

Kinematics of Fluid Motion





Contents

- 3.1 Steady and Unsteady Flow, Streamlines, and Streamtubes
- 3.2 One-, Two-, and Three-Dimensional Flows
- 3.3 Velocity and Acceleration
- 3.4 Circulation, Vorticity, and Rotation

Objectives

- treat kinematics of idealized fluid motion along streamlines and flowfields
- learn how to describe motion in terms of displacement, velocities, and accelerations without regard to the forces that cause the motion
- distinguish between rotational and irrotational regions of flow based on the flow property vorticity





2/88



1) Lagrangian views

Joseph Louis Lagrange (1736-1813), Italian mathematician

~ Each fluid particle is labeled by its spatial coordinates at some initial time.

- ~ Then fluid variables (path, density, velocity, and others) of each individual particle are traced as time passes.
- ~ used in the dynamic analyses of solid particles











- Difficulties of Lagrangian description for fluid motion
- We cannot easily define and identify particles of fluid as they move around.
- A fluid is a continuum, so interactions between parcels of fluid are not easy to describe as are interactions between distinct objects in solid mechanics.
- The fluid parcels continually deform as they move in the flow.





- Path line
 - ~ the position is plotted as a function of time = trajectory of the particle
- \rightarrow path line
 - ~ since path line is <u>tangent to the instantaneous velocity</u> at each point along the path, changes in the particle location over an infinitesimally small time are given by

$$dx = udt; dy = vdt; dz = wdt$$

This means that

 $u = \frac{dx}{dt}; v = \frac{dy}{dt} w = \frac{dz}{dt}$ (3.1a) $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dt}{1}$ (3.1b)



The acceleration components are



- ~ attention is focused on particular points in the space filled by the fluid
- ~ motion of individual particles is no longer traced
- \rightarrow A finite volume called a <u>control volume</u> (flow domain) is defined, through which fluid flows in and out.
- ~The values and variations of the velocity, density, and other fluid variables are determined as a function of space and time within the control volume.
- \rightarrow flow field











Define velocity field as a vector field variable in Cartesian coordinates

 $\vec{v} = \vec{v}(x, y, z, t) \tag{E1}$

Acceleration field

 $\vec{a} = \vec{a}(x, y, z, t) \tag{E2}$

Define the pressure field as a scalar field variable

p = p(x, y, z, t)	(E3

where

$$\vec{v} = u\vec{e_x} + v\vec{e_y} + w\vec{e_z}$$
(E4)

$$\vec{e_x}, \vec{e_y}, \vec{e_z}$$
 = unit vectors





Substitute (E4) into (E1) to expand the velocity field

$$\vec{v} = (u, v, w) = u(x, y, z, t)\vec{e_x} + v(x, y, z, t)\vec{e_y} + w(x, y, z, t)\vec{e_z}$$
 (E5)

Difference between two descriptions

Imagine a person standing beside a river, measuring its properties.

- Lagrangian approach: he throws in a prove that moves downstream with the river flow
- ^L Eulerian approach: he anchors the probe at a fixed location in the river
- ~ Eulerian approach is practical for most fluid engineering problems (experiments).





nction, we would have to give the velocity of all the pieces of matter in the flow as a function of time and

Such a description, in terms of material point, is ated a Lagrangian description of the flow. The identifying co-ordinates are called Lagrangian or somelines material, co-ordinates. Given the Lagrangian biological point of the call of the second second biological differentiation with respect to time.



7. A pressure gauge is attached to one of the moving paint. (Learnee in) To make what we might call a Lagrangian measure

To make what we might call a Lagrangian measurement, we can imagine attaching an instrument like a pressure gauge to a fluid material point (Fig. 7). This tert of measurement is attempted in the atmosphere with billoons of neutral bowarey. If the balloon does indeed move faithfully with the air, it gives the Lagrangin displacement, i.e. the displacement of an ideatised fluid "element." Such Lagrangian measurements are actually very difficult, particularly in the laboratory. We usually prefer to make measurements at ponts fixed an instrument at a faced location.



(Eulerian)

5 EULERIAN AND LAGRANGIAN DESCRIPTIONS



Classically, the idea of a field, such as an electric, magnetic, or temperature field, is defined by how the response of a test body or probe, like the anenometer in Fig. 8, varies with time at each point in some spatial co-ordinate system. In Fig. 8 the fixed anenometer probes in laboratory co-ordinates. We will always use

••	
• •• •	

solid points and solid arrows to indicate such probing positions, fixed in our laboratory, and the velocities measured there.

In Fig. 9 we have a grid of points fixed in space with an arrow at each to indicate the velocity at sach point. A description like this which gives the spatial velocity distribution in laboratory co-ordinates is called an *Eulerian* description of the flow.

Relation Between Eulerian and Lagrangian Frames

Although the physical field is the same, the Eulerian and Lagrangian representations are not the same, be-



t a <u>fixed position</u> in the flow
 Duiterian velocities
 grangian velocities
 S EULERIAN AND L





11/88

- Two types of flow
- \rightarrow In Eulerian view, two types of flow can be identified.
- 1) Steady flow
 - ~ The fluid variables at any point do not vary (change) with time.
 - \sim The fluid variables may be a function of a position in the space.
 - → non-uniform flow
 - \rightarrow In Eulerian view, steady flow still can have accelerations

(advective acceleration).





2) Unsteady flow

- \sim The fluid variables will vary with time at the spatial points in the flow.
- Fig. 3.1

when value is being opened or closed \rightarrow unsteady flow when value opening is fixed \rightarrow steady flow











[Re] Mathematical expressions

Let fluid variables (pressure, velocity, density, discharge, depth) = F

$$\frac{\partial F}{\partial t} = 0 \rightarrow \text{steady flow}$$
$$\frac{\partial F}{\partial t} \neq 0 \rightarrow \text{unsteady flow}$$
$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \rightarrow \text{uniform flow}$$
$$\frac{\partial F}{\partial x} \neq 0 \rightarrow \text{non-uniform (varied) flow}$$











- Flow lines
- 1) Streamline

~ The curves drawn at an instant of time in such a way that the <u>tangent</u> <u>at any point</u> is in the <u>direction of the velocity</u> vector at that point are called instantaneous streamlines, and they continually evolve in time in an unsteady flow.











Individual fluid particles must travel on paths whose tangent is always in the direction of the fluid velocity at any point.

 \rightarrow In an unsteady (turbulent) flow, path lines are not coincident with the instantaneous streamlines.

2) Path line

- ~ trajectory of the particle
- 3) Streak line

~ current location of all particles which have passed through a fixed point in space





- ~ The streak lines can be used to trace the travel of a pollutant downstream from a smoke stack or other discharge.
- ~ In steady flow, Lagrangian path lines are the same as the Eulerian streamlines, and both are the same as the streak lines, because the streamlines are then fixed in space and path lines, streak lines and streamlines are tangent to the steady velocities.
- → In a steady flow, all the particles on a streamlines that passes through a point in space also passed through or will pass through that point as well.
 ~ In an unsteady flow, the path lines, the streak lines, and the instantaneous streamlines are not coincident.





• In a steady flow, streamlines can be defined integrating Eq. (3.1) in space. [Ex] A fluid flow has the following velocity components; u = 1 m / s, v = 2xm / s. Find an equation for the streamlines of this flow.

Sol.:
$$\frac{dy}{dx} = \frac{v}{u} = \frac{2x}{1}$$
 (a)
Integrating (a) gives
 $y = x^2 + c$

- Stream tube
- ~ aggregation of streamlines drawn through a closed curve in a steady flow forming a boundary across which fluid particles cannot pass because the velocity is always tangent to the boundary





- ~ may be treated as if isolated from the adjacent fluid
- ~ many of the equations developed for a small streamtube will apply equally well to a streamline.

- [Re] How to shoot flow lines?
- 1) streamline: shoot bunch of reflectors instantly $(\Delta t \rightarrow 0)$
- 2) path line: shoot only one reflector with long time exposure
- 3) streak line: shoot dye injecting from on slot with instant exposure



















- 11 -











[Re] Substantial (total) derivative

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

$$\uparrow \qquad = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$$
total
derivative
$$\uparrow \qquad \uparrow$$
local advective (convective)
derivative derivative





- steady flow: $\frac{\partial F}{\partial t} = 0$
- uniform flow: $u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} = 0$





- One-dimensional flow
 - ~ All fluid particles are assumed uniform over any cross section.
 - ~ The change of fluid variables perpendicular to (across) a streamline is negligible compared to the change along the streamline.
 - ~ powerful, simple
- ~ pipe flow, flow in a stream tube average fluid properties are used at each section

F = f(x)











30/88

- Two-dimensional flow
 - ~ flow fields defined by streamlines in a single plane (unit width)
 - ~ flows over weir and about wing Fig. 3.4
 - ~ assume end effects on weir and wing is negligible

v = f(x, z)











32/88

- Three-dimensional flow
 - ~ The flow fields defined by streamlines in space.
 - ~ axisymmetric three-dim. flow Fig. 3.5
 - \rightarrow streamlines = stream surfaces

v = f(x, y, z)











34/88

3.3 Velocity and Acceleration

- velocity, vacceleration, \vec{a}
 - ~ vector:
 - direction known or assumed
- ♦ <u>One-dimensional flow</u> along a streamline

Select a fixed point O as a reference point and define the displacement s of a fluid particle along the streamline in the direction of motion. \rightarrow In time dt the particle will cover a differential distance ds along the streamline.





3.3 Velocity and Acceleration






1) Velocity

- magnitude of velocity
$$v = \frac{ds}{dt}$$

where s = displacement

- direction of velocity = tangent to the streamline

2) Acceleration

- acceleration along (tangent to) the streamline = a_s
- acceleration (normal to) the streamline = a_r





$$a_{s} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$$

$$a_{r} = -\frac{v^{2}}{r} \leftarrow \text{particle mechanics}$$
(3.2)
(3.3)

where r = radius of curvature of the streamline at s





[Re] Uniform circular motion

 \rightarrow Particle moves in a circle with a constant speed.

- direction of $\Delta \vec{v}$: pointing inward, approximately toward the center of circle

Apply similar triangles OPP' and P'QQ'

$$\therefore \quad \frac{\overline{PP'}}{r} = \frac{\Delta v}{v}$$











Now, approximate $\overline{PP'}$ (chord length) as $\widehat{PP'}$ (arc length) when θ is small.

 $\therefore \frac{\Delta v}{v} = \frac{v \Delta t}{r}$ $\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$ $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$ $a_r = -a = -\frac{v^2}{r}$ (along a radius <u>inward toward the center</u> of the circle)

 a_r = radial (centripetal) acceleration= constant in magnitude directed radially inward





• Angular velocity, ω

$$v = r\omega$$

$$a_r = \frac{(r\omega)^2}{r} = r\omega^2$$

[IP 3.1] p. 96

Along a <u>straight streamline</u>, $v = 3\sqrt{x^2 + y^2}$ m/s

Calculate velocity and acceleration at the point (8.6)











$$a_s = v \frac{dv}{ds}, \qquad a_r = -v^2 / r$$

We observe that

- 1) The streamline is straight. $\rightarrow a_r = 0$
- 2) The displacement *s* is give as

$$\rightarrow s = \sqrt{x^2 + y^2}$$





Therefore, v = 3s

At (8.6)
$$\rightarrow s = 10 \rightarrow v = 3(10) = 30 \text{ m/s}$$

 $a_s = v \frac{dv}{ds} = 3s(3s)' = 9s = 90 \text{ m/s}^2$

[IP 3.2]

The fluid at the wall of the tank moves along the circular streamline with a <u>constant tangential velocity</u> component, $v_s = 1.04 \text{ m/s}$. Calculate the tangential and radial components of acceleration at any point on the streamline.











1) $a_s = 0$ \rightarrow Because tangential velocity is constant, $a_s = v \frac{dv}{ds} = 0$

2)
$$a_r$$

 $a_r = -\frac{v^2}{r} = -\frac{(1.04)^2}{2} = -0.541 \text{ m/sec}^2$

 \rightarrow directed toward the center of circle





* Flowfield

The velocities are everywhere different in magnitude and direction at different points in the flowfield and at different times. \rightarrow three-dimensional flow

At each point, each velocity has components *u*, *v*, *w* which are parallel to the *x*-, *y*-, and *z*- axes.

In Eulerian view,

$$u = u(x, y, z, t), v = v(x, y, z, t), w = w(x, y, z, t)$$











In Lagrangian view, velocities can be described in terms of displacement and time as

$$u = \frac{dx}{dt}, \ v = \frac{dy}{dt}, \ w = \frac{dz}{dt}$$
(3.4)

where x, y, and z are the actual coordinates of a fluid particle that is being tracked

 \rightarrow The velocity at a point is the same in both the Eulerian and the Lagrangian view.

The acceleration components are

$$a_x = \frac{du}{dt}, \ a_y = \frac{dv}{dt}, \ a_z = \frac{dw}{dt}$$

(3.5)





• Total (substantial, material) derivatives (App. 6)

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$
$$dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$
$$dw = \frac{\partial w}{\partial t}dt + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$





• Substituting these relationships in Eq. (3.5) yields





(3.6)

For steady flow, $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$, however convective acceleration is not zero.

- 2-D steady flow
 - (i) Cartesian coordinate; P(x, y)
 - $\vec{r} = \vec{i}x + \vec{j}y$ $\vec{v} = \vec{i}u + \vec{j}v$ u = u(x, y) - horizontal v = v(x, y) - vertical $u = \frac{dx}{dt}; \quad v = \frac{dy}{dt}$











(ii) Polar coordinate $P(r, \theta)$ θ = radian = $\frac{s}{r}$ where s = arc length, r = radius







[Re] Conversion

$$x = r\cos\theta, \ y = r\sin\theta$$

 $r = \sqrt{x^2 + y^2}, \ \theta = \arctan\frac{y}{x}$





• Total derivative in polar coordinates

$$dv_{r} = \frac{\partial v_{r}}{\partial r} dr + \frac{\partial v_{r}}{\partial \theta} d\theta$$

$$dv_{t} = \frac{\partial v_{t}}{\partial r} dr + \frac{\partial v_{t}}{\partial \theta} d\theta$$

$$\frac{dv_{r}}{dt} = \frac{\partial v_{r}}{\partial r} \frac{dr}{dt} + \frac{\partial v_{r}}{\partial \theta} \frac{d\theta}{dt} = \frac{\partial v_{r}}{\partial r} v_{r} + \frac{\partial v_{r}}{\partial \theta} \frac{1}{r} \frac{\partial s}{\partial t} = \frac{\partial v_{r}}{\partial r} v_{r} + \frac{\partial v_{r}}{\partial \theta} \frac{1}{r} v_{t}$$

$$\frac{dv_{t}}{dt} = \frac{\partial v_{t}}{\partial r} \frac{dr}{dt} + \frac{\partial v_{t}}{\partial \theta} \frac{d\theta}{dt} = \frac{\partial v_{t}}{\partial r} v_{r} + \frac{\partial v_{t}}{\partial \theta} \frac{1}{r} v_{t}$$





• Acceleration in polar coordinates \rightarrow Hydrodynamics (Lamb, 1959)

$$a_r = a_{rr} + a_{rt}$$

where a_{rr} = acceleration due to variation of v_r in r – direction;

 a_{rt} = acceleration due to variation of v_t in r – direction





 $\omega = \frac{v_t}{r}$

$$a_t = a_{tt} + a_{tr}$$

$$a_{t} = \frac{dv_{t}}{dt} + \omega v_{r} = v_{r} \frac{\partial v_{t}}{\partial r} + v_{t} \frac{1}{r} \frac{\partial v_{t}}{\partial \theta} + \frac{v_{r} v_{t}}{r}$$

$$\downarrow \qquad \downarrow$$

$$a_{tt} \qquad a_{tr}$$

[IP 3.3] p. 99

For circular streamline along which $v_t = 1.04 \text{ m/s}$, r = 2 m(radius of curvatures)

Calculate
$$\begin{pmatrix} u, v, v_t, v_r \\ a_x, a_y, a_t, a_r \end{pmatrix}$$
 at $P(2 \text{ m}, 60^\circ)$





[Sol]

Determine P(x, y)

 $x = 2 \cos 60^\circ = 1$ $y = 2 \sin 60^\circ = \sqrt{3}$

- 1) Velocity
- Polar coordinate

$$v_t = 1.04 \text{ m/s}$$

 $v_r = \frac{dr}{dt} = 0$











Cartesian coordinate

Apply similar triangles

$$v_t : u = r : y, \quad r = \sqrt{x^2 + y^2} = 2$$

$$u = \frac{-v_t y}{r} = -v_t \sin 60^\circ = -(1.04) \sin 60^\circ = -0.90 \text{ m/s}$$

$$v = \frac{v_t x}{r} = v_t \cos 60^\circ = (1.04) \cos 60^\circ = 0.52 \text{ m/s}$$





- 2) Accelerations
- Cartesian coordinate

$$a_{x} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1.04 y}{r} \frac{\partial}{\partial x} \left(-\frac{1.04 y}{r} \right) + \frac{1.04 x}{r} \frac{\partial}{\partial y} \left(-\frac{1.04 y}{r} \right)$$
$$= \frac{-(1.04)^{2}}{r^{2}} x = -\frac{1.082}{4} x$$
$$a_{y} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1.04 y}{r} \frac{\partial}{\partial x} \left(\frac{1.04 x}{r} \right) + \frac{1.04 x}{r} \frac{\partial}{\partial y} \left(\frac{1.04 x}{r} \right)$$
$$= -\frac{1.082}{4} y$$





At ,
$$P(1,\sqrt{3})$$
, $a_x = -\frac{1.082}{4}(1) = -0.27 \text{ m/s}^2$
 $a_y = -\frac{1.082}{4}\sqrt{3} = -0.47 \text{ m/s}^2$

•Polar coordinate

$$a_r = v_r \frac{\partial v_r}{\partial r} + v_t \frac{\partial v_r}{r \partial \theta} - \frac{v_t^2}{r} = -\frac{(1.04)^2}{2} = -0.54 \text{ m/s}^2$$

 \rightarrow direction toward the center of circle

$$a_{t} = v_{r} \frac{\partial v_{t}}{\partial r} + v_{t} \frac{\partial v_{t}}{r \partial \theta} + \frac{v_{r} v_{t}}{r} = 1.04 \frac{\partial}{r \partial \theta} (1.04) = 0 \text{ m/s}^{2}$$





[Re]

$$a_x^2 + a_y^2 = -(0.27)^2 + (0.47)^2 = 0.073 + 0.22 = 0.29$$

 $a_r^2 = (-0.54)^2 = 0.29$

$$\therefore a_r^2 + a_t^2 = a_x^2 + a_y^2$$





3.4.1 Circulation

As shown in IP 3.3, tangential components of the velocity cause the fluid in a flow a swirl.

Eddy, whirl,

vortex

- \rightarrow A measure of swirl can be defined as circulation.
- Circulation, Γ
- = <u>line integral of the tangential component of velocity</u> around a closed curve fixed in the flow (circle and squares)











$$d\Gamma = (V \cos a)dl$$
(3.8)

$$\Gamma = \oint_C d\Gamma = \oint (V \cos \alpha)dl = \oint \vec{V} \cdot \vec{dl}$$
(3.9)

where \vec{dl} = elemental vector of size dl and direction tangent to the control surface at each point

[Re] vector dot product

 $\vec{a} \cdot \vec{b} = ab\cos\alpha$





[Cf] Integral of normal component of velocity \rightarrow Continuity equation

$$\oint_{C.S.} \rho \vec{V} \cdot \vec{n} \, dA = 0 \rightarrow \text{Ch. 4}$$

 \vec{n} = unit normal vector

- Point value of the circulation in a flow for square of differential size
- \rightarrow proceed from A counterclockwise around the boundary of the element











$$d\Gamma \cong \begin{bmatrix} \text{mean velocity} \\ \text{along } AB \end{bmatrix} dx \cos 0^{\circ} + \begin{bmatrix} \text{mean velocity} \\ \text{along } BC \end{bmatrix} dy \cos 0^{\circ} \\ + \begin{bmatrix} \text{mean velocity} \\ \text{along } CD \end{bmatrix} dx \cos 180^{\circ} + \begin{bmatrix} \text{mean velocity} \\ \text{along } DA \end{bmatrix} dy \cos 180^{\circ} \\ \begin{bmatrix} \text{mean velocity} \\ \text{along } AB \end{bmatrix} dx \cos 0^{\circ} = \begin{bmatrix} u - \frac{\partial u}{\partial y} \frac{dy}{2} \end{bmatrix} dx \\ \begin{bmatrix} \text{mean velocity} \\ \text{along } CD \end{bmatrix} dx \cos 180^{\circ} = \begin{bmatrix} u + \frac{\partial u}{\partial y} \frac{dy}{2} \end{bmatrix} (-dx) \\ d\Gamma \cong \begin{bmatrix} u - \frac{\partial u}{\partial y} \frac{dy}{2} \end{bmatrix} dx + \begin{bmatrix} v + \frac{\partial v}{\partial x} \frac{dx}{2} \end{bmatrix} dy - \begin{bmatrix} u + \frac{\partial u}{\partial y} \frac{dy}{2} \end{bmatrix} dx - \begin{bmatrix} v - \frac{\partial v}{\partial x} \frac{dx}{2} \end{bmatrix} dy \end{bmatrix}$$





Expanding the products and retaining only the terms of lowest order (largest magnitude) gives

$$d\Gamma = udx - \frac{\partial u}{\partial y}\frac{dxdy}{2} + vdy + \frac{\partial v}{\partial x}\frac{dxdy}{2} - udx - \frac{\partial u}{\partial y}\frac{dxdy}{2} - vdy + \frac{\partial v}{\partial x}\frac{dxdy}{2}$$
$$d\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

where dx dy is the area inside the control surface




3.4.2 Vorticity, ξ

- ~ measure of the rotational movement
- ~ differential circulation per unit area enclosed

$$\xi = \frac{d\Gamma}{dxdy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

[Cf] continuity:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For polar coordinates

$$\xi = \frac{\partial v_t}{\partial r} + \frac{v_t}{r} - \frac{\partial v_r}{r\partial \theta}$$



73/88

(3.11)

(3.10)



3.4.3 Angular rotation

If the fluid element tends to rotate, two lines will tend to rotate also.

 \rightarrow For the instant their average angular velocity can be calculated.

Consider counterclockwise rotation for vertical line AB

$$\widehat{d\theta_{V}} = \frac{ds}{dy} = \frac{dudt}{dy} = du\frac{dt}{dy}$$
$$= -\left[\left(u + \frac{\partial u}{\partial y}\frac{dy}{2}\right) - \left(u + \frac{\partial u}{\partial y}\frac{dy}{2}\right)\right]\frac{dt}{dy} = -\frac{\partial u}{\partial y}dt$$
$$\therefore \omega_{V} = \frac{d\theta_{V}}{dt} = -\frac{\partial u}{\partial y}$$

where $\omega = rate of rotation$











For horizontal line

$$\widehat{d\theta_{H}} = +\left[\left(v + \frac{\partial v}{\partial x}\frac{dx}{2}\right) - \left(v - \frac{\partial v}{\partial x}\frac{dx}{2}\right)\right]\frac{dt}{dx} = +\frac{\partial v}{\partial x}dt$$
$$\omega_{H} = \frac{d\theta_{H}}{dt} = \frac{\partial v}{\partial x}$$

Consider average rotation

$$\omega = \frac{1}{2}(\omega_V + \omega_H) = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega$$







rotational flow ~ flow possesses vorticity $\rightarrow \xi \neq 0$

irrotational flow ~ flow possesses no vorticity, no net rotation $\rightarrow \xi = 0$

= <u>potential flow</u> (velocity potential exists) ← Ch. 5

 Actually flow fields can possess zones of <u>both irrotational and rotational</u> <u>flows</u>.

free vortex flow \rightarrow irrotational flow, bath tub, hurricane, morning glory spillway forced vortex flow \rightarrow rotational flow, rotating cylinder















Irrotational flow ← gravity, pressure force

















[IP 3.4] Calculate the vorticity of two-dimensional flowfield described by the equations $v_t = \omega r$ and $v_r = 0$

- \rightarrow Forced vortex
- \rightarrow A cylindrical container is rotating at an angular velocity ω .











[Sol]

$$\xi = \frac{\partial v_t}{\partial r} + \frac{v_t}{r} - \frac{\partial v_r}{r\partial \theta}$$
$$\xi = \frac{\partial}{\partial r}(\omega r) + \frac{\omega r}{r} - \frac{\partial}{r\partial \theta}(0) = \omega + \omega - 0 = 2\omega \neq 0$$

- → rotational flow (forced vortex) possessing a constant vorticity over the whole flow field
- \rightarrow streamlines are concentric circles





[IP 3.5] When a viscous, incompressible fluid flow between two plates

and the flow is laminar and two-dimensional, the velocity profile is

parabolic,
$$u = U_c \left(1 - \frac{y^2}{b^2}\right)$$
.
2D Poiseuille flow

Calculate τ and ω (rotation)











[Sol]

$$\frac{\partial v}{\partial x} = 0; \quad \frac{\partial u}{\partial y} = -U_c \frac{2y}{b^2}$$
1) $\omega = \frac{1}{2}(\omega_v + \omega_H) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{1}{2}\left(-U_c \frac{2y}{b^2}\right) = \left(\frac{U_c}{b^2}\right)y$

$$2) \quad \tau = \mu \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy} = \left(-\frac{2\mu U_c}{b^2}\right) y$$
$$\tau = -2\mu\omega = -\mu\xi$$

 \rightarrow rotation and vorticity are large where shear stress is large.





Homework Assignment # 3

Due: 1 week from today

- Prob. 3.3
- Prob. 3.5
- Prob. 3.6
- Prob. 3.10
- Prob. 3.12

Prob. 3.15



