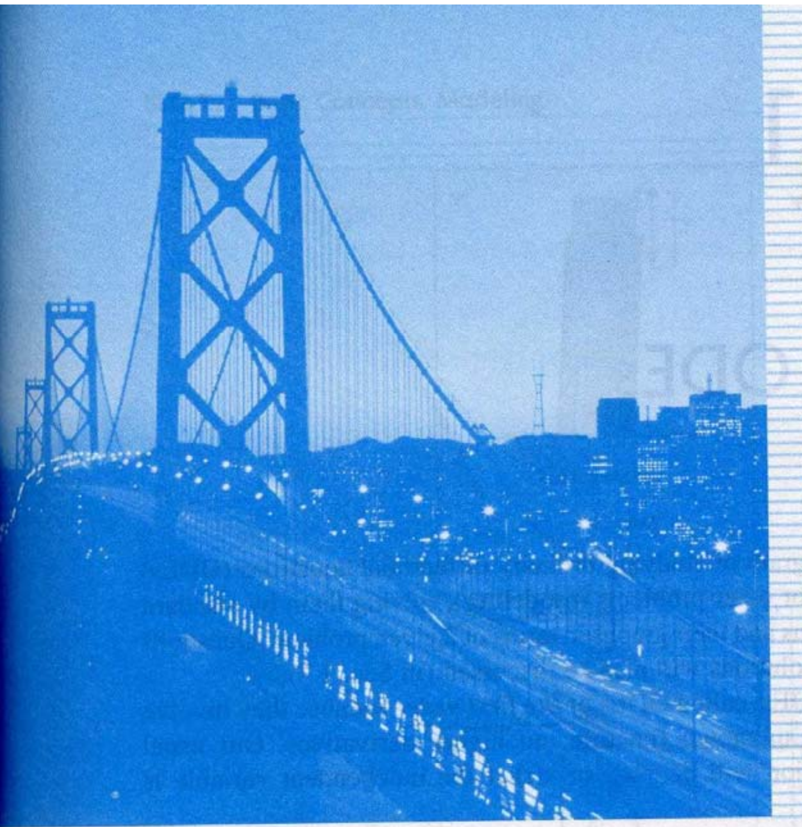


PART A

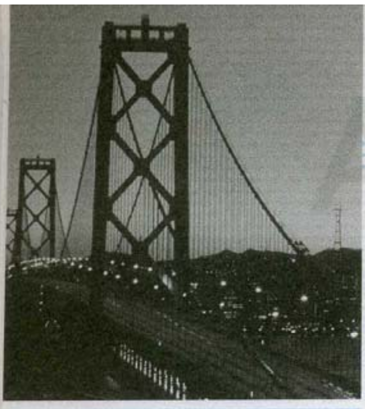
Ordinary Differential Equations (ODEs)

Differential equations are of basic importance in engineering mathematics because many physical laws and relations appear mathematically in the form of a differential equation. In Part A we shall consider various physical and geometric problems that lead to differential equations, with emphasis on ***modeling***, that is, the transition from the physical situation to a "mathematical model." In this chapter the model will be a differential equation, and as we proceed we shall explain the most important standard methods for solving such equations.

常微分方程式

Part A concerns ***ordinary*** **differential equations (ODEs)**, whose unknown functions depend on a *single* variable. *Partial* differential equations (PDEs), involving unknown functions of *several* variables, follow in Part C.

ODEs are very well suited for computers. ***Numeric methods for ODEs can be studied directly after Chaps.* 1 *or* 2.** See Secs. 21.1–21.3, which are independent of the other sections on numerics.



CHAPTER 1

First-Order ODEs

In this chapter we begin our program of studying ordinary differential equations (ODEs) by deriving them from physical or other problems **(modeling)**, solving them by standard methods, and interpreting solutions and their graphs in terms of a given problem. Questions of existence and uniqueness of solutions will also be discussed (in Sec. 1.7).

We begin with the simplest ODEs, called ODEs *of the first order* because they involve only the first derivative of the unknown function, no higher derivatives. Our usual notation for the unknown function will be $y(x)$, or $y(t)$ if the independent variable is time t.

$y(t)$ or $y(x)$

If you wish, use your computer algebra system (CAS) for checking solutions, but make sure that you gain a conceptual understanding of the basic terms, such as ODE, direction field, and initial value problem.

COMMENT. ***Numerics for first-order ODEs can be studied immediately after this chapter.*** See Secs. 21.1–21.2, which are independent of other sections on numerics.

Prerequisite: Integral calculus.
Sections that may be omitted in a shorter course: 1.6, 1.7.
References and Answers to Problems: App. 1 Part A, and App. 2

1.1 Basic Concepts. Modeling

If we want to solve an engineering problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, equations, and so forth. Such an expression is known as a mathematical **model** of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling.** We shall illustrate this process by various examples and problems because modeling requires experience. (Your computer may help you in solving but hardly in setting up models.)

Since many physical concepts, such as velocity and acceleration, are derivatives, a model is very often an equation containing derivatives of an unknown function. Such a model is called a **differential equation.** Of course, we then want to find a solution (a function that satisfies the equation), explore its properties, graph it, find values of it, and interpret it in physical terms so that we can understand the behavior of the physical system in our given problem. However, before we can turn to methods of solution we must first define basic concepts needed throughout this chapter.

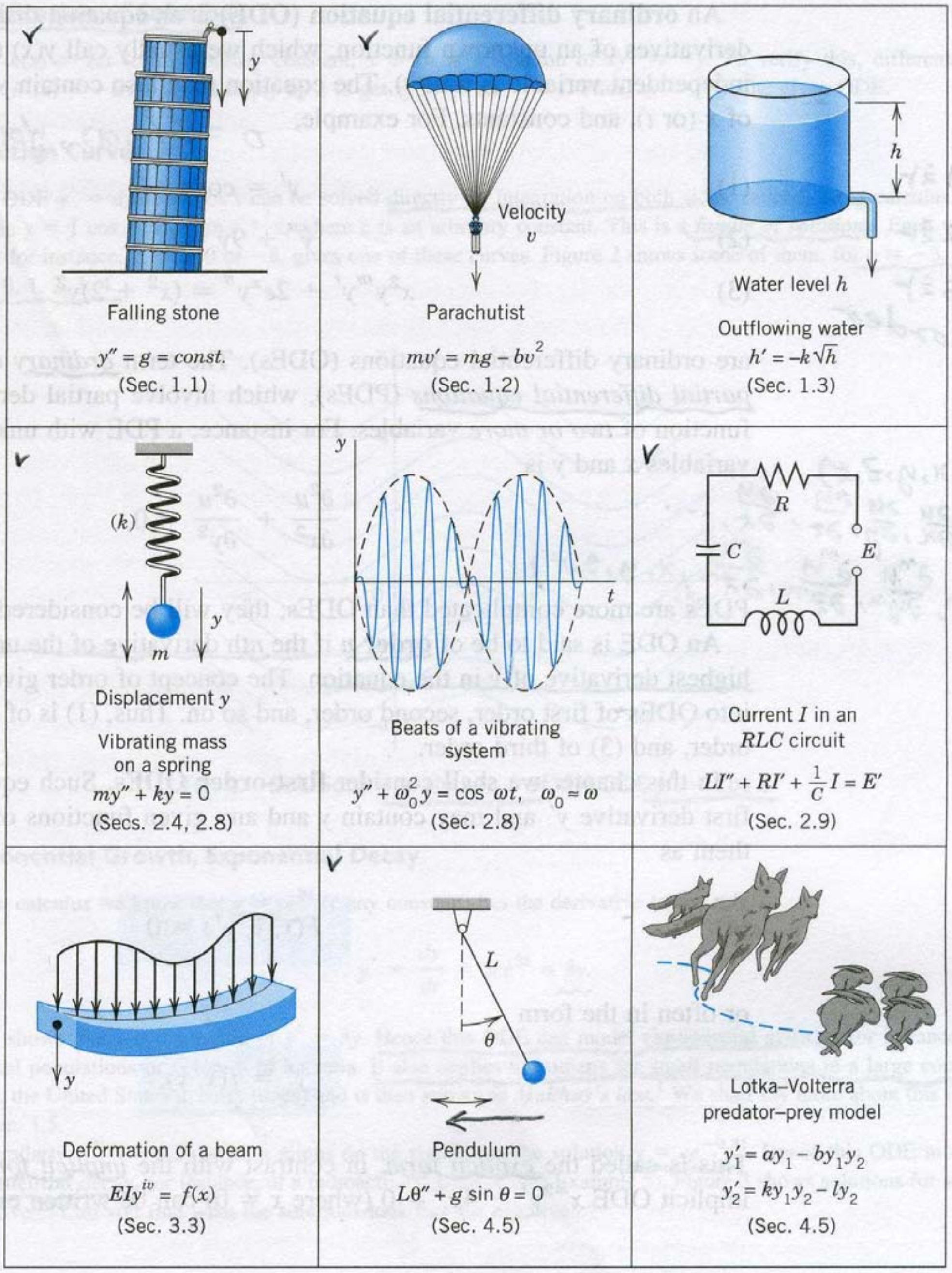


Fig. 1. Some applications of differential equations

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

$0 = F(y(x), y'(x) \cdots y^n(x), x)$

1차 (1) $$y' = \cos x,$$

2차 (2) $$y'' + 9y = 0,$$

3차 (3) $$x^2y'''y' + 2e^x y'' = (x^2 + 2)y^2$$

highst order

are ordinary differential equations (ODEs). The term *ordinary* distinguishes them from *partial differential equations* (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function u of two variables x and y is

for $u(x, y, z, t)$

$0 = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial y}{\partial t}, \cdots \frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial y^n}, \frac{\partial^n u}{\partial z^n}, \frac{\partial^n u}{\partial t^n}, x, y, z, t\right)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PDEs are more complicated than ODEs; they will be considered in Chap. 12.

An ODE is said to be of **order** n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

In this chapter we shall consider **first-order ODEs.** Such equations contain only the first derivative y' and may contain y and any given functions of x. Hence we can write them as

(4) $$F(x, y, y') = 0$$

or often in the form

$$y' = f(x, y).$$

This is called the *explicit form,* in contrast with the *implicit form* (4). For instance, the implicit ODE $x^{-3}y' - 4y^2 = 0$ (where $x \neq 0$) can be written explicitly as $y' = 4x^3y^2$.

Concept of Solution

A function

$$y = h(x)$$

is called a **solution** of a given ODE (4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

Here, **open interval** $a < x < b$ means that the endpoints a and b are not regarded as points belonging to the interval. Also, $a < x < b$ includes *infinite intervals* $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

EXAMPLE 1 Verification of Solution

代入

$y = h(x) = c/x$ (c an arbitrary constant, $x \neq 0$) is a solution of $xy' = -y$. To verify this, differentiate, $y' = h'(x) = -c/x^2$, and multiply by x to get $xy' = -c/x = -y$. Thus, $xy' = -y$, the given ODE. ■

EXAMPLE 2 Solution Curves

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions***. Each value of c, for instance, 2.75 or 0 or -8, gives one of these curves. Figure 2 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$. ■

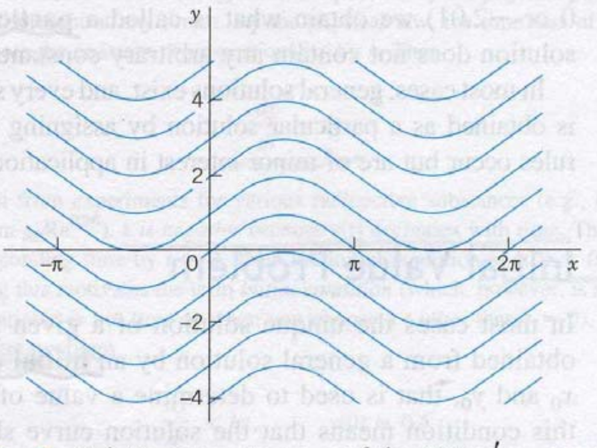


Fig. 2. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

EXAMPLE 3 Exponential Growth, Exponential Decay

From calculus we know that $y = ce^{3t}$ (c any constant) has the derivative (chain rule!)

$$y' = \frac{dy}{dt} = 3ce^{3t} = 3y.$$

This shows that y is a solution of $y' = 3y$. Hence this ODE can model **exponential growth**, for instance, of animal populations or colonies of bacteria. It also applies to humans for small populations in a large country (e.g., the United States in early times) and is then known as ***Malthus's law***.[1] We shall say more about this topic in Sec. 1.5.

Similarly, $y' = -0.2y$ (with a minus on the right!) has the solution $y = ce^{-0.2t}$. Hence this ODE models **exponential decay**, for instance, of a radioactive substance (see Example 5). Figure 3 shows solutions for some positive c. Can you find what the solutions look like for negative c? ■

Fig. 3. Solutions of $y' = -0.2y$ in Example 3

[1] Named after the English pioneer in classic economics, THOMAS ROBERT MALTHUS (1766–1834).

一般解

We see that each ODE in these examples has a solution that contains an arbitrary constant c. Such a solution containing an arbitrary constant c is called a **general solution** of the ODE.

(We shall see that c is sometimes not completely arbitrary but must be restricted to some interval to avoid complex expressions in the solution.)

We shall develop methods that will give general solutions *uniquely* (perhaps except for notation). Hence we shall say *the* general solution of a given ODE (instead of *a* general solution).

特殊解

Geometrically, the general solution of an ODE is a family of infinitely many solution curves, one for each value of the constant c. If we choose a specific c (e.g., $c = 6.45$ or 0 or -2.01) we obtain what is called a **particular solution** of the ODE. A particular solution does not contain any arbitrary constants.

In most cases, general solutions exist, and every solution not containing an arbitrary constant is obtained as a particular solution by assigning a suitable value to c. Exceptions to these rules occur but are of minor interest in applications; see Prob. 16 in Problem Set 1.1.

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an **initial condition** $y(x_0) = y_0$, with given values x_0 and y_0, that is used to determine a value of the arbitrary constant c. Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the xy-plane. An ODE together with an initial condition is called an **initial value problem.** Thus, if the ODE is explicit, $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{5}$$

EXAMPLE 4 Initial Value Problem

Solve the initial value problem

$$y' = \frac{dy}{dx} = 3y, \qquad y(0) = 5.7.$$

Solution. The general solution is $y(x) = ce^{3x}$; see Example 3. From this solution and the initial condition we obtain $y(0) = ce^0 = c = 5.7$. Hence the initial value problem has the solution $y(x) = 5.7e^{3x}$. This is a particular solution. ■

Modeling

The general importance of modeling to the engineer and physicist was emphasized at the beginning of this section. We shall now consider a basic physical problem that will show the typical steps of modeling in detail: Step 1 the transition from the physical situation (the physical system) to its mathematical formulation (its mathematical model); Step 2 the solution by a mathematical method; and Step 3 the physical interpretation of the result. This may be the easiest way to obtain a first idea of the nature and purpose of differential equations and their applications. Realize at the outset that your ***computer*** (your ***CAS***) may perhaps give you a hand in Step 2, but Steps 1 and 3 are basically your work. And Step 2

computer: Step2 only

input
→ output

requires a solid knowledge and good understanding of solution methods available to you—*you* have to choose the method for your work by hand or by the computer. Keep this in mind, and always check computer results for errors (which may result, for instance, from false inputs).

EXAMPLE 5 Radioactivity. Exponential Decay

Given an amount of a radioactive substance, say, 0.5 g (gram), find the amount present at any later time.

Physical Information. Experiments show that at each instant a radioactive substance decomposes at a rate proportional to the the amount present.

Step 1. Setting up a mathematical model (a differential equation) of the physical process. Denote by $y(t)$ the amount of substance still present at any time t. By the physical law, the time rate of change $y'(t) = dy/dt$ is proportional to $y(t)$. Denote the constant of proportionality by k. Then

$$\frac{dy}{dt} = ky. \tag{6}$$

The value of k is known from experiments for various radioactive substances (e.g., $k = -1.4 \cdot 10^{-11} \text{sec}^{-1}$, approximately, for radium ${}_{88}\text{Ra}^{226}$). *k is negative* because $y(t)$ decreases with time. The given initial amount is 0.5 g. Denote the corresponding time by $t = 0$. Then the initial condition is $y(0) = 0.5$. This is the instant at which the process begins; this motivates the term *initial condition* (which, however, is also used more generally when the independent variable is not time or when you choose a t other than $t = 0$). Hence the model of the process is the **initial value problem**

$$\frac{dy}{dt} = ky, \qquad y(0) = 0.5. \tag{7}$$

Step 2. Mathematical solution. As in Example 3 we conclude that the ODE (6) models exponential decay and has the general solution (with arbitrary constant c but definite given k)

$$y(t) = ce^{kt}. \tag{8}$$

We now use the initial condition to determine c. Since $y(0) = c$ from (8), this gives $y(0) = c = 0.5$. Hence the particular solution governing this process is

$$y(t) = 0.5e^{kt} \qquad \text{(Fig. 4).} \tag{9}$$

Always check your result—it may involve human or computer errors! Verify by differentiation (chain rule!) that your solution (9) satisfies (7) as well as $y(0) = 0.5$:

check!

$$\frac{dy}{dt} = 0.5ke^{kt} = k \cdot 0.5e^{kt} = ky, \qquad y(0) = 0.5e^0 = 0.5.$$

Step 3. Interpretation of result. Formula (9) gives the amount of radioactive substance at time t. It starts from the correct given initial amount and decreases with time because k (the constant of proportionality, depending on the kind of substance) is negative. The limit of y as $t \to \infty$ is zero. ■

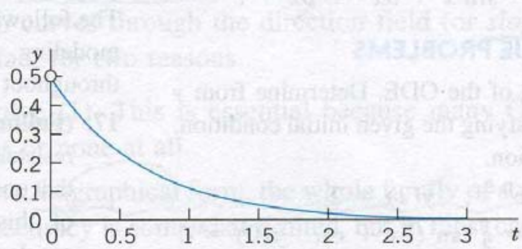


Fig. 4. Radioactivity (Exponential decay, $y = 0.5\, e^{kt}$, with $k = -1.5$ as an example)

EXAMPLE 6 A Geometric Application

Geometric problems may also lead to initial value problems. For instance, find the curve through the point (1, 1) in the xy-plane having at each of its points the slope $-y/x$.

Solution. The slope y' should equal $-y/x$. This gives the ODE $y' = -y/x$. Its general solution is $y = c/x$ (see Example 1). This is a family of hyperbolas with the coordinate axes as asymptotes.

Now, for the curve to pass through (1, 1), we must have $y = 1$ when $x = 1$. Hence the initial condition is $y(1) = 1$. From this condition and $y = c/x$ we get $y(1) = c/1 = 1$; that is, $c = 1$. This gives the particular solution $y = 1/x$ (drawn somewhat thicker in Fig. 5). ■

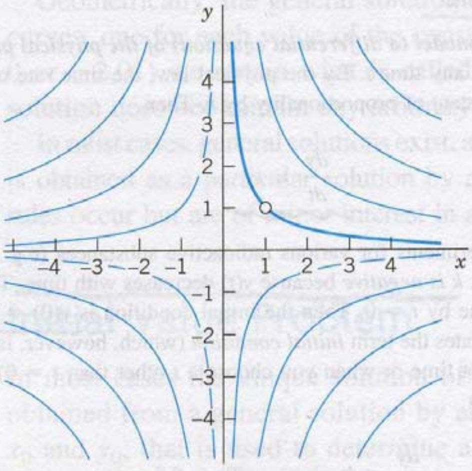


Fig. 5. Solutions of $y' = -y/x$ (hyperbolas)

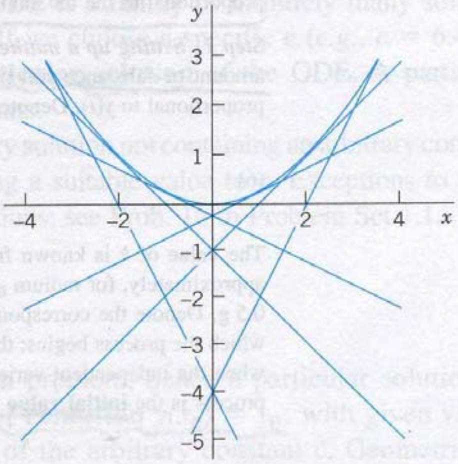


Fig. 6. Particular solutions and singular solution in Problem 16

PROBLEM SET 1.1

1–4 CALCULUS

Solve the ODE by integration.

1. $y' = -\sin \pi x$ **2.** $y' = e^{-3x}$

3. $y' = xe^{x^2/2}$ **4.** $y' = \cosh 4x$

5–9 VERIFICATION OF SOLUTION

State the order of the ODE. Verify that the given function is a solution. (a, b, c are arbitrary constants.)

5. $y' = 1 + y^2, \quad y = \tan(x + c)$

6. $y'' + \pi^2 y = 0, \quad y = a \cos \pi x + b \sin \pi x$

7. $y'' + 2y' + 10y = 0, \quad y = 4e^{-x} \sin 3x$

8. $y' + 2y = 4(x+1)^2, \quad y = 5e^{-2x} + 2x^2 + 2x + 1$

9. $y''' = \cos x, \quad y = -\sin x + ax^2 + bx + c$

10–14 INITIAL VALUE PROBLEMS

Verify that y is a solution of the ODE. Determine from y the particular solution satisfying the given initial condition. Sketch or graph this solution.

10. $y' = 0.5y, \quad y = ce^{0.5x} \quad y(2) = 2$

11. $y' = 1 + 4y^2, \quad y = \frac{1}{2}\tan(2x + c), \quad y(0) = 0$

12. $y' = y - x, \quad y = ce^x + x + 1, \quad y(0) = 3$

13. $y' + 2xy = 0, \quad y = ce^{-x^2} \quad y(1) = 1/e$

14. $y' = y \tan x, \quad y = c \sec x, \quad y(0) = \frac{1}{2}\pi$

15. **(Existence)** **(A)** Does the ODE $y'^2 = -1$ have a (real) solution?

(B) Does the ODE $|y'| + |y| = 0$ have a general solution?

16. **(Singular solution)** An ODE may sometimes have an additional solution that cannot be obtained from the general solution and is then called a *singular solution.* The ODE $y'^2 - xy' + y = 0$ is of the kind. Show by differentiation and substitution that it has the general solution $y = cx - c^2$ and the singular solution $y = x^2/4$. Explain Fig. 6.

17–22 MODELING, APPLICATIONS

The following problems will give you a first impression of modeling. Many more problems on modeling follow throughout this chapter.

17. **(Falling body)** If we drop a stone, we can assume air resistance ("drag") to be negligible. Experiments show that under that assumption the acceleration $y'' = d^2y/dt^2$ of this motion is constant (equal to the so-called acceleration of gravity $g = 9.80 \text{ m/sec}^2 = 32 \text{ ft/sec}^2$). State this as an ODE for $y(t)$, the distance fallen as a function of time t. Solve the ODE to get the familiar law of free fall, $y = gt^2/2$.

18. (Falling body) If in Prob. 17 the stone starts at $t = 0$ from initial position y_0 with initial velocity $v = v_0$, show that the solution is $y = gt^2/2 + v_0 t + y_0$. How long does a fall of 100 m take if the body falls from rest? A fall of 200 m? (Guess first.)

19. (Airplane takeoff) If an airplane has a run of 3 km, starts with a speed 6 m/sec, moves with constant acceleration, and makes the run in 1 min, with what speed does it take off?

20. (Subsonic flight) The efficiency of the engines of subsonic airplanes depends on air pressure and usually is maximum near about 36 000 ft. Find the air pressure $y(x)$ at this height without calculation. *Physical information.* The rate of change $y'(x)$ is proportional to the pressure, and at 18 000 ft the pressure has decreased to half its value y_0 at sea level.

21. (Half-life) The half-life of a radioactive substance is the time in which half of the given amount disappears. Hence it measures the rapidity of the decay. What is the half-life of radium ${}_{88}\mathrm{Ra}^{226}$ (in years) in Example 5?

22. (Interest rates) Show by algebra that the investment $y(t)$ from a deposit y_0 after t years at an interest rate r is

$$y_a(t) = y_0[1 + r]^t \quad \text{(Interest compounded annually)}$$

$$y_d(t) = y_0[1 + (r/365)]^{365t} \quad \text{(Interest compounded daily).}$$

Recall from calculus that

$$[1 + (1/n)]^n \to e \text{ as } n \to \infty;$$

hence $[1 + (r/n)]^{nt} \to e^{rt}$; thus

$$y_c(t) = y_0 e^{rt} \quad \text{(Interest compounded continuously).}$$

What ODE does the last function satisfy? Let the initial investment be \$1000 and $r = 6\%$. Compute the value of the investment after 1 year and after 5 years using each of the three formulas. Is there much difference?

1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields

A first-order ODE

$$y' = f(x, y) \tag{1}$$

has a simple geometric interpretation. From calculus you know that the derivative $y'(x)$ of $y(x)$ is the slope of $y(x)$. Hence a solution curve of (1) that passes through a point (x_0, y_0) must have at that point the slope $y'(x_0)$ equal to the value of f at that point; that is,

x_0 에서의 기울기 $\rightarrow$ $$y'(x_0) = f(x_0, y_0).$$ $\leftarrow$ x_0, y_0 에서의 함숫값

Read this paragraph again before you go on, and think about it.

It follows that you can indicate directions of solution curves of (1) by drawing short straight-line segments (*lineal elements*) in the xy-plane (as in Fig. 7a) and then fitting (approximate) solution curves through the direction field (or *slope field*) thus obtained. This method is important for two reasons.

1. You need not solve (1). This is essential because many ODEs have complicated solution formulas or none at all.
2. The method shows, in graphical form, the whole family of solutions and their typical properties. The accuracy is somewhat limited, but in most cases this does not matter.

Let us illustrate this method for the ODE

$$y' = xy. \tag{2}$$

Direction Fields by a CAS (Computer Algebra System). A CAS plots lineal elements at the points of a square grid, as in Fig. 7a for (2), into which you can fit solution curves. Decrease the mesh size of the grid in regions where $f(x, y)$ varies rapidly.

Direction Fields by Using Isoclines (the Older Method). Graph the curves $f(x, y) = k = const$, called *isoclines* (meaning *curves of equal inclination*). For (2) these are the hyperbolas $f(x, y) = xy = k = const$ (and the coordinate axes) in Fig. 7b. By (1), these are the curves along which the derivative y' is constant. These are not yet solution curves—don't get confused. Along each isocline draw many parallel line elements of the corresponding slope k. This gives the direction field, into which you can now graph approximate solution curves.

We mention that for the ODE (2) in Fig. 7 we would not need the method, because we shall see in the next section that ODEs such as (2) can easily be solved exactly. For the time being, let us verify by substitution that (2) has the general solution

$$y(x) = ce^{x^2/2} \qquad (c \text{ arbitrary}).$$

Indeed, by differentiation (chain rule!) we get $y' = x(ce^{x^2/2}) = xy$. Of course, knowing the solution, we now have the advantage of obtaining a feel for the accuracy of the method by comparing with the exact solution. The particular solution in Fig. 7 through $(x, y) = (1, 2)$ must satisfy $y(1) = 2$. Thus, $2 = ce^{1/2}$, $c = 2/\sqrt{e} = 1.213$, and the particular solution is $y(x) = 1.213e^{x^2/2}$.

A famous ODE for which we *do* need direction fields is

$$y' = 0.1(1 - x^2) - \frac{x}{y}. \tag{3}$$

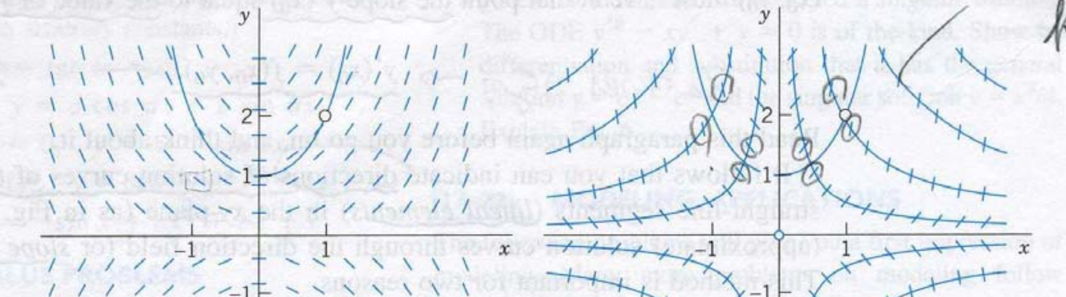
(It is related to the van der Pol equation of electronics, which we shall discuss in Sec. 4.5.) The direction field in Fig. 8 shows lineal elements generated by the computer. We have also added the isoclines for $k = -5, -3, \frac{1}{4}, 1$ as well as three typical solution curves, one that is (almost) a circle and two spirals approaching it from inside and outside.

곡선의 기울기)

k

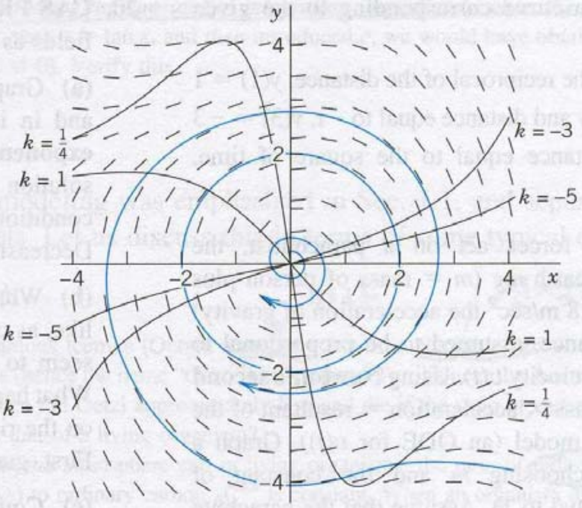
solution curve의 기울기

x, y grid



(a) By a CAS (b) By isoclines

Fig. 7. Direction field of $y' = xy$



3 typical solutions

Fig. 8. Direction field of $y' = 0.1(1 - x^2) - \dfrac{x}{y}$

On Numerics

Direction fields give "all" solutions, but with limited accuracy. If we need accurate numeric values of a solution (or of several solutions) for which we have no formula, we can use a **numeric method.** If you want to get an idea of how these methods work, go to Sec. 21.1 and study the first two pages on the **Euler–Cauchy method,** which is typical of more accurate methods later in that section, notably of the classical **Runge–Kutta method.** It would make little sense to interrupt the present flow of ideas by including such methods here; indeed, it would be a duplication of the material in Sec. 21.1. For an excursion to that section you need no extra prerequisites; Sec. 1.1 just discussed is sufficient.

H.W. 8, 14

PROBLEM SET 1.2

1–10 DIRECTION FIELDS, SOLUTION CURVES

Graph a direction field (by a CAS or by hand). In the field graph approximate solution curves through the given point or points (x, y) by hand.

1. $y' = e^x - y$, $(0, 0)$, $(0, 1)$

2. $4yy' = -9x$, $(2, 2)$

3. $y' = 1 + y^2$, $(\frac{1}{4}\pi, 1)$

4. $y' = y - 2y^2$, $(0, 0)$, $(0, 0.25)$, $(0, 0.5)$, $(0, 1)$

5. $y' = x^2 - 1/y$, $(1, -2)$

6. $y' = 1 + \sin y$, $(-1, 0)$, $(1, -4)$

7. $y' = y^3 + x^3$, $(0, 1)$

8. $y' = 2xy + 1$, $(-1, 2)$, $(0, 0)$, $(1, -2)$

9. $y' = y \tanh x - 2$, $(-1, -2)$, $(1, 0)$, $(1, 2)$

10. $y' = e^{y/x}$, $(1, 1)$, $(2, 2)$, $(3, 3)$

11–15 ACCURACY

Direction fields are very useful because you can see solutions (as many as you want) without solving the ODE, which may be difficult or impossible in terms of a formula. To get a feel for the accuracy of the method, graph a field, sketch solution curves in it, and compare them with the exact solutions.

11. $y' = \sin \frac{1}{2}\pi x$

12. $y' = 1/x^2$

13. $y' = -2y$ (Sol. $y = ce^{-2x}$)

14. $y' = 3y/x$ (Sol. $y = cx^3$)

15. $y' = -\ln x$

16–18 MOTIONS

A body moves on a straight line, with velocity as given, and $y(t)$ is its distance from a fixed point 0 and t time. Find a model of the motion (an ODE). Graph a direction field.

In it sketch a solution curve corresponding to the given initial condition.

16. Velocity equal to the reciprocal of the distance, $y(1) = 1$

17. Product of velocity and distance equal to $-t$, $y(3) = -3$

18. Velocity plus distance equal to the square of time, $y(0) = 6$

19. (Skydiver) Two forces act on a parachutist, the attraction by the earth mg (m = mass of person plus equipment, $g = 9.8 \text{ m/sec}^2$ the acceleration of gravity) and the air resistance, assumed to be proportional to the square of the velocity $v(t)$. Using **Newton's second law** of motion (mass × acceleration = resultant of the forces), set up a model (an ODE for $v(t)$). Graph a direction field (choosing m and the constant of proportionality equal to 1). Assume that the parachute opens when $v = 10$ m/sec. Graph the corresponding solution in the field. What is the limiting velocity?

20. CAS PROJECT. Direction Fields. Discuss direction fields as follows.

(a) Graph a direction field for the ODE $y' = 1 - y$ and in it the solution satisfying $y(0) = 5$ showing **exponential approach.** Can you see the limit of any solution directly from the ODE? For what initial condition will the solution be increasing? Constant? Decreasing?

(b) What do the solution curves of $y' = -x^3/y^3$ look like, as concluded from a direction field. How do they seem to differ from circles? What are the isoclines? What happens to those curves when you drop the minus on the right? Do they look similar to familiar curves? First, guess.

(c) Compare, as best as you can, the old and the computer methods, their advantages and disadvantages. Write a short report.

1.3 Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

$$g(y)y' = f(x) \qquad (1)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

$$\int g(y)\, y'\, dx = \int f(x)\, dx + c. \qquad (2)$$

On the left we can switch to y as the variable of integration. By calculus, $y'\, dx = dy$, so that

$$\int g(y)\, dy = \int f(x)\, dx + c. \qquad (3)$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables,** and (1) is called a **separable equation,** because in (3) the variables are now separated: x appears only on the right and y only on the left.

EXAMPLE 1 A Separable ODE

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1 + y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

積分常数는 적분할시에 바로넣을것

It is very important to introduce the constant of integration immediately when the integration is performed. If we wrote arctan $y = x$, then $y = \tan x$, and *then* introduced c, we would have obtained $y = \tan x + c$, which is not a solution (when $c \neq 0$). Verify this. ■

Modeling

The importance of modeling was emphasized in Sec. 1.1, and separable equations yield various useful models. Let us discuss this in terms of some typical examples.

EXAMPLE 2 Radiocarbon Dating[2]

미라 신석기시대

In September 1991 the famous Iceman (Oetzi), a mummy from the Neolithic period of the Stone Age found in the ice of the Oetztal Alps (hence the name "Oetzi") in Southern Tyrolia near the Austrian–Italian border, caused a scientific sensation. When did Oetzi approximately live and die if the ratio of carbon ${}_6C^{14}$ to carbon ${}_6C^{12}$ in this mummy is 52.5% of that of a living organism?

cosmic : 宇宙의

Physical Information. In the atmosphere and in living organisms, the ratio of radioactive carbon ${}_6C^{14}$ (made radioactive by cosmic rays) to ordinary carbon ${}_6C^{12}$ is constant. When an organism dies, its absorption of ${}_6C^{14}$ by breathing and eating terminates. Hence one can estimate the age of a fossil by comparing the radioactive carbon ratio in the fossil with that in the atmosphere. To do this, one needs to know the half-life of ${}_6C^{14}$, which is 5715 years (*CRC Handbook of Chemistry and Physics,* 83rd ed., Boca Raton: CRC Press, 2002, page 11–52, line 9).

Solution. ***Modeling.*** Radioactive decay is governed by the ODE $y' = ky$ (see Sec. 1.1, Example 5). By separation and integration (where t is time and y_0 is the initial ratio of ${}_6C^{14}$ to ${}_6C^{12}$)

$$\frac{dy}{y} = k\,dt, \qquad \ln|y| = kt + c, \qquad y = y_0 e^{kt}.$$

Next we use the half-life $H = 5715$ to determine k. When $t = H$, half of the original substance is still present. Thus,

$$y_0 e^{kH} = 0.5y_0, \qquad e^{kH} = 0.5, \qquad k = \frac{\ln 0.5}{H} = -\frac{0.693}{5715} = -0.0001213.$$

Finally, we use the ratio 52.5% for determining the time t when Oetzi died (actually, was killed),

$$e^{kt} = e^{-0.0001213t} = 0.525, \qquad t = \frac{\ln 0.525}{-0.0001213} = 5312. \qquad \textit{Answer:} \quad \text{About 5300 years ago.}$$

Other methods show that radiocarbon dating values are usually too small. According to recent research, this is due to a variation in that carbon ratio because of industrial pollution and other factors, such as nuclear testing. ■

EXAMPLE 3 Mixing Problem

염수

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 9 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.

Solution. ***Step 1. Setting up a model.*** Let $y(t)$ denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = \text{Salt inflow rate} - \text{Salt outflow rate} \qquad \textbf{"Balance law".}$$

1%의 양들가 유출됨

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is 10/1000 = 0.01 (= 1%) of the total brine content in the tank, hence 0.01 of the salt content $y(t)$, that is, $0.01y(t)$. Thus the model is the ODE

$$y' = 50 - 0.01y = -0.01(y - 5000). \tag{4}$$

1%의 양이 유출된 salt

[2]Method by WILLARD FRANK LIBBY (1908–1980), American chemist, who was awarded for this work the 1960 Nobel Prize in chemistry.

Step 2. Solution of the model. The ODE (4) is separable. Separation, integration, and taking exponents on both sides gives

$$\frac{dy}{y-5000} = -0.01\,dt, \qquad \ln|y-5000| = -0.01t + c^*, \qquad y - 5000 = ce^{-0.01t}.$$

Initially the tank contains 100 lb of salt. Hence $y(0) = 100$ is the initial condition that will give the unique solution. Substituting $y = 100$ and $t = 0$ in the last equation gives $100 - 5000 = ce^0 = c$. Hence $c = -4900$. Hence the amount of salt in the tank at time t is

$$y(t) = 5000 - 4900e^{-0.01t}. \tag{5}$$

This function shows an exponential approach to the limit 5000 lb; see Fig. 9. Can you explain physically that $y(t)$ should increase with time? That its limit is 5000 lb? Can you see the limit directly from the ODE?

The model discussed becomes more realistic in problems on pollutants in lakes (see Problem Set 1.5, Prob. 27) or drugs in organs. These types of problems are more difficult because the mixing may be imperfect and the flow rates (in and out) may be different and known only very roughly. ■

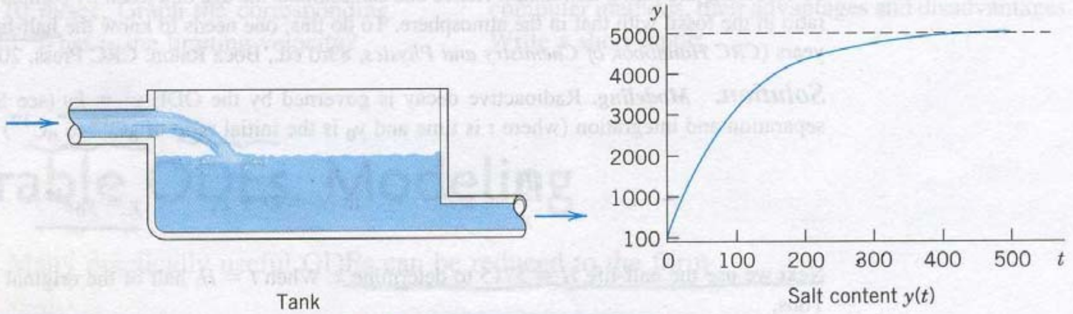


Fig. 9. Mixing problem in Example 3

EXAMPLE 4 Heating an Office Building (Newton's Law of Cooling[3])

ON OFF 65°F
6AM 10pM 2AM

Suppose that in Winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was 50°F at 10 P.M. and had dropped to 40°F by 6 A.M. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Physical information. Experiments show that the time rate of change of the temperature T of a body B (which conducts heat well, as, for example, a copper ball does) is proportional to the difference between T and the temperature of the surrounding medium (**Newton's law of cooling**).

Solution. ***Step 1. Setting up a model.*** Let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's law). Then by Newton's law,

$$\frac{dT}{dt} = k(T - T_A). \tag{6}$$

Such experimental laws are derived under idealized assumptions that rarely hold exactly. However, even if a model seems to fit the reality only poorly (as in the present case), it may still give valuable qualitative information. To see how good a model is, the engineer will collect experimental data and compare them with calculations from the model.

[3]Sir ISAAC NEWTON (1642–1727), great English physicist and mathematician, became a professor at Cambridge in 1669 and Master of the Mint in 1699. He and the German mathematician and philosopher GOTTFRIED WILHELM LEIBNIZ (1646–1716) invented (independently) the differential and integral calculus. Newton discovered many basic physical laws and created the method of investigating physical problems by means of calculus. His *Philosophiae naturalis principia mathematica* (*Mathematical Principles of Natural Philosophy,* 1687) contains the development of classical mechanics. His work is of greatest importance to both mathematics and physics.

Step 2. General solution. We cannot solve (6) because we do not know T_A, just that it varied between 50°F and 40°F, so we follow the ***Golden Rule:*** *If you cannot solve your problem, try to solve a simpler one.* We solve (6) with the unknown function T_A replaced with the average of the two known values, or 45°F. For physical reasons we may expect that this will give us a reasonable approximate value of T in the building at 6 A.M.

야간중 최기온도
중간값 45°

For constant $T_A = 45$ (or any other *constant* value) the ODE (6) is separable. Separation, integration, and taking exponents gives the general solution

$$\frac{dT}{T-45} = k\,dt, \qquad \ln|T-45| = kt + c^*, \qquad T(t) = 45 + ce^{kt} \qquad (c = e^{c^*}).$$

Step 3. Particular solution. We choose 10 P.M. to be $t = 0$. Then the given initial condition is $T(0) = 70$ and yields a particular solution, call it T_p. By substitution,

$$T(0) = 45 + ce^0 = 70, \qquad c = 70 - 45 = 25, \qquad T_p(t) = 45 + 25e^{kt}.$$

10PM 4시간 뒤를

Step 4. Determination of k. We use $T(4) = 65$, where $t = 4$ is 2 A.M. Solving algebraically for k and inserting k into $T_p(t)$ gives (Fig. 10)

$$T_p(4) = 45 + 25e^{4k} = 65, \qquad e^{4k} = 0.8, \qquad k = \tfrac{1}{4}\ln 0.8 = -0.056, \qquad T_p(t) = 45 + 25e^{-0.056t}.$$

Step 5. Answer and interpretation. 6 A.M. is $t = 8$ (namely, 8 hours after 10 P.M.), and

$$T_p(8) = 45 + 25e^{-0.056 \cdot 8} = 61[°\text{F}].$$

Hence the temperature in the building dropped 9°F, a result that looks reasonable. ■

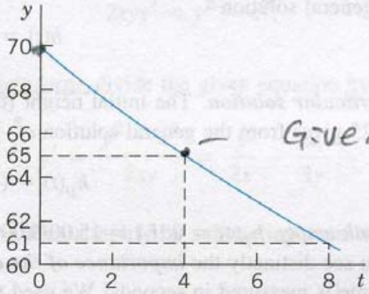


Fig. 10. Particular solution (temperature) in Example 4

EXAMPLE 5 Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law)

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 11). You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Physical information. Under the influence of gravity the outflowing water has velocity

$$(7) \qquad v(t) = 0.600\sqrt{2gh(t)} \qquad \textbf{(Torricelli's law}^4\textbf{)},$$

where $h(t)$ is the height of the water above the hole at time t, and $g = 980 \text{ cm/sec}^2 = 32.17 \text{ ft/sec}^2$ is the acceleration of gravity at the surface of the earth.

Solution. ***Step 1. Setting up the model.*** To get an equation, we relate the decrease in water level $h(t)$ to the outflow. The volume ΔV of the outflow during a short time Δt is

$$\Delta V = Av\,\Delta t \qquad (A = \text{Area of hole}).$$

[4]EVANGELISTA TORRICELLI (1608–1647), Italian physicist, pupil and successor of GALILEO GALILEI (1564–1642) at Florence. The "contraction factor" 0.600 was introduced by J. C. BORDA in 1766 because the stream has a smaller cross section than the area of the hole.

ΔV must equal the change ΔV^* of the volume of the water in the tank. Now

$$\Delta V^* = -B\,\Delta h \qquad (B = \text{Cross-sectional area of tank})$$

where Δh (> 0) is the decrease of the height $h(t)$ of the water. The minus sign appears because the volume of the water in the tank decreases. Equating ΔV and ΔV^* gives

$$-B\,\Delta h = Av\,\Delta t.$$

We now express v according to Torricelli's law and then let Δt (the length of the time interval considered) approach 0—this is a ***standard way*** of obtaining an ODE as a model. That is, we have

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}\,v = -\frac{A}{B}\,0.600\sqrt{2gh(t)},$$

and by letting $\Delta t \to 0$ we obtain the ODE

$$\frac{dh}{dt} = -26.56\,\frac{A}{B}\,\sqrt{h},$$

where $26.56 = 0.600\,\sqrt{2 \cdot 980}$. This is our model, a first-order ODE.

Step 2. General solution. Our ODE is separable. A/B is constant. Separation and integration gives

$$\frac{dh}{\sqrt{h}} = -26.56\,\frac{A}{B}\,dt \qquad \text{and} \qquad 2\sqrt{h} = c^* - 26.56\,\frac{A}{B}\,t.$$

Dividing by 2 and squaring gives $h = (c - 13.28At/B)^2$. Inserting $13.28A/B = 13.28 \cdot 0.5^2\pi/100^2\pi = 0.000332$ yields the general solution

$$h(t) = (c - 0.000332t)^2.$$

Step 3. Particular solution. The initial height (the initial condition) is $h(0) = 225$ cm. Substitution of $t = 0$ and $h = 225$ gives from the general solution $c^2 = 225$, $c = 15.00$ and thus the particular solution (Fig. 11)

$$h_p(t) = (15.00 - 0.000332t)^2.$$

Step 4. Tank empty. $h_p(t) = 0$ if $t = 15.00/0.000332 = 45\,181$ [sec] $= 12.6$ [hours].

Here you see distinctly the ***importance of the choice of units***—we have been working with the Cgs system, in which time is measured in seconds! We used $g = 980$ cm/sec^2.

Step 5. Checking. Check the result. ■

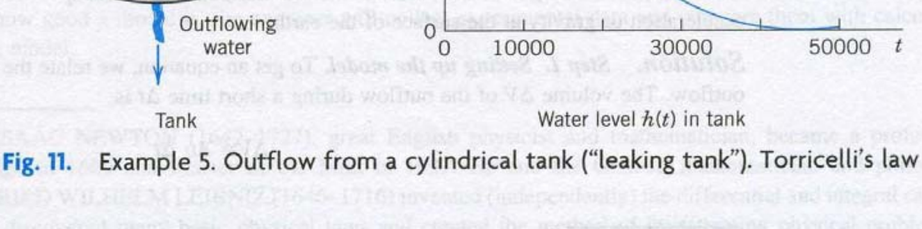


Fig. 11. Example 5. Outflow from a cylindrical tank ("leaking tank"). Torricelli's law

Extended Method: Reduction to Separable Form

Certain nonseparable ODEs can be made separable by transformations that introduce for y a new unknown function. We discuss this technique for a class of ODEs of practical importance, namely, for equations

$$y' = f\left(\frac{y}{x}\right). \tag{8}$$

Here, f is any (differentiable) function of y/x, such as $\sin(y/x)$, $(y/x)^4$, and so on. (Such an ODE is sometimes called a *homogeneous ODE*, a term we shall not use but reserve for a more important purpose in Sec. 1.5.)

$y/x = u$

The form of such an ODE suggests that we set $y/x = u$; thus,

$$y = ux \quad \text{and by product differentiation} \quad y' = u'x + u. \tag{9}$$

Substitution into $y' = f(y/x)$ then gives $u'x + u = f(u)$ or $u'x = f(u) - u$. We see that this can be separated:

$$\frac{du}{f(u) - u} = \frac{dx}{x}. \tag{10}$$

$u' = \dfrac{du}{dx}$

EXAMPLE 6 Reduction to Separable Form

Solve

$$2xyy' = y^2 - x^2.$$

Solution. To get the usual explicit form, divide the given equation by $2xy$,

$$y' = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}.$$

Now substitute y and y' from (9) and then simplify by subtracting u on both sides,

$$u'x + u = \frac{u}{2} - \frac{1}{2u}, \qquad u'x = -\frac{u}{2} - \frac{1}{2u} = \frac{-u^2 - 1}{2u}.$$

$C = e^{c^*}$

You see that in the last equation you can now separate the variables,

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}. \quad \text{By integration,} \quad \ln(1 + u^2) = -\ln|x| + c^* = \ln\left|\frac{1}{x}\right| + c^*.$$

Take exponents on both sides to get $1 + u^2 = c/x$ or $1 + (y/x)^2 = c/x$. Multiply the last equation by x^2 to obtain (Fig. 12)

$$x^2 + y^2 = cx. \qquad \text{Thus} \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

This general solution represents a family of circles passing through the origin with centers on the x-axis. ■

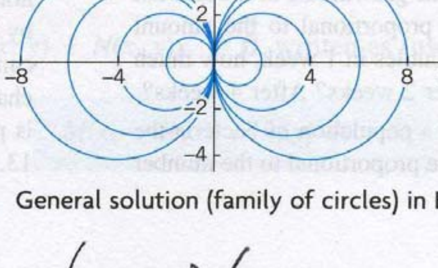


Fig. 12. General solution (family of circles) in Example 6

H.W. 6, 26

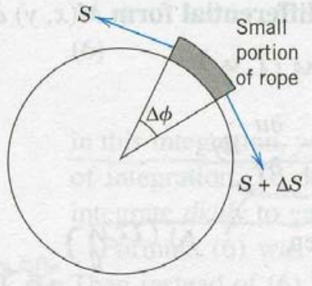


Fig. 13. Problem 32

33. (Mixing) A tank contains 800 gal of water in which 200 lb of salt is dissolved. Two gallons of fresh water runs in per minute, and 2 gal of the mixture in the tank, kept uniform by stirring, runs out per minute. How much salt is left in the tank after 5 hours?

34. WRITING PROJECT. Exponential Increase, Decay, Approach. Collect, order, and present all the information on the ODE $y' = ky$ and its applications from the text and the problems. Add examples of your own.

35. CAS EXPERIMENT. Graphing Solutions. A CAS can usually graph solutions even if they are given by integrals that cannot be evaluated by the usual methods of calculus. Show this as follows.

(A) Graph the curves for the seven initial value problems $y' = e^{-x^2/2}$, $y(0) = 0, \pm 1, \pm 2, \pm 3$, common axes. Are these curves congruent? Why?

(B) Experiment with approximate curves of nth partial sums of the Maclaurin series obtained by termwise integration of that of y in (A); graph them and describe qualitatively the accuracy for a fixed interval $0 \leqq x \leqq b$ and increasing n, and then for fixed n and increasing b.

(C) Experiment with $y' = \cos(x^2)$ as in (B).

(D) Find an initial value problem with solution $y = e^{x^2}\int_0^x e^{-t^2}\,dt$ and experiment with it as in (B).

36. TEAM PROJECT. Torricelli's Law. Suppose that the tank in Example 5 is hemispherical, of radius R, initially full of water, and has an outlet of 5 cm² cross-sectional area at the bottom. (Make a sketch.) Set up the model for outflow. Indicate what portion of your work in Example 5 you can use (so that it can become part of the general method independent of the shape of the tank). Find the time t to empty the tank (a) for any R, (b) for $R = 1$ m. Plot t as function of R. Find the time when $h = R/2$ (a) for any R, (b) for $R = 1$ m.

1.4 Exact ODEs. Integrating Factors

We remember from calculus that if a function $u(x, y)$ has continuous partial derivatives, its **differential** (also called its *total differential*) is

全微分

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

From this it follows that if $u(x, y) = c = const$, then $du = 0$.

For example, if $u = x + x^2y^3 = c$, then

$$du = (1 + 2xy^3)\,dx + 3x^2y^2\,dy = 0$$

or

$$y' = \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2},$$

an ODE that we can solve by going backward. This idea leads to a powerful solution method as follows.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'\,dx$ as in Sec. 1.3)

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact,** that is, this form is the differential

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \tag{2}$$

M(x,y)

N(x,y)

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

$$u(x, y) = c. \tag{3}$$

This is called an **implicit solution,** in contrast with a solution $y = h(x)$ as defined in Sec. 1.1, which is also called an *explicit solution,* for distinction. Sometimes an implicit solution can be converted to explicit form. (Do this for $x^2 + y^2 = 1$.) If this is not possible, your CAS may graph a figure of the **contour lines** (3) of the function $u(x, y)$ and help you in understanding the solution.

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

$$\text{(a)}\quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)}\quad \frac{\partial u}{\partial y} = N. \tag{4}$$

From this we can derive a formula for checking whether (1) is exact or not, as follows.

Exactness의 check 방법

Let M and N be continuous and have continuous first partial derivatives in a region in the xy-plane whose boundary is a closed curve without self-intersections. Then by partial differentiation of (4) (see App. 3.2 for notation),

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

By the assumption of continuity the two second partial derivatives are equal. Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{5}$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation. (We shall prove this in Sec. 10.2 in another context. Some calculus books (e.g., Ref. [GR11] also contain a proof.)

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

(6)의 결론에
$\frac{\partial u}{\partial y}$ 적용
$\Rightarrow \frac{dk}{dy}$ 구함
$\Rightarrow$ 적분하여 k를 구함

$$u = \int M\,dx + k(y); \tag{6}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k.

Formula (6) was obtained from (4a). Instead of (4a) we may equally well use (4b). Then instead of (6) we first have by integration with respect to y

$$u = \int N\,dy + l(x). \tag{6*}$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

EXAMPLE 1 An Exact ODE

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{7}$$

Solution. *Step 1.* *Test for exactness.* Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y). \tag{8}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$\frac{\partial u}{\partial y} \triangleq N$

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$\sin(x+y) + y^3 + y^2 + c^* = c$

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

$c - c^* \to c$

Step 3. Checking an implicit solution. We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = \cos(x + y)\,dx + (\cos(x + y) + 3y^2 + 2y)\,dy = 0. \tag{9}$$

This completes the check. ■

EXAMPLE 2 An Initial Value Problem

Solve the initial value problem

$$(\cos y \sinh x + 1)\,dx - \sin y \cosh x\,dy = 0, \qquad y(1) = 2. \tag{10}$$

Solution. You may verify that the given ODE is exact. We find u. For a change, let us use (6*),

$$u = -\int \sin y \cosh x\,dy + l(x) = \cos y \cosh x + l(x).$$

From this, $\partial u/\partial x = \cos y \sinh x + dl/dx = M = \cos y \sinh x + 1$. Hence $dl/dx = 1$. By integration, $l(x) = x + c^*$. This gives the general solution $u(x, y) = \cos y \cosh x + x = c$. From the initial condition, $\cos 2 \cosh 1 + 1 = 0.358 = c$. Hence the answer is $\cos y \cosh x + x = 0.358$. Figure 14 shows the particular solutions for $c = 0, 0.358$ (thicker curve), 1, 2, 3. Check that the answer satisfies the ODE. (Proceed as in Example 1.) Also check that the initial condition is satisfied. ■

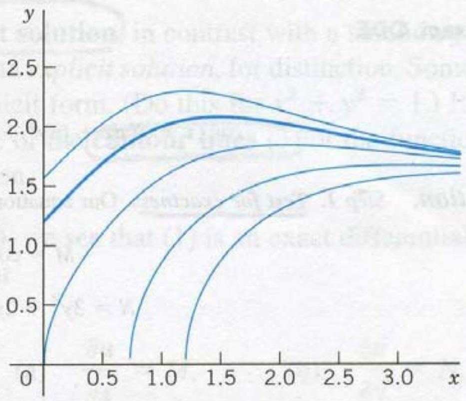


Fig. 14. Particular solu.ions in Example 2

EXAMPLE 3 WARNING! Breakdown in the Case of Nonexactness

The equation $-y\,dx + x\,dy = 0$ is not exact because $M = -y$ and $N = x$, so that in (5), $\partial M/\partial y = -1$ but $\partial N/\partial x = 1$. Let us show that in such a case the present method does not work. From (6),

$$u = \int M\,dx + k(y) = -xy + k(y), \qquad \text{hence} \qquad \frac{\partial u}{\partial y} = -x + \frac{dk}{dy}.$$

$-x + \dfrac{dk}{dy} = x$

$\dfrac{dk}{dy} = 2x$

y만의 함수이므로
불가능

Now, $\partial u/\partial y$ should equal $N = x$, by (4b). However, this is impossible because $k(y)$ can depend only on y. Try (6*); it will also fail. Solve the equation by another method that we have discussed. ■

Reduction to Exact Form. Integrating Factors

The ODE in Example 3 is $-y\,dx + x\,dy = 0$. It is not exact. However, if we multiply it by $1/x^2$, we get an exact equation [check exactness by (5)!],

$$\frac{-y\,dx + x\,dy}{x^2} = -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = d\left(\frac{y}{x}\right) = 0. \tag{11}$$

Integration of (11) then gives the general solution $y/x = c = const.$

This example gives the idea. All we did was multiply a given nonexact equation, say,

$$P(x, y)\, dx + Q(x, y)\, dy = 0, \tag{12}$$

by a function F that, in general, will be a function of both x and y. The result was an equation

$$FP\, dx + FQ\, dy = 0 \tag{13}$$

that is exact, so we can solve it as just discussed. Such a function $F(x, y)$ is then called an **integrating factor** of (12).

EXAMPLE 4 Integrating Factor

The integrating factor in (11) is $F = 1/x^2$. Hence in this case the exact equation (13) is

$$FP\, dx + FQ\, dy = \frac{-y\, dx + x\, dy}{x^2} = d\left(\frac{y}{x}\right) = 0. \qquad \text{Solution} \qquad \frac{y}{x} = c.$$

These are straight lines $y = cx$ through the origin.

It is remarkable that we can readily find other integrating factors for the equation $-y\, dx + x\, dy = 0$, namely, $1/y^2$, $1/(xy)$, and $1/(x^2 + y^2)$, because

$$\frac{-y\, dx + x\, dy}{y^2} = d\left(\frac{x}{y}\right), \quad \frac{-y\, dx + x\, dy}{xy} = -d\left(\ln \frac{x}{y}\right), \quad \frac{-y\, dx + x\, dy}{x^2 + y^2} = d\left(\arctan \frac{y}{x}\right). \tag{14}$$ ■

How to Find Integrating Factors

In simpler cases we may find integrating factors by inspection or perhaps after some trials, keeping (14) in mind. In the general case, the idea is the following.

For $M\, dx + N\, dy = 0$ the exactness condition (4) is $\partial M/\partial y = \partial N/\partial x$. Hence for (13), $FP\, dx + FQ\, dy = 0$, the exactness condition is

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ). \tag{15}$$

By the product rule, with subscripts denoting partial derivatives, this gives

$$F_y P + F P_y = F_x Q + F Q_x.$$

쉽게 풀려고
노력하라!!

In the general case, this would be complicated and useless. So we follow the *Golden Rule: If you cannot solve your problem, try to solve a simpler one*—the result may be useful (and may also help you later on). Hence we look for an integrating factor depending only on *one* variable; fortunately, in many practical cases, there are such factors, as we shall see. Thus, let $F = F(x)$. Then $F_y = 0$, and $F_x = F' = dF/dx$, so that (15) becomes

$$FP_y = F'Q + FQ_x.$$

FQ로 나누고
재정리

Dividing by FQ and reshuffling terms, we have

$$\frac{1}{F}\frac{dF}{dx} = R, \qquad \text{where} \qquad R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right). \tag{16}$$

This proves the following theorem.

THEOREM 1

Integrating Factor $F(x)$

If (12) is such that the right side R of (16), depends only on x, then (12) has an integrating factor $F = F(x)$, which is obtained by integrating (16) and taking exponents on both sides,

$$F(x) = \exp \int R(x)\, dx. \tag{17}$$

Similarly, if $F^* = F^*(y)$, then instead of (16) we get

$$\frac{1}{F^*}\frac{dF^*}{dy} = R^*, \qquad \text{where} \qquad R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \tag{18}$$

and we have the companion

THEOREM 2

Integrating Factor $F^*(y)$

If (12) is such that the right side R^* of (18) depends only on y, then (12) has an integrating factor $F^* = F^*(y)$, which is obtained from (18) in the form

$$F^*(y) = \exp \int R^*(y)\, dy. \tag{19}$$

EXAMPLE 5 **Application of Theorems 1 and 2. Initial Value Problem**

Using Theorem 1 or 2, find an integrating factor and solve the initial value problem

$$(e^{x+y} + ye^y)\, dx + (xe^y - 1)\, dy = 0, \quad y(0) = -1 \tag{20}$$

Solution. ***Step 1. Nonexactness.*** The exactness check fails:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^{x+y} + ye^y) = e^{x+y} + e^y + ye^y \qquad \text{but} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y - 1) = e^y.$$

Step 2. Integrating factor. General solution. Theorem 1 fails because R [the right side of (16)] depends on both x and y,

$$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y).$$

Try Theorem 2. The right side of (18) is

$$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{e^{x+y} + ye^y}(e^y - e^{x+y} - e^y - ye^y) = -1.$$

Hence (19) gives the integrating factor $F^*(y) = e^{-y}$. From this result and (20) you get the exact equation

$$(e^x + y)\, dx + (x - e^{-y})\, dy = 0.$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$

Test for exactness; you will get 1 on both sides of the exactness condition. By integration, using (4a),

$$u = \int (e^x + y)\, dx = e^x + xy + k(y).$$

k(y)

Differentiate this with respect to y and use (4b) to get

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N = x + e^{-y}, \qquad \frac{dk}{dy} = -e^{-y}, \qquad k = e^{-y} + c^*.$$

Hence the general solution is

$$u(x, y) = e^x + xy + e^{-y} = c.$$

Step 3. Particular solution. The initial condition $y(0) = 1$ gives $u(0, -1) = 1 + 0 + e = 3.72$. Hence the answer is $e^x + xy + e^{-y} = 1 + e = 3.72$. Figure 15 shows several particular solutions obtained as level curves of $u(x, y) = c$, obtained by a CAS, a convenient way in cases in which it is impossible or difficult to cast a solution into explicit form. Note the curve that (nearly) satisfies the initial condition.

Step 4. Checking. Check by substitution that the answer satisfies the given equation as well as the initial condition. ■

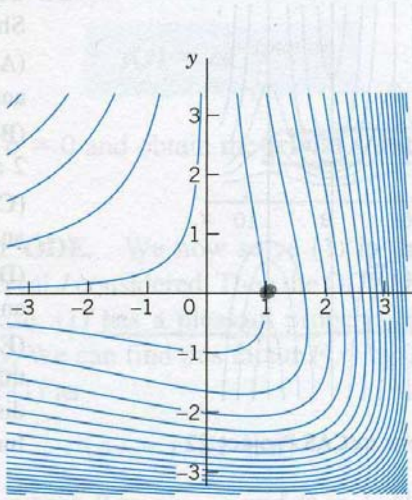


Fig. 15. Particular solutions in Example 5

H.W. 8 , 20

PROBLEM SET 1.4

1–20 EXACT ODEs. INTEGRATING FACTORS

Test for exactness. If exact, solve. If not, use an integrating factor as given or find it by inspection or from the theorems in the text. Also, if an initial condition is given, determine the corresponding particular solution.

1. $x^3\,dx + y^3\,dy = 0$ **2.** $(x - y)(dx - dy) = 0$

3. $-\pi \sin \pi x \sinh y\,dx + \cos \pi x \cosh y\,dy = 0$

4. $(e^y - ye^x)\,dx + (xe^y - e^x)\,dy = 0$

5. $9x\,dx + 4y\,dy = 0$

6. $e^x(\cos y\,dx - \sin y\,dy) = 0$

7. $e^{-2\theta}\,dr - 2re^{-2\theta}\,d\theta = 0$

8. $(2x + 1/y - y/x^2)\,dx + (2y + 1/x - x/y^2)\,dy = 0$

9. $(-y/x^2 + 2\cos 2x)\,dx + (1/x - 2\sin 2y)\,dy = 0$

10. $-2xy \sin(x^2)\,dx + \cos(x^2)\,dy = 0$

11. $-y\,dx + x\,dy = 0$

12. $(e^{x+y} - y)\,dx + (xe^{x+y} + 1)\,dy = 0$

13. $-3y\,dx + 2x\,dy = 0, \quad F(x, y) = y/x^4$

14. $(x^4 + y^2)\,dx - xy\,dy = 0, \quad y(2) = 1$

15. $e^{2x}(2\cos y\,dx - \sin y\,dy) = 0, \quad y(0) = 0$

16. $-\sin xy\,(y\,dx + x\,dy) = 0, \quad y(1) = \pi$

17. $(\cos \omega x + \omega \sin \omega x)\,dx + e^x\,dy = 0,\ y(0) = 1$

18. $(\cos xy + x/y)\,dx + (1 + (x/y)\cos xy)\,dy = 0$

19. $e^{-y}\,dx + e^{-x}(-e^{-y} + 1)\,dy = 0, \quad F = e^{x+y}$

20. $(\sin y \cos y + x \cos^2 y)\,dx + x\,dy = 0$

21. Under what conditions for the constants A, B, C, D is $(Ax + By)\,dx + (Cx + Dy)\,dy = 0$ exact? Solve the exact equation.

22. CAS PROJECT. Graphing Particular Solutions Graph particular solutions of the following ODE, proceeding as explained.

$$y \cos x \, dx + \frac{1}{y} \, dy = 0 \qquad (21)$$

(a) Test for exactness. If necessary, find an integrating factor. Find the general solution $u(x, y) = c$.

(b) Solve (21) by separating variables. Is this simpler than **(a)**?

(c) Graph contours $u(x, y) = c$ by your CAS. (Cf. Fig. 16.)

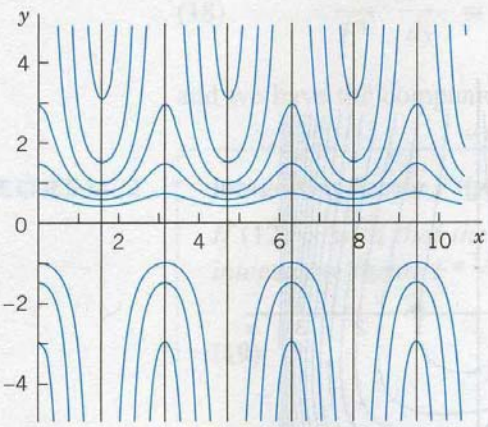


Fig. 16. Particular solutions in CAS Project 22

(d) In another graph show the solution curves satisfying $y(0) = \pm 1, \pm 2, \pm 3, \pm 4$. Compare the quality of **(c)** and **(d)** and comment.

(e) Do the same steps for another nonexact ODE of your choice.

23. WRITING PROJECT. Working Backward. Start from solutions $u(x, y) = c$ of your choice, find a corresponding exact ODE, destroy exactness by a multiplication or division. This should give you a feel for the form of ODEs you can reach by the method of integrating factors. (Working backward is useful in other areas, too; Euler and other great masters frequently did it.)

24. TEAM PROJECT. Solution by Several Methods. Show this as indicated. Compare the amount of work.

(A) $e^y(\sinh x \, dx + \cosh x \, dy) = 0$ as an exact ODE and by separation.

(B) $(1 + 2x) \cos y \, dx + dy/\cos y = 0$ by Theorem 2 and by separation.

(C) $(x^2 + y^2) \, dx - 2xy \, dy = 0$ by Theorem 1 or 2 and by separation with $v = y/x$.

(D) $3x^2 y \, dx + 4x^3 \, dy = 0$ by Theorems 1 and 2 and by separation.

(E) Search the text and the problems for further ODEs that can be solved by more than one of the methods discussed so far. Make a list of these ODEs. Find further cases of your own.

1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

Linear ODEs or ODEs that can be transformed to linear form are models of various phenomena, for instance, in physics, biology, population dynamics, and ecology, as we shall see. A first-order ODE is said to be ***linear*** if it can be written

$$y' + p(x)y = r(x). \qquad \textbf{(1)}$$

p, r 은 independent 변수(x)의 함수

The defining feature of this equation is that it is linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be *any* given functions of x. If in an application the independent variable is time, we write t instead of x.

If the first term is $f(x)y'$ (instead of y'), divide the equation by $f(x)$ to get the **"standard form"** (1), with y' as the first term, which is practical.

For instance, $y' \cos x + y \sin x = x$ is a linear ODE, and its standard form is $y' + y \tan x = x \sec x$.

The function $r(x)$ on the right may be a force, and the solution $y(x)$ a displacement in a motion or an electrical current or some other physical quantity. In engineering, $r(x)$ is frequently called the **input,** and $y(x)$ is called the **output** or the *response* to the input (and, if given, to the initial condition).

Homogeneous Linear ODE. We want to solve (1) in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0 \tag{2}$$

and is called **homogeneous.** By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

$\Rightarrow |y| = e^{c*}\, e^{-\int p(x)\,dx}$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\int p(x)\,dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0); \tag{3}$$

here we may also choose $c = 0$ and obtain the **trivial solution** $y(x) = 0$ for all x in that interval.

$y' + py = r$

Nonhomogeneous Linear ODE. We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called **nonhomogeneous.** It turns out that in this case, (1) has a pleasant property; namely, it has an integrating factor depending only on x. We can find this factor $F(x)$ by Theorem 1 in the last section. For this purpose we write (1) as

$$(py - r)\,dx + dy = 0.$$

This is $P\,dx + Q\,dy = 0$, where $P = py - r$ and $Q = 1$. Hence the right side of (16) in Sec. 1.4 is simply $1(p - 0) = p$, so that (16) becomes

$$\frac{1}{F}\frac{dF}{dx} = p(x).$$

Separation and integration gives

$$\frac{dF}{F} = p\,dx \qquad \text{and} \qquad \ln|F| = \int p\,dx.$$

Taking exponents on both sides, we obtain the desired integrating factor $F(x)$,

$$F(x) = e^{\int p\,dx}.$$

We now multiply (1) on both sides by this F. Then by the product rule,

$$e^{\int p\,dx}(y' + py) = (e^{\int p\,dx}y)' = e^{\int p\,dx}r.$$

By integrating the second and third of these three expressions with respect to x we get

$$e^{\int p\,dx}y = \int e^{\int p\,dx}\,r\,dx + c.$$

Dividing this equation by $e^{\int p\,dx}$ and denoting the exponent $\int p\,dx$ by h, we obtain

$$y(x) = e^{-h}\left(\int e^{h}r\,dx + c\right), \qquad h = \int p(x)\,dx. \tag{4}$$

$h = \int p\,dx$

(The constant of integration in h does not matter; see Prob. 2.) Formula (4) is the general solution of (1) in the form of an integral. Solving (1) is now reduced to the evaluation of an integral. In cases in which this cannot be done by the usual methods of calculus, one may have to use a numeric method for integrals (Sec. 19.5) or for the ODE itself (Sec. 21.1).

The structure of (4) is interesting. The only quantity depending on a given initial condition is c. Accordingly, writing (4) as a sum of two terms,

$$(4^*) \qquad y(x) = e^{-h}\int e^{h} r\, dx + ce^{-h}$$

we see the following:

(5) Total Output = Response to the Input r + Response to the Initial Data.

EXAMPLE 1 First-Order ODE, General Solution

Solve the linear ODE

$y' + py = r$

$$y' - y = e^{2x}.$$

Solution. Here,

$$p = -1, \qquad r = e^{2x}, \qquad h = \int p\, dx = -x$$

and from (4) we obtain the general solution

h e $r(x)$

$$y(x) = e^{x}\left(\int e^{-x}e^{2x}\, dx + c\right) = e^{x}(e^{x} + c) = ce^{x} + e^{2x}.$$

e^{-h}

From (4*) and (5) we see that the response to the input is e^{2x}.

In simpler cases, such as the present, we may not need the general formula (4), but may wish to proceed directly, multiplying the given equation by $e^{h} = e^{-x}$. This gives

e^{-x}; integrating factor

$$(y' - y)e^{-x} = (ye^{-x})' = e^{2x}e^{-x} = e^{x}.$$

Integrating on both sides, we obtain the same result as before:

$$ye^{-x} = e^{x} + c, \qquad \text{hence} \qquad y = e^{2x} + ce^{x}. \qquad \blacksquare$$

EXAMPLE 2 First-Order ODE, Initial Value Problem

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$\int p\, dx = \int \tan x\, dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^{h} = \sec x, \qquad e^{-h} = \cos x, \qquad e^{h}r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2\int \sin x\, dx + c\right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$. $\blacksquare$

Reduction to Linear Form. Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**[5]

$$y' + p(x)y = g(x)y^a \qquad (a \text{ any real number}). \tag{6}$$

If $a = 0$ or $a = 1$, Equation (6) is linear. Otherwise it is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute y' from (6), obtaining

$$u' = (1 - a)y^{-a}y' = (1 - a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' = (1 - a)(g - py^{1-a}),$$

where $y^{1-a} = u$ on the right, so that we get the linear ODE

$$u' + (1 - a)pu = (1 - a)g. \tag{7}$$

; linear

For further ODEs reducible to linear from, see Ince's classic [A11] listed in App. 1. See also Team Project 44 in Problem Set 1.5.

EXAMPLE 4 Logistic Equation

Solve the following Bernoulli equation, known as the **logistic equation** (or **Verhulst equation**[6]):

$$y' = Ay - By^2 \tag{8}$$

Solution. Write (8) in the form (6), that is,

$$y' - Ay = -By^2$$

to see that $a = 2$, so that $u = y^{1-a} = y^{-1}$. Differentiate this u and substitute y' from (8),

$$u' = -y^{-2}y' = -y^{-2}(Ay - By^2) = B - Ay^{-1}.$$

The last term is $-Ay^{-1} = -Au$. Hence we have obtained the linear ODE

---

[5]JAKOB BERNOULLI (1654–1705), Swiss mathematician, professor at Basel, also known for his contribution to elasticity theory and mathematical probability. The method for solving Bernoulli's equation was discovered by the Leibniz in 1696. Jakob Bernoulli's students included his nephew NIKLAUS BERNOULLI (1687–1759), who contributed to probability theory and infinite series, and his youngest brother JOHANN BERNOULLI (1667–1748), who had profound influence on the development of calculus, became Jakob's successor at Basel, and had among his students GABRIEL CRAMER (see Sec. 7.7) and LEONHARD EULER (see Sec. 2.5). His son DANIEL BERNOULLI (1700–1782) is known for his basic work in fluid flow and the kinetic theory of gases.

[6]PIERRE-FRANÇOIS VERHULST, Belgian statistician, who introduced Eq. (8) as a model for human population growth in 1838.

$$u' + Au = B.$$

The general solution is [by (4)]

$$u = ce^{-At} + B/A.$$

Since $u = 1/y$, this gives the general solution of (8),

$$y = \frac{1}{u} = \frac{1}{ce^{-At} + B/A} \qquad \text{(Fig. 18).} \tag{9}$$

Directly from (8) we see that $y \equiv 0$ ($y(t) = 0$ for all t) is also a solution. ■

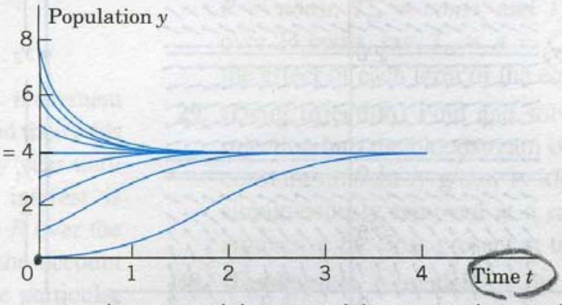


Fig. 18. Logistic population model. Curves (9) in Example 4 with $A/B = 4$

Population Dynamics

✓ B=0 : growing

The logistic equation (8) plays an important role in **population dynamics**, a field that models the evolution of populations of plants, animals, or humans over time t. If $B = 0$, then (8) is $y' = dy/dt = Ay$. In this case its solution (9) is $y = (1/c)e^{At}$ and gives exponential growth, as for a small population in a large country (the United States in early times!). This is called ***Malthus's law.*** (See also Example 3 in Sec. 1.1.)

✓ $y' > 0$ if $y < A/B$

✓ $y' < 0$ if $y > A/B$

The term $-By^2$ in (8) is a "braking term" that prevents the population from growing without bound. Indeed, if we write $y' = Ay[1 - (B/A)y]$, we see that if $y < A/B$, then $y' > 0$, so that an initially small population keeps growing as long as $y < A/B$. But if $y > A/B$, then $y' < 0$ and the population is decreasing as long as $y > A/B$. The limit is the same in both cases, namely, A/B. See Fig. 18.

We see that in the logistic equation (8) the independent variable t does not occur explicitly. An ODE $y' = f(t, y)$ in which t does not occur explicitly is of the form

$$y' = f(y) \tag{10}$$

$y' + py = r$ 에서 p, r 이 constant 인 경우

✓ 시간 변수가 수식에 없다.

and is called an **autonomous ODE.** Thus the logistic equation (8) is autonomous.

Equation (10) has constant solutions, called **equilibrium solutions** or **equilibrium points.** These are determined by the zeros of $f(y)$, because $f(y) = 0$ gives $y' = 0$ by (10); hence $y = const.$ These zeros are known as **critical points** of (10). An equilibrium solution is called **stable** if solutions close to it for some t remain close to it for all further t. It is called **unstable** if solutions initially close to it do not remain close to it as t increases. For instance, $y = 0$ in Fig. 18 is an unstable equilibrium solution, and $y = 4$ is a stable one.

EXAMPLE 5 **Stable and Unstable Equilibrium Solutions. "Phase Line Plot"**

The ODE $y' = (y-1)(y-2)$ has the stable equilibrium solution $y_1 = 1$ and the unstable $y_2 = 2$, as the direction field in Fig. 19 suggests. The values y_1 and y_2 are the zeros of the parabola $f(y) = (y-1)(y-2)$ in the figure. Now, since the ODE is autonomous, we can "condense" the direction field to a "phase line plot" giving y_1 and y_2, and the direction (upward or downward) of the arrows in the field, and thus giving information about the stability or instability of the equilibrium solutions. ■

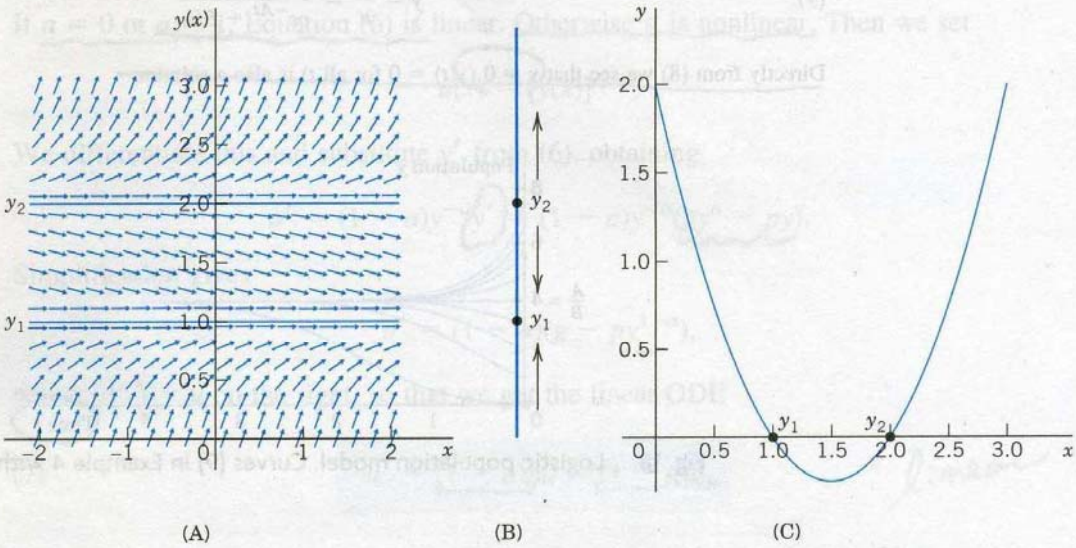


Fig. 19. Example 5. (A) Direction field. (B) "Phase line". (C) Parabola $f(y)$

A few further population models will be discussed in the problem set. For some more details of population dynamics, see C. W. Clark, *Mathematical Bioeconomics*, New York, Wiley, 1976.

Further important applications of linear ODEs follow in the next section.

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PROBLEM SET 1.5

1. **(CAUTION!)** Show that $e^{-\ln x} = 1/x$ (not $-x$) and $e^{-\ln(\sec x)} = \cos x$.
2. **(Integration constant)** Give a reason why in (4) you may choose the constant of integration in $\int p\,dx$ to be zero.

3–17 GENERAL SOLUTION. INITIAL VALUE PROBLEMS

Find the general solution. If an initial condition is given, find also the corresponding particular solution and graph or sketch it. (Show the details of your work.)

3. $y' + 3.5y = 2.8$
4. $y' = 4y + x$
5. $y' + 1.25y = 5, \quad y(0) = 6.6$
6. $x^2y' + 3xy = 1/x, \quad y(1) = -1$
7. $y' + ky = e^{2kx}$
8. $y' + 2y = 4\cos 2x, \quad y(\tfrac{1}{4}\pi) = 2$
9. $y' = 6(y - 2.5)\tanh 1.5x$
10. $y' + 4x^2y = (4x^2 - x)e^{-x^2/2}$
11. $y' + 2y\sin 2x = 2e^{\cos 2x}, \quad y(0) = 0$
12. $y'\tan x = 2y - 8, \quad y(\tfrac{1}{2}\pi) = 0$
13. $y' + 4y\cot 2x = 6\cos 2x, \quad y(\tfrac{1}{4}\pi) = 2$
14. $y' + y\tan x = e^{-0.01x}\cos x, \quad y(0) = 0$
15. $y' + y/x^2 = 2xe^{1/x}, \quad y(1) = 13.86$
16. $y'\cos^2 x + 3y = 1, \quad y(\tfrac{1}{4}\pi) = \tfrac{4}{3}$
17. $x^3y' + 3x^2y = 5\sinh 10x$

14. **(Family $g(x, y) = c$)** Show that if a family is given as $g(x, y) = c$, then the orthogonal trajectories can be obtained from the following ODE, and use the latter to solve Prob. 6 written in the form $g(x, y) = c$.

$$\frac{d\tilde{y}}{dx} = \frac{\partial g/\partial \tilde{y}}{\partial g/\partial x}$$

15. **(Cauchy–Riemann equations)** Show that for a family $u(x, y) = c = const$ the orthogonal trajectories $v(x, y) = c^* = const$ can be obtained from the following *Cauchy–Riemann equations* (which are basic in complex analysis in Chap. 13) and use them to find the orthogonal trajectories of $e^x \sin y = const$. (Here, subscripts denote partial derivatives.)

$$u_x = v_y, \qquad u_y = -v_x$$

16–20 APPLICATIONS

16. **(Fluid flow)** Suppose that the *streamlines* of the flow (paths of the particles of the fluid) in Fig. 24 are $\Psi(x, y) = xy = const.$ Find their orthogonal trajectories (called **equipotential lines,** for reasons given in Sec. 18.4).

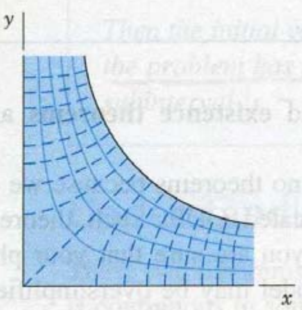

Fig. 24. Flow in a channel in Problem 16

17. **(Electric field)** Let the electric equipotential lines (curves of constant potential) between two concentric cylinders (Fig. 22) be given by $u(x, y) = x^2 + y^2 = c$. Use the method in the text to find their orthogonal trajectories (the curves of electric force).

18. **(Electric field)** The lines of electric force of two opposite charges of the same strength at $(-1, 0)$ and $(1, 0)$ are the circles through $(-1, 0)$ and $(1, 0)$. Show that these circles are given by $x^2 + (y - c)^2 = 1 + c^2$. Show that the **equipotential lines** (orthogonal trajectories of those circles) are the circles given by $(x + c^*)^2 + \tilde{y}^2 = c^{*2} - 1$ (dashed in Fig. 25).

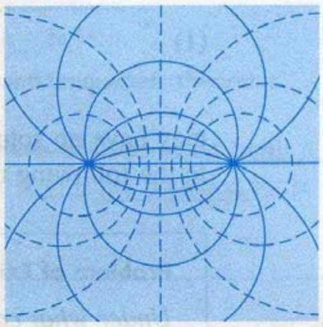

Fig. 25. Electric field in Problem 18

19. **(Temperature field)** Let the **isotherms** (curves of constant temperature) in a body in the upper half-plane $y > 0$ be given by $4x^2 + 9y^2 = c$. Find the orthogonal trajectories (the curves along which heat will flow in regions filled with heat-conducting material and free of heat sources or heat sinks).

20. **TEAM PROJECT. Conic Sections. (A)** State the main steps of the present method of obtaining orthogonal trajectories.

(B) Find conditions under which the orthogonal trajectories of families of ellipses $x^2/a^2 + y^2/b^2 = c$ are again conic sections. Illustrate your result graphically by sketches or by using your CAS. What happens if $a \to 0$? If $b \to 0$?

(C) Investigate families of hyperbolas $x^2/a^2 - y^2/b^2 = c$ in a similar fashion.

(D) Can you find more complicated curves for which you get ODEs that you can solve? Give it a try.

1.7 Existence and Uniqueness of Solutions

The initial value problem

$$|y'| + |y| = 0, \qquad y(0) = 1$$

has no solution because $y = 0$ (that is, $y(x) = 0$ for all x) is the only solution of the ODE. The initial value problem

$$y' = 2x, \qquad y(0) = 1$$

has precisely one solution, namely, $y = x^2 + 1$. The initial value problem

$$xy' = y - 1, \qquad y(0) = 1$$

has infinitely many solutions, namely, $y = 1 + cx$, where c is an arbitrary constant because $y(0) = 1$ for all c.

From these examples we see that an **initial value problem**

$$(1) \qquad y' = f(x, y), \qquad y(x_0) = y_0$$

may have no solution, precisely one solution, or more than one solution. This fact leads to the following two fundamental questions.

Problem of Existence

Under what conditions does an initial value problem of the form (1) *have at least one solution* (*hence one or several solutions*)?

Problem of Uniqueness

Under what conditions does that problem have at most one solution (*hence excluding the case that is has more than one solution*)?

Theorems that state such conditions are called **existence theorems** and **uniqueness theorems,** respectively.

Of course, for our simple examples we need no theorems because we can solve these examples by inspection; however, for complicated ODEs such theorems may be of considerable practical importance. Even when you are sure that your physical or other system behaves uniquely, occasionally your model may be oversimplified and may not give a faithful picture of the reality.

THEOREM 1 **Existence Theorem**

Let the right side $f(x, y)$ of the ODE in the initial value problem

$$(1) \qquad y' = f(x, y), \qquad y(x_0) = y_0$$

be continuous at all points (x, y) in some rectangle

$$R: |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and bounded in R; that is, there is a number K such that

$$(2) \qquad |f(x, y)| \leqq K \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at least one solution $y(x)$. This solution exists at least for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$; here, α is the smaller of the two numbers a and b/K.*

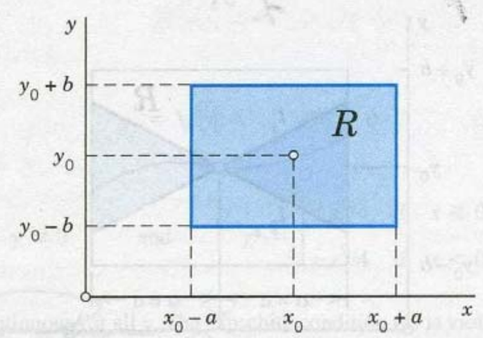

Fig. 26. Rectangle R in the existence and uniqueness theorems

(*Example of Boundedness.* The function $f(x, y) = x^2 + y^2$ is bounded (with $K = 2$) in the square $|x| < 1$, $|y| < 1$. The function $f(x, y) = \tan(x + y)$ is not bounded for $|x + y| < \pi/2$. Explain!)

THEOREM 2 **Uniqueness Theorem**

Let f and its partial derivative $f_y = \partial f/\partial y$ *be continuous for all* (x, y) *in the rectangle* R (Fig. 26) *and bounded, say,*

$$\text{(3)} \qquad \text{(a)} \quad |f(x, y)| \leqq K, \qquad \text{(b)} \quad |f_y(x, y)| \leqq M \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at most one solution* $y(x)$. *Thus, by Theorem* 1, *the problem has precisely one solution. This solution exists at least for all x in that subinterval* $|x - x_0| < \alpha$.

Understanding These Theorems

These two theorems take care of almost all practical cases. Theorem 1 says that if $f(x, y)$ is continuous in some region in the xy-plane containing the point (x_0, y_0), then the initial value problem (1) has at least one solution.

Theorem 2 says that if, moreover, the partial derivative $\partial f/\partial y$ of f with respect to y exists and is continuous in that region, then (1) can have at most one solution; hence, by Theorem 1, it has precisely one solution.

Read again what you have just read—these are entirely new ideas in our discussion.

Proofs of these theorems are beyond the level of this book (see Ref. [A11] in App. 1); however, the following remarks and examples may help you to a good understanding of the theorems.

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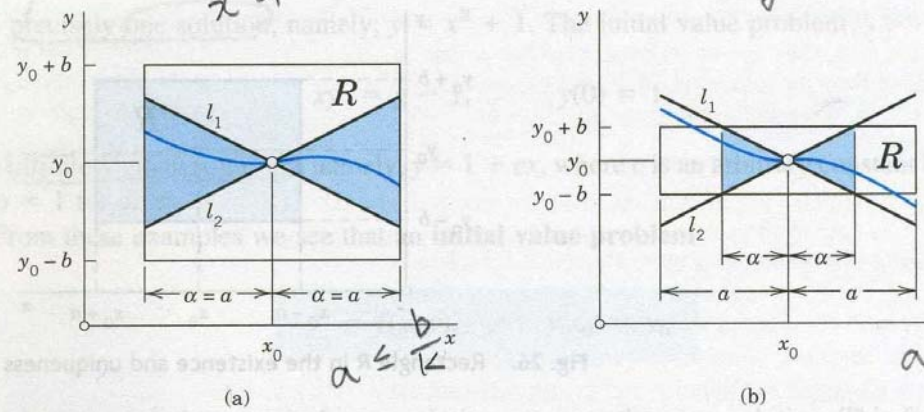
Since $y' = f(x, y)$, the condition (2) implies that $|y'| \leqq K$; that is, the slope of any solution curve $y(x)$ in R is at least $-K$ and at most K. Hence a solution curve that passes through the point (x_0, y_0) must lie in the colored region in Fig. 27 on the next page bounded by the lines l_1 and l_2 whose slopes are $-K$ and K, respectively. Depending on the form of R, two different cases may arise. In the first case, shown in Fig. 27a, we have $b/K \geqq a$ and therefore $\alpha = a$ in the existence theorem, which then asserts that the solution exists for all x between $x_0 - a$ and $x_0 + a$. In the second case, shown in Fig. 27b, we have $b/K < a$. Therefore, $\alpha = b/K < a$, and all we can conclude from the theorems is that the solution exists for all x between $x_0 - b/K$ and $x_0 + b/K$. For larger or smaller x's the solution curve may leave the rectangle R, and since nothing is assumed about f outside R, nothing can be concluded about the solution for those larger or smaller x's; that is, for such x's the solution may or may not exist—we don't know.

x의 능동적 active y의 능동적 active



$\alpha \leq \frac{b}{K}$ $\alpha > \frac{b}{K}$

Fig. 27. The condition (2) of the existence theorem. (a) First case. (b) Second case

Let us illustrate our discussion with a simple example. We shall see that our choice of a rectangle R with a large base (a long x-interval) will lead to the case in Fig. 27b.

EXAMPLE 1 Choice of a Rectangle

Consider the initial value problem

$$y' = 1 + y^2, \qquad y(0) = 0$$

and take the rectangle R; $|x| < 5$, $|y| < 3$. Then $a = 5$, $b = 3$, and

$$|f(x, y)| = |1 + y^2| \leq K = 10,$$

$$\left|\frac{\partial f}{\partial y}\right| = 2|y| \leq M = 6,$$

y의 능동적인 bounded 된 성질

$$\alpha = \frac{b}{K} = 0.3 < a.$$

Indeed, the solution of the problem is $y = \tan x$ (see Sec. 1.3, Example 1). This solution is discontinuous at $\pm\pi/2$, and there is no *continuous* solution valid in the entire interval $|x| < 5$ from which we started. ■

The conditions in the two theorems are sufficient conditions rather than necessary ones, and can be lessened. In particular, by the mean value theorem of differential calculus we have

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1) \left.\frac{\partial f}{\partial y}\right|_{y=\tilde{y}}$$

where (x, y_1) and (x, y_2) are assumed to be in R, and $\tilde{y}$ is a suitable value between y_1 and y_2. From this and (3b) it follows that

$$(4) \qquad |f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1|.$$

It can be shown that (3b) may be replaced by the weaker condition (4), which is known as a **Lipschitz condition.**[7] However, continuity of $f(x, y)$ is not enough to guarantee the *uniqueness* of the solution. This may be illustrated by the following example.

EXAMPLE 2 Nonuniqueness

The initial value problem

$$y' = \sqrt{|y|}, \qquad y(0) = 0$$

has the two solutions

$$y \equiv 0 \qquad \text{and} \qquad y^* = \begin{cases} x^2/4 & \text{if } x \geqq 0 \\ -x^2/4 & \text{if } x < 0 \end{cases}$$

although $f(x, y) = \sqrt{|y|}$ is continuous for all y. The Lipschitz condition (4) is violated in any region that includes the line $y = 0$, because for $y_1 = 0$ and positive y_2 we have

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, \qquad (\sqrt{y_2} > 0) \tag{5}$$

Lipschitz 条件 不成立

and this can be made as large as we please by choosing y_2 sufficiently small, whereas (4) requires that the quotient on the left side of (5) should not exceed a fixed constant M. ■

HW. 4, 6

PROBLEM SET 1.7

1. **(Vertical strip)** If the assumptions of Theorems 1 and 2 are satisfied not merely in a rectangle but in a vertical infinite strip $|x - x_0| < a$, in what interval will the solution of (1) exist?

2. **(Existence?)** Does the initial value problem $(x - 1)y' = 2y$, $y(1) = 1$ have a solution? Does your result contradict our present theorems?

3. **(Common points)** Can two solution curves of the same ODE have a common point in a rectangle in which the assumptions of the present theorems are satisfied?

4. **(Change of initial condition)** What happens in Prob. 2 if you replace $y(1) = 1$ with $y(1) = k$?

5. **(Linear ODE)** If p and r in $y' + p(x)y = r(x)$ are continuous for all x in an interval $|x - x_0| \leqq a$, show that $f(x, y)$ in this ODE satisfies the conditions of our present theorems, so that a corresponding initial value problem has a unique solution. Do you actually need these theorems for this ODE?

6. **(Three possible cases)** Find all initial conditions such that $(x^2 - 4x)y' = (2x - 4)y$ has no solution, precisely one solution, and more than one solution.

7. **(Length of x-interval)** In most cases the solution of an initial value problem (1) exists in an x-interval larger than that guaranteed by the present theorems. Show this fact for $y' = 2y^2$, $y(1) = 1$ by finding the best possible α (choosing b optimally) and comparing the result with the actual solution.

8. **PROJECT. Lipschitz Condition.** (A) State the definition of a Lipschitz condition. Explain its relation to the existence of a partial derivative. Explain its significance in our present context. Illustrate your statements by examples of your own.

   (B) Show that for a ***linear*** ODE $y' + p(x)y = r(x)$ with continuous p and r in $|x - x_0| \leqq a$ a Lipschitz condition holds. This is remarkable because it means that for a *linear* ODE the continuity of $f(x, y)$ guarantees not only the existence but also the uniqueness of the solution of an initial value problem. (Of course, this also follows directly from (4) in Sec. 1.5.)

   (C) Discuss the uniqueness of solution for a few simple ODEs that you can solve by one of the methods considered, and find whether a Lipschitz condition is satisfied.

9. **(Maximum α)** What is the largest possible α in Example 1 in the text?

10. **CAS PROJECT. Picard Iteration.** (A) Show that by integrating the ODE in (1) and observing the initial condition you obtain

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{6}$$

[7] RUDOLF LIPSCHITZ (1832–1903), German mathematician. Lipschitz and similar conditions are important in modern theories, for instance, in partial differential equations.

This form (6) of (1) suggests **Picard's iteration method**[8], which is defined by

$$(7)\quad y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \quad n = 1, 2, \cdots.$$

It gives approximations $y_1, y_2, y_3, \cdots$ of the unknown solution y of (1). Indeed, you obtain y_1 by substituting $y = y_0$ on the right and integrating—this is the first step—, then y_2 by substituting $y = y_1$ on the right and integrating—this is the second step—, and so on. Write a program of the iteration that gives a printout of the first approximations $y_0, y_1, \cdots, y_N$ as well as their graphs on common axes. Try your program on two initial value problems of your own choice.

(B) Apply the iteration to $y' = x + y$, $y(0) = 0$. Also solve the problem exactly.

(C) Apply the iteration to $y' = 2y^2$, $y(0) = 1$. Also solve the problem exactly.

(D) Find all solutions of $y' = 2\sqrt{y}$, $y(1) = 0$. Which of them does Picard's iteration approximate?

(E) Experiment with the conjecture that Picard's iteration converges to the solution of the problem for any initial choice of y in the integrand in (7) (leaving y_0 outside the integral as it is). Begin with a simple ODE and see what happens. When you are reasonably sure, take a slightly more complicated ODE and give it a try.

H.W 12 20 26

CHAPTER 1 REVIEW QUESTIONS AND PROBLEMS

1. Explain the terms *ordinary differential equation (ODE), partial differential equation (PDE), order, general solution,* and *particular solution.* Give examples. Why are these concepts of importance?
2. What is an initial condition? How is this condition used in an initial value problem?
3. What is a homogeneous linear ODE? A nonhomogeneous linear ODE? Why are these equations simpler than nonlinear ODEs?
4. What do you know about direction fields and their practical importance?
5. Give examples of mechanical problems that lead to ODEs.
6. Why do electric circuits lead to ODEs?
7. Make a list of the solution methods considered. Explain each method with a few short sentences and illustrate it by a typical example.
8. Can certain ODEs be solved by more than one method? Give three examples.
9. What are integrating factors? Explain the idea. Give examples.
10. Does every first-order ODE have a solution? A general solution? What do you know about uniqueness of solutions?

11–14 DIRECTION FIELDS

Graph a direction field (by a CAS or by hand) and sketch some of the solution curves. Solve the ODE exactly and compare.

11. $y' = 1 + 4y^2$

12. $y' = 3y - 2x$

13. $y' = 4y - y^2$

14. $y' = 16x/y$

15–26 GENERAL SOLUTION

Find the general solution. Indicate which method in this chapter you are using. Show the details of your work.

15. $y' = x^2(1 + y^2)$

16. $y' = x(y - x^2 + 1)$

17. $yy' + xy^2 = x$

18. $-\pi \sin \pi x \cosh 3y\, dx + 3 \cos \pi x \sinh 3y\, dy = 0$

19. $y' + y \sin x = \sin x$

20. $y' - y = 1/y$

21. $3 \sin 2y\, dx + 2x \cos 2y\, dy = 0$

22. $xy' = x \tan (y/x) + y$

23. $(y \cos xy - 2x)\, dx + (x \cos xy + 2y)\, dy = 0$

24. $xy' = (y - 2x)^2 + y$ (Set $y - 2x = z$.)

25. $\sin (y - x)\, dx + [\cos (y - x) - \sin (y - x)]\, dy = 0$

26. $xy' = (y/x)^3 + y$

27–32 INITIAL VALUE PROBLEMS

Solve the following initial value problems. Indicate the method used. Show the details of your work.

27. $yy' + x = 0, \quad y(3) = 4$

28. $y' - 3y = -12y^2, \quad y(0) = 2$

29. $y' = 1 + y^2, \quad y(\frac{1}{4}\pi) = 0$

30. $y' + \pi y = 2b \cos \pi x, \quad y(0) = 0$

31. $(2xy^2 - \sin x)\, dx + (2 + 2x^2y)\, dy = 0, \; y(0) = 1$

32. $[2y + y^2/x + e^x(1 + 1/x)]\, dx + (x + 2y)\, dy = 0,$ $y(1) = 1$

[8] EMILE PICARD (1856–1941), French mathematician, also known for his important contributions to complex analysis (see Sec. 16.2 for his famous theorem). Picard used his method to prove Theorems 1 and 2 as well as the convergence of the sequence (7) to the solution of (1). In precomputer times the iteration was of little *practical* value because of the integrations.

33–43 APPLICATIONS, MODELING

33. (Heat flow) If the isotherms in a region are $x^2 - y^2 = c$, what are the curves of heat flow (assuming orthogonality)?

34. (Law of cooling) A thermometer showing 10°C is brought into a room whose temperature is 25°C. After 5 minutes it shows 20°C. When will the thermometer practically reach the room temperature, say, 24.9°C?

35. (Half-life) If 10% of a radioactive substance disintegrates in 4 days, what is its half-life?

36. (Half-life) What is the half-life of a substance if after 5 days, 0.020 *g* is present and after 10 days, 0.015 g?

37. (Half-life) When will 99% of the substance in Prob. 35 have disintegrated?

38. (Air circulation) In a room containing 20 000 ft^3 of air, 600 ft^3 of fresh air flows in per minute, and the mixture (made practically uniform by circulating fans) is exhausted at a rate of 600 cubic feet per minute (cfm). What is the amount of fresh air $y(t)$ at any time if $y(0) = 0$? After what time will 90% of the air be fresh?

39. (Electric field) If the equipotential lines in a region of the xy-plane are $4x^2 + y^2 = c$, what are the curves of the electrical force? Sketch both families of curves.

40. (Chemistry) In a bimolecular reaction $A + B \rightarrow M$, a moles per liter of a substance A and b moles per liter of a substance B are combined. Under constant temperature the rate of reaction is

$$y' = k(a - y)(b - y) \qquad \textbf{(Law of mass action);}$$

that is, y' is proportional to the product of the concentrations of the substances that are reacting, where $y(t)$ is the number of moles per liter which have reacted after time t. Solve this ODE, assuming that $a \neq b$.

41. (Population) Find the population $y(t)$ if the birth rate is proportional to $y(t)$ and the death rate is proportional to the square of $y(t)$.

42. (Curves) Find all curves in the first quadrant of the xy-plane such that for every tangent, the segment between the coordinate axes is bisected by the point of tangency. (Make a sketch.)

43. (Optics) Lambert's law of absorption[9] states that the absorption of light in a thin transparent layer is proportional to the thickness of the layer and to the amount of light incident on that layer. Formulate this law as an ODE and solve it.

SUMMARY OF CHAPTER 1
First-Order ODEs

This chapter concerns **ordinary differential equations (ODEs) of first order** and their applications. These are equations of the form

$$(1) \qquad F(x, y, y') = 0 \qquad \text{or in explicit form} \qquad y' = f(x, y)$$

involving the derivative $y' = dy/dx$ of an unknown function y, given functions of x, and, perhaps, y itself. If the independent variable x is time, we denote it by t.

In Sec. 1.1 we explained the basic concepts and the process of **modeling,** that is, of expressing a physical or other problem in some mathematical form and solving it. Then we discussed the method of direction fields (Sec. 1.2), solution methods and models (Secs. 1.3–1.6), and, finally, ideas on existence and uniqueness of solutions (Sec. 1.7).

[9] JOHANN HEINRICH LAMBERT (1728–1777), German physicist and mathematician.

A first-order ODE usually has a **general solution,** that is, a solution involving an arbitrary constant, which we denote by c. In applications we usually have to find a unique solution by determining a value of c from an **initial condition** $y(x_0) = y_0$. Together with the ODE this is called an **initial value problem**

$$(2) \qquad y' = f(x, y), \qquad y(x_0) = y_0 \qquad (x_0, y_0 \text{ given numbers})$$

and its solution is a **particular solution** of the ODE. Geometrically, a general solution represents a family of curves, which can be graphed by using **direction fields** (Sec. 1.2). And each particular solution corresponds to one of these curves.

A **separable ODE** is one that we can put into the form

$$(3) \qquad g(y)\, dy = f(x)\, dx \qquad \text{(Sec. 1.3)}$$

by algebraic manipulations (possibly combined with transformations, such as $y/x = u$) and solve by integrating on both sides.

An **exact ODE** is of the form

$$(4) \qquad M(x, y)\, dx + N(x, y)\, dy = 0 \qquad \text{(Sec. 1.4)}$$

where $M\, dx + N\, dy$ is the **differential**

$$du = u_x\, dx + u_y\, dy$$

of a function $u(x, y)$, so that from $du = 0$ we immediately get the implicit general solution $u(x, y) = c$. This method extends to nonexact ODEs that can be made exact by multiplying them by some function $F(x, y)$, called an **integrating factor** (Sec. 1.4).

Linear ODEs

$$(5) \qquad y' + p(x)y = r(x)$$

are very important. Their solutions are given by the integral formula (4), Sec. 1.5. Certain nonlinear ODEs can be transformed to linear form in terms of new variables. This holds for the **Bernoulli equation**

$$y' + p(x)y = g(x)y^a \qquad \text{(Sec. 1.5).}$$

Applications and **modeling** are discussed throughout the chapter, in particular in Secs. 1.1, 1.3, 1.5 (**population dynamics,** etc.), and 1.6 (**trajectories**).

Picard's **existence** and **uniqueness theorems** are explained in Sec. 1.7 (and *Picard's iteration* in Problem Set 1.7).

Numeric methods for first-order ODEs can be studied in Secs. 21.1 and 21.2 immediately after this chapter, as indicated in the chapter opening.