60**	2.11245E-06
0.53	0.70194403	1.43	0.92364149	2.33	0.99009692	3.23	0.99938105	**4.65**	1.65968E-06
0.54	0.70540148	1.44	0.9250663	2.34	0.99035813	3.24	0.99940235	**4.70**	1.30081E-06
0.55	0.70884031	1.45	0.92647074	2.35	0.99061329	3.25	0.99942297	**4.75**	1.01708E-06
0.56	0.71226028	1.46	0.92785496	2.36	0.99086253	3.26	0.99944294	**4.80**	7.93328E-07
0.57	0.71566115	1.47	0.92921912	2.37	0.99110596	3.27	0.99946226	**4.85**	6.17307E-07
0.58	0.71904269	1.48	0.93056338	2.38	0.99134368	3.28	0.99948096	**4.90**	4.79183E-07
0.59	0.72240468	1.49	0.93188788	2.39	0.99157581	3.29	0.99949906	**4.95**	3.71068E-07
0.60	0.72574688	1.50	0.9331928	2.40	0.99180246	3.30	0.99951658	**5.00**	2.86652E-07
0.61	0.7290691	1.51	0.93447829	2.41	0.99202374	3.31	0.99953352	**5.10**	1.69827E-07
0.62	0.73237111	1.52	0.93574451	2.42	0.99223975	3.32	0.99954991	**5.20**	9.96443E-08
0.63	0.73565271	1.53	0.93699164	2.43	0.99245059	3.33	0.99956577	**5.30**	5.79013E-08
0.64	0.7389137	1.54	0.93821982	2.44	0.99265637	3.34	0.99958111	**5.40**	3.33204E-08
0.65	0.74215389	1.55	0.93942924	2.45	0.99285719	3.35	0.99959594	**5.50**	1.89896E-08
0.66	0.74537309	1.56	0.94062006	2.46	0.99305315	3.36	0.99961029	**5.60**	1.07176E-08
0.67	0.7485711	1.57	0.94179244	2.47	0.99324435	3.37	0.99962416	**5.70**	5.99037E-09
0.68	0.75174777	1.58	0.94294657	2.48	0.99343088	3.38	0.99963757	**5.80**	3.31575E-09
0.69	0.75490291	1.59	0.9440826	2.49	0.99361285	3.39	0.99965054	**5.90**	1.81751E-09
0.70	0.75803635	1.60	0.94520071	2.50	0.99379033	3.40	0.99966307	**6.00**	9.86588E-10
0.71	0.76114793	1.61	0.94630107	2.51	0.99396344	3.41	0.99967519	**6.10**	5.30342E-10
0.72	0.7642375	1.62	0.94738386	2.52	0.99413226	3.42	0.99968689	**6.20**	2.82316E-10
0.73	0.76730491	1.63	0.94844925	2.53	0.99429687	3.43	0.99969821	**6.30**	1.48823E-10
0.74	0.77035	1.64	0.94949742	2.54	0.99445738	3.44	0.99970914	**6.40**	7.76885E-11
0.75	0.77337265	1.65	0.95052853	2.55	0.99461385	3.45	0.99971971	**6.50**	4.01600E-11
0.76	0.77637271	1.66	0.95154277	2.56	0.99476639	3.46	0.99972991	**6.60**	2.05579E-11
0.77	0.77935005	1.67	0.95254032	2.57	0.99491507	3.47	0.99973977	**6.70**	1.04210E-11
0.78	0.78230456	1.68	0.95352134	2.58	0.99505998	3.48	0.99974929	**6.80**	5.23093E-12
0.79	0.78523612	1.69	0.95448602	2.59	0.9952012	3.49	0.99975849	**6.90**	2.60014E-12
0.80	0.7881446	1.70	0.95543454	2.60	0.99533881	3.50	0.99976737	**7.00**	1.27987E-12
0.81	0.79102991	1.71	0.95636706	2.61	0.99547289	3.51	0.99977595	**7.10**	6.23834E-13
0.82	0.79389195	1.72	0.95728378	2.62	0.99560351	3.52	0.99978423	**7.20**	3.01092E-13
0.83	0.79673061	1.73	0.95818486	2.63	0.99573076	3.53	0.99979222	**7.30**	1.43885E-13
0.84	0.79954581	1.74	0.95907049	2.64	0.9958547	3.54	0.99979994	**7.40**	6.80567E-14
0.85	0.80233746	1.75	0.95994084	2.65	0.99597541	3.55	0.99980738	**7.50**	3.18634E-14
0.86	0.80510548	1.76	0.9607961	2.66	0.99609297	3.56	0.99981457	**7.60**	1.47660E-14
0.87	0.8078498	1.77	0.96163643	2.67	0.99620744	3.57	0.99982151	**7.70**	6.77236E-15
0.88	0.81057035	1.78	0.96246202	2.68	0.99631889	3.58	0.9998282	**7.80**	3.10862E-15
0.89	0.81326706	1.79	0.96327304	2.69	0.9964274	3.59	0.99983466	**7.90**	0.00000E+00

* Note: For x>=4.0, 1-PHI(x) is given instead.

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Probability Distribution Models in Matlab® Statistics Toolbox

Full Name	Short	Parameters	Probability Density/Mass Function	Mean	Variance
Binomial	*bino*	$0<p<1$ n integer	$\binom{n}{x} p^x (1-p)^{(n-x)}, \quad x=0,1,\ldots,n$	np	$np(1-p)$
Geometric	*geo*	$0<p<1$	$p(1-p)^x, \quad x=0.1,2,\ldots$	$(1-p)/p$	$(1-p)/p^2$
Hypergeometric	*hyge*	$0<K,N\le M$ K,N,M integers	$\binom{K}{x}\binom{M-K}{N-x}\binom{M}{N}^{-1}, \quad K+N-M\le x\le K$	$\frac{NK}{M}$	$N\frac{K}{M}\frac{M-K}{M}\frac{M-N}{M-1}$
Negative Binomial	*nbin*	$0<p<1$ r integer	$\binom{r+x-1}{x} p^r (1-p)^x, \quad x=0,1,...$	$r(1-p)/p$	$r(1-p)/p^2$
Poisson	*poiss*	$0<\lambda$	$\frac{\lambda^x}{x!}e^{-\lambda}, \quad x=0,1,...$	λ	λ
Beta	*beta*	$0<a,b$	$B(a,b)^{-1}x^{a-1}(1-x)^{b-1}, \quad 0\le x\le 1$	$a/(a+b)$	$ab/(a+b+1)/(a+b)^2$
Chisquare	*chi2*	$0<\nu$	$x^{(\nu-2)/2}e^{-x/2}2^{-\nu/2}\Gamma(\nu/2)^{-1}, \quad 0<x$	ν	2ν
Exponential	*exp*	$0<\mu$	$\mu^{-1}e^{-x/\mu}, \quad 0<x$	μ	μ^2
F	*f*	$0<\nu_1,\nu_2$	$\frac{\Gamma[(\nu_1+\nu_2)/2](\nu_1/\nu_2)^{\nu_1/2}x^{\nu_1/2-1}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)[1+(\nu_1/\nu_2)x]^{(\nu_1+\nu_2)/2}}, \quad 0<x$	$v_2/(v_2-2)$	$\frac{2v_2^2(v_1+v_2-2)}{v_1(v_2-2)^2(v_2-4)}$
Gamma	*gam*	$0<a,b$	$b^{-a}\Gamma(a)^{-1}x^{a-1}e^{-x/b}, \quad 0<x$	ab	ab^2
Lognormal	*logn*	$\lambda, 0<\zeta$	$x^{-1}\xi^{-1}(2\pi)^{-1/2}\exp[-(\ln x-\lambda)^2/2\xi^2], \quad 0<x$	$e^{(\lambda+0.5\xi^2)}$	$e^{(2\lambda+2\xi^2)}-e^{(2\lambda+\xi^2)}$
Normal	*norm*	$\mu, 0<\sigma$	$\sigma^{-1}(2\pi)^{-1/2}\exp[-(x-\mu)^2/2\sigma^2]$	μ	σ^2
Rayleigh	*rayl*	$0<b$	$xb^{-2}\exp(-x^2/2b^2), \quad 0<x$	$b\sqrt{\pi/2}$	$(4-\pi)b^2/2$
T	*t*	$0<\nu$	$(v\pi)^{-1/2}\Gamma((v+1)/2)\Gamma(v/2)^{-1}(1+x^2/v)^{-(v+1)/2}$	0	$v/(v-2)$
Uniform	*unif*	$a<b$	$(b-a)^{-1}, \quad a\le x\le b$	$(a+b)/2$	$(b-a)^2/12$
Weibull	*weib*	$0<a,b$	$abx^{b-1}e^{-ax^b}, \quad 0<x$	$a^{-1/b}\Gamma(1+b^{-1})$	$a^{-2/b}[\Gamma(1+2b^{-1})-\Gamma^2(1+b^{-1})]$

Use *shortname***pdf**( ) to compute the probability density/mass function; *shortname***cdf**( ) to compute cumulative distribution function; *shortname***fit**( ) to estimate parameters from data; *shortname***rnd**( ) to generate random numbers; *shortname***stat**( ) to compute mean and variance for specified parameters; and *shortname***inv**( ) to compute the inverse cumulative probability. Use Matlab® help to learn more about these commands.