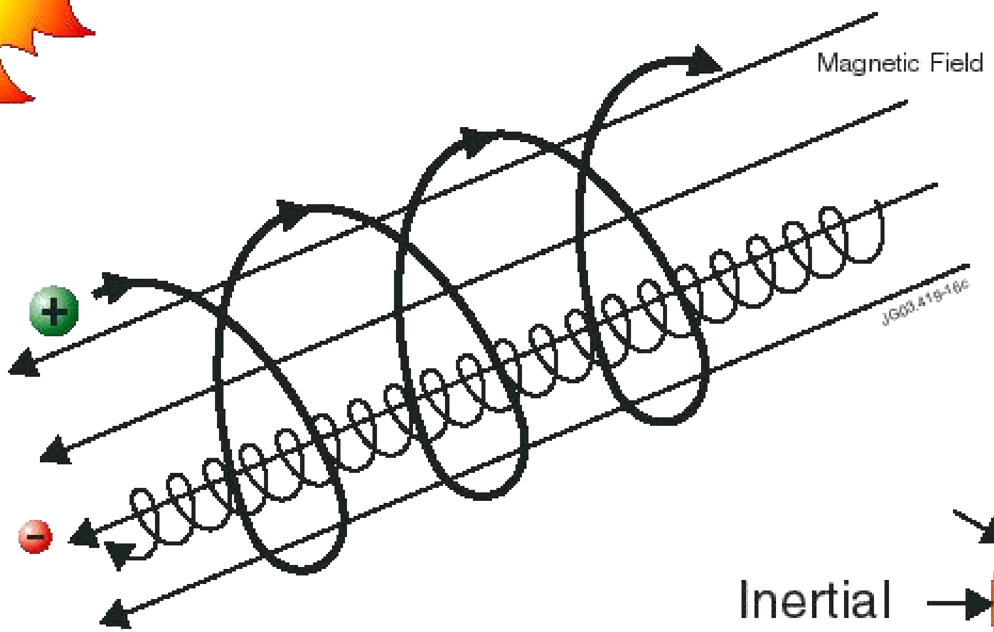
Introduction to Nuclear Fusion

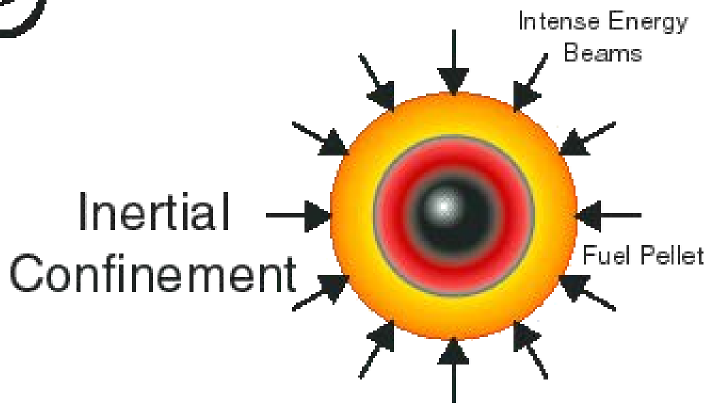
Prof. Dr. Yong-Su Na

To build a sun on earth

Magnetic Confinement

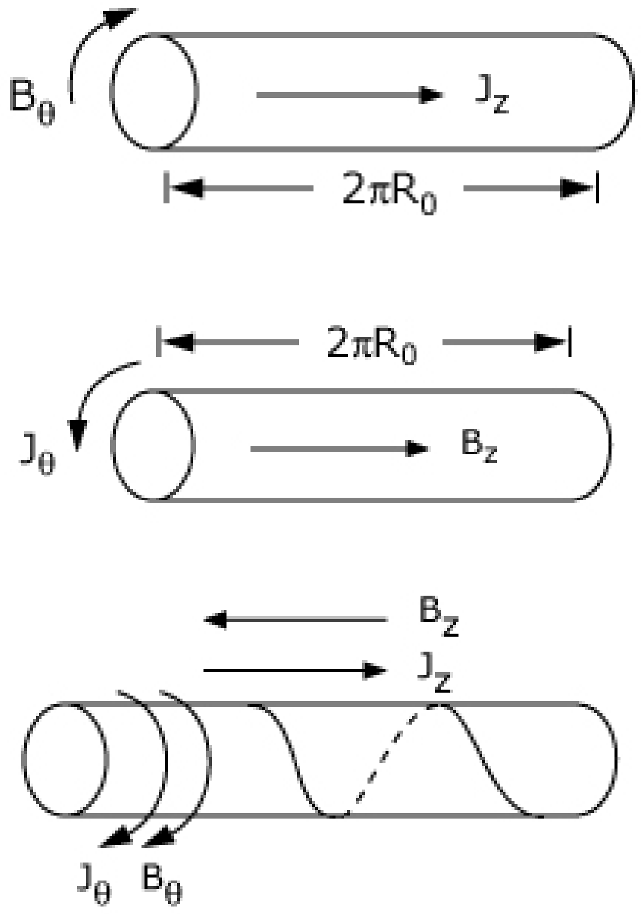


- Open magnetic confinement
- Closed magnetic confinement

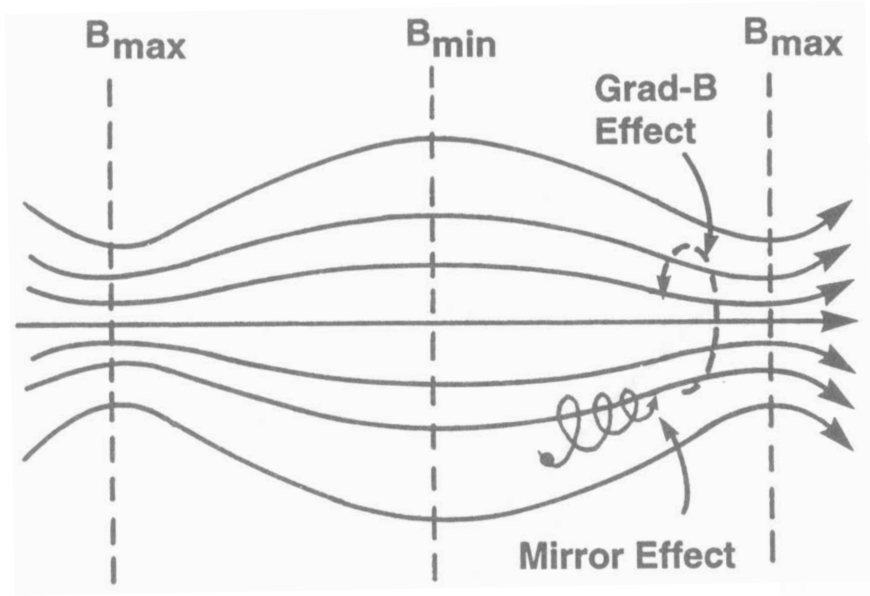


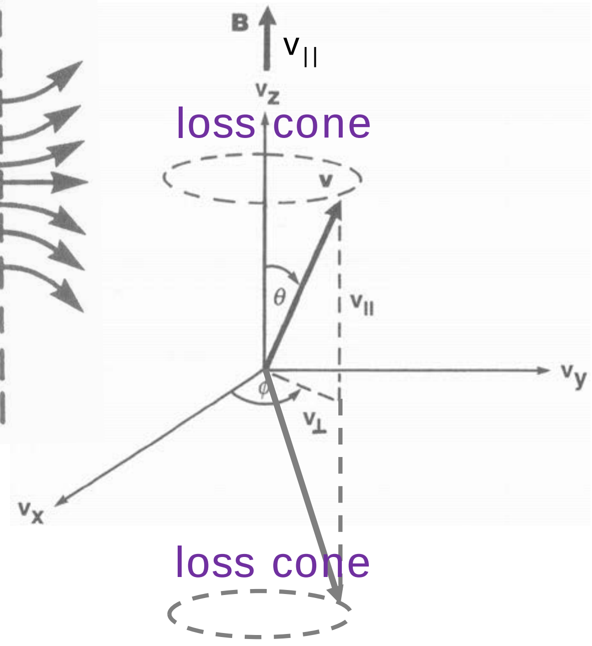
What is closed magnetic confinement?

Open Magnetic System



$$\sin^2\theta \geq \frac{B_{\min}}{B_{\max}}$$

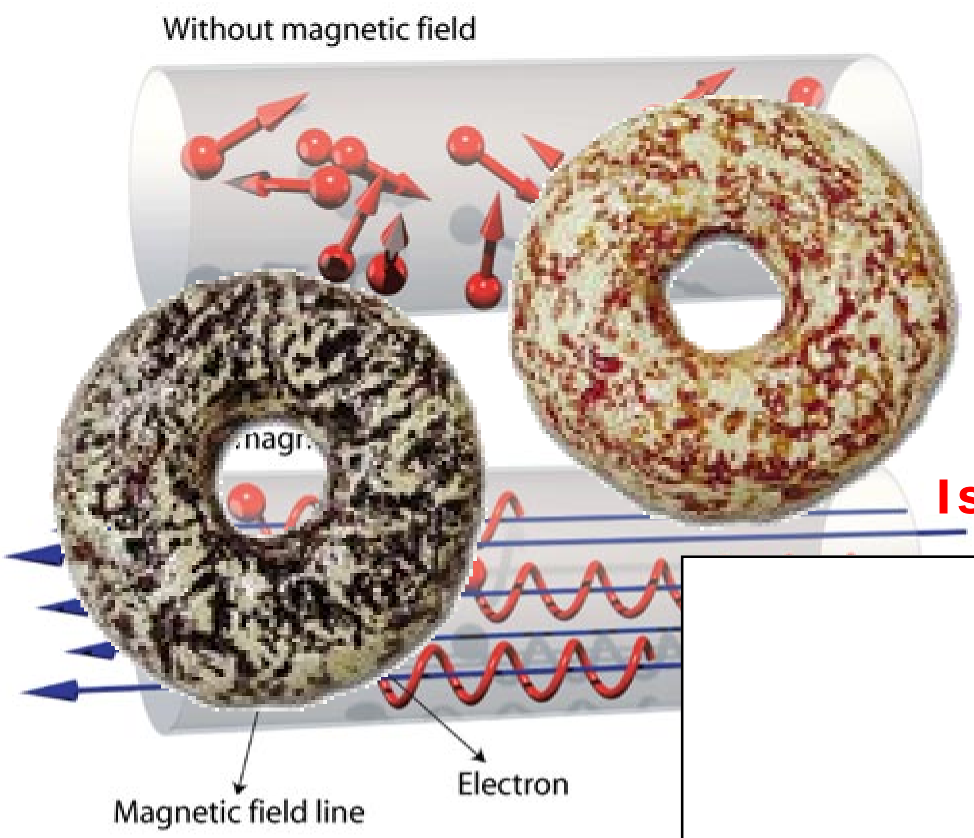




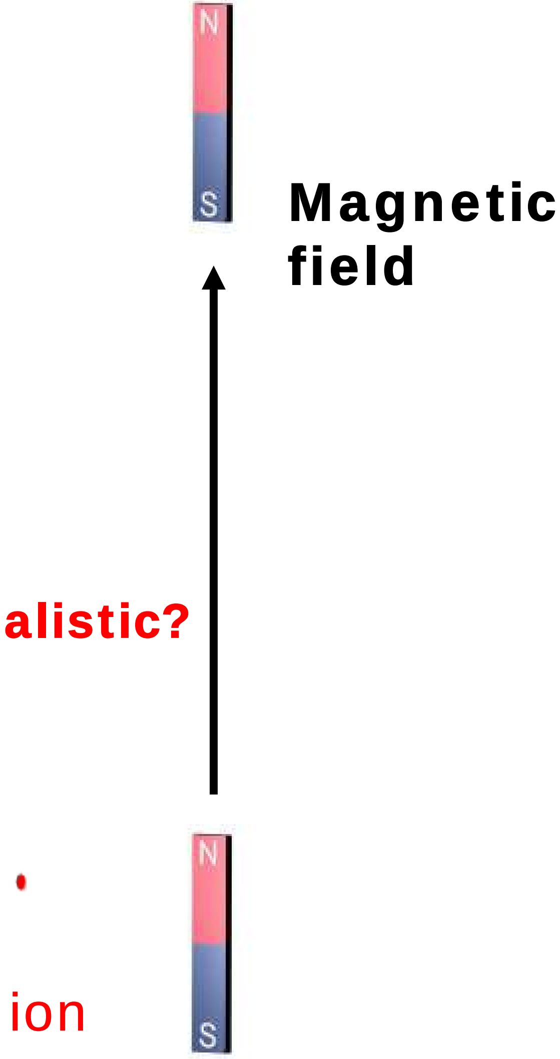
- Suffering from end losses

J.P. Freidberg, "Ideal Magneto-Hydro-Dynamics", lecture note
A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Open Magnetic System

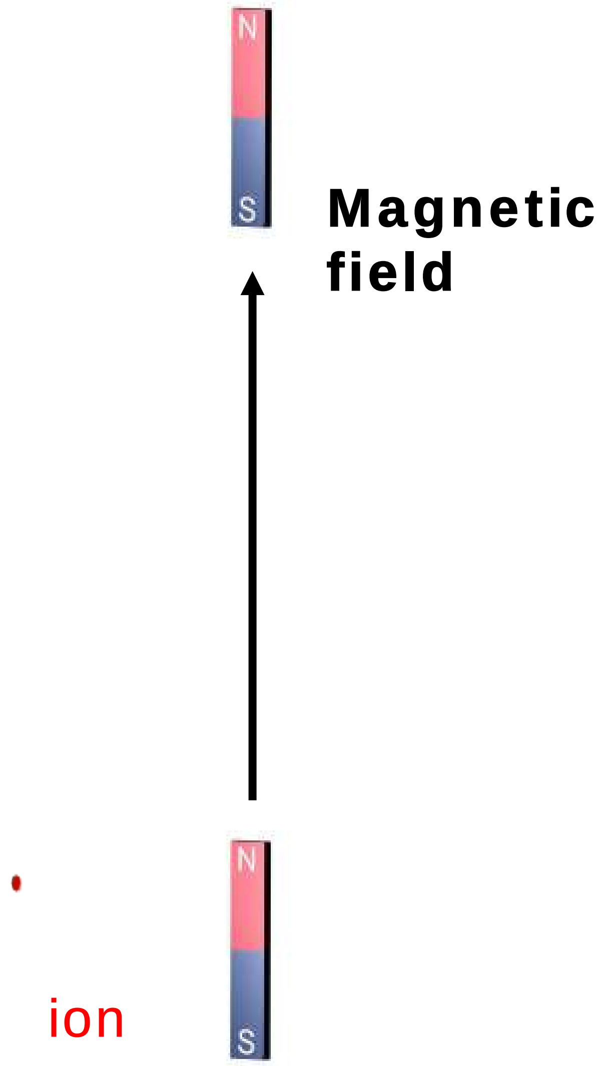


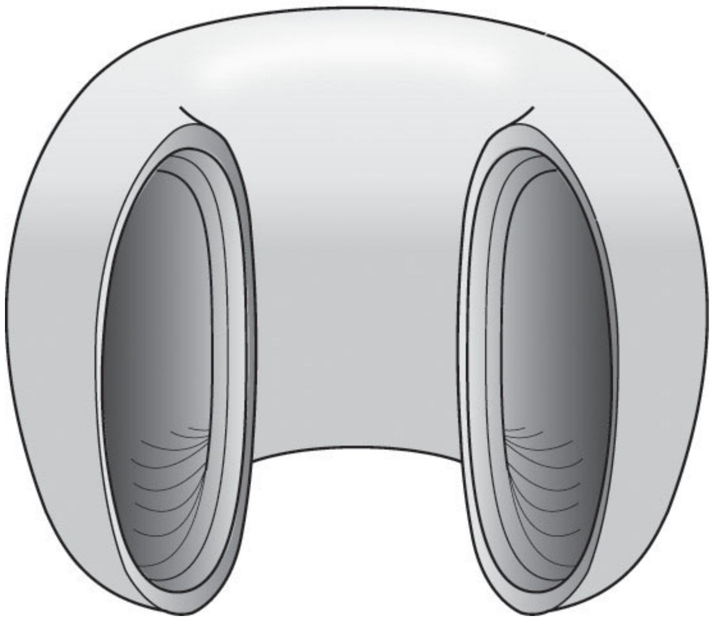
Is this motion realistic?



Dunkin donuts (2010)

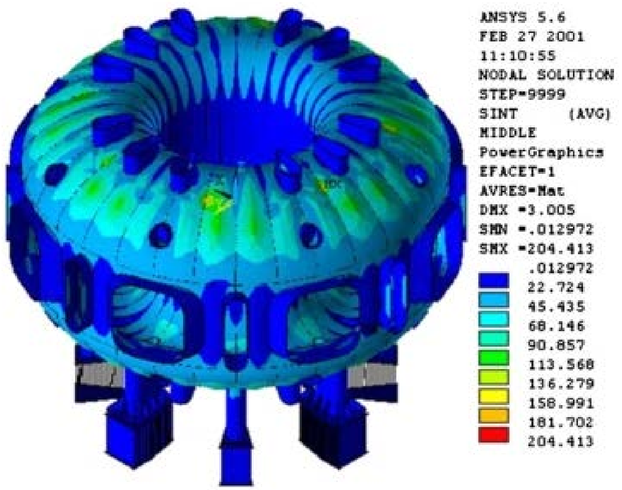
Closed Magnetic System

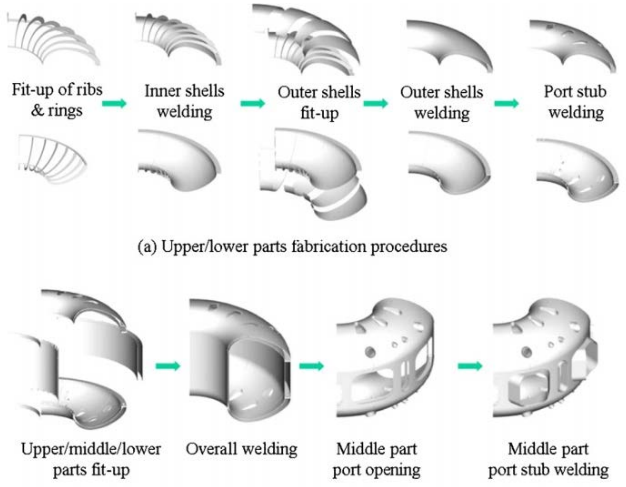




Donut-shaped vacuum vessel

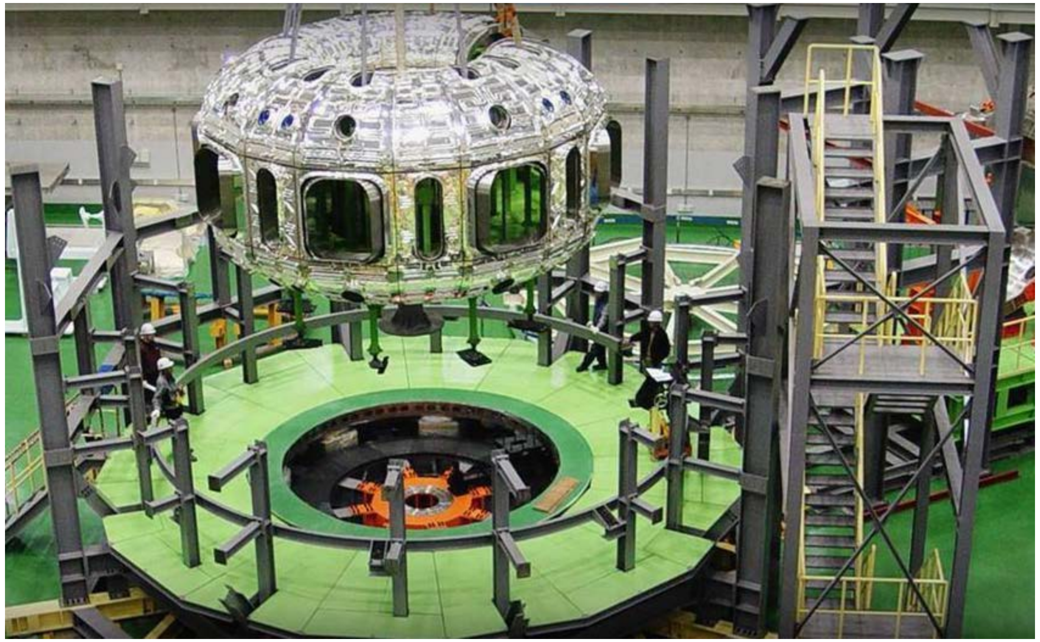
Closed Magnetic System



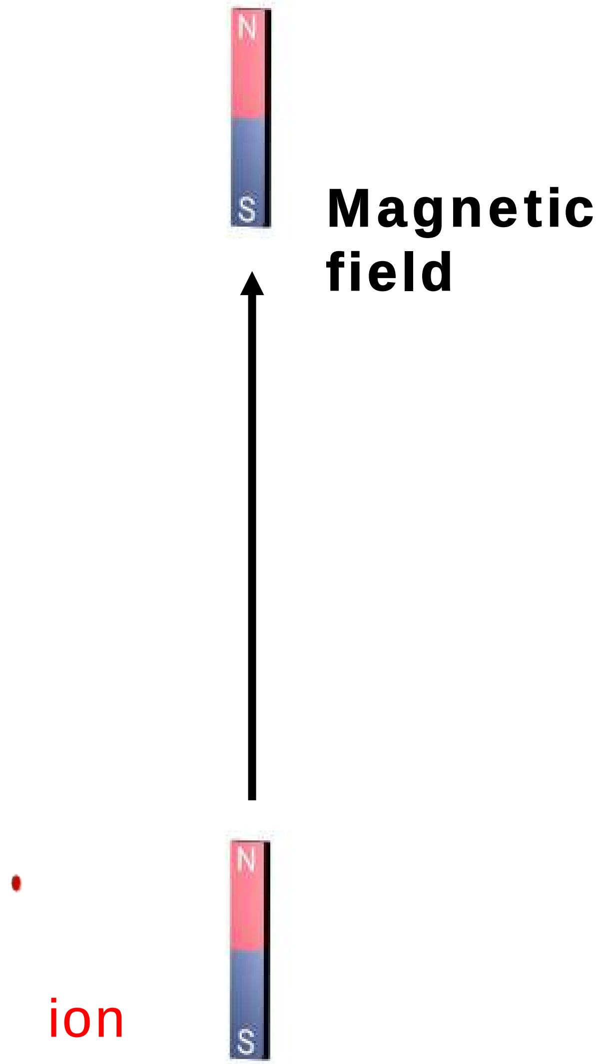


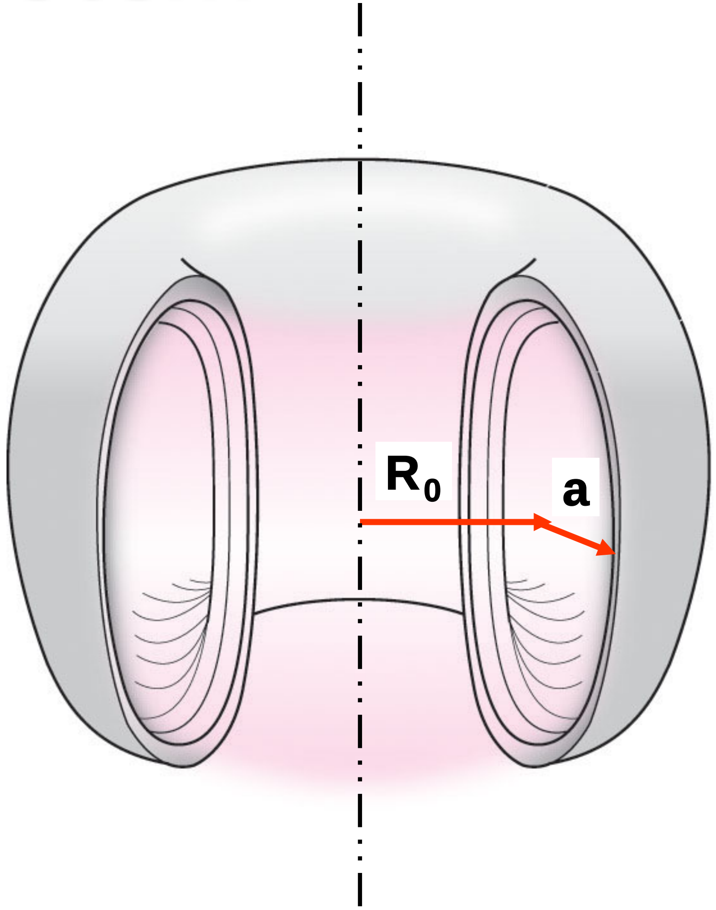
(a) Upper/lower parts fabrication procedures

(b) Quadrant fabrication procedure



Closed Magnetic System

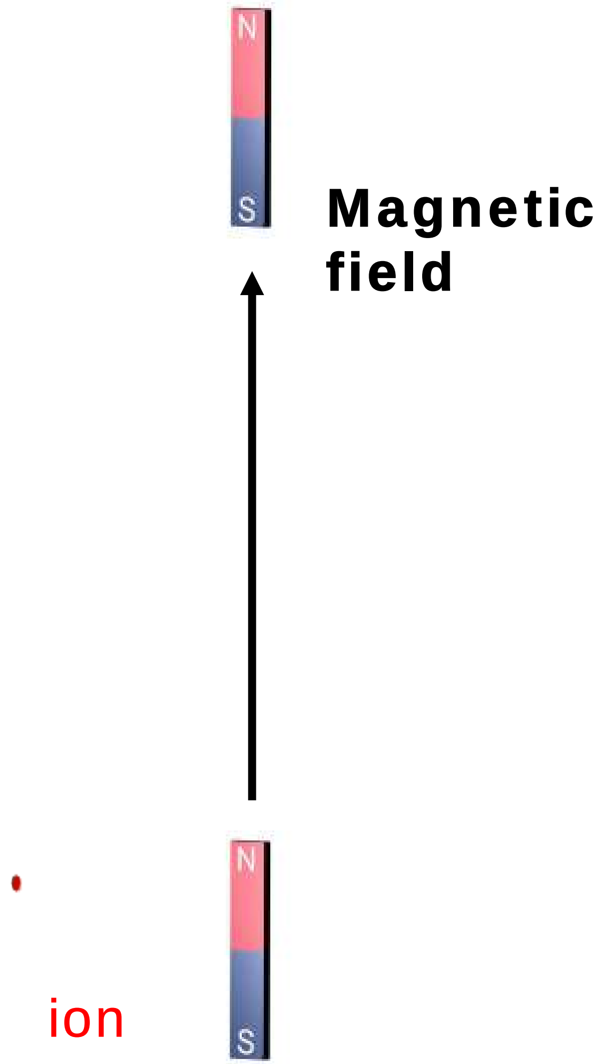


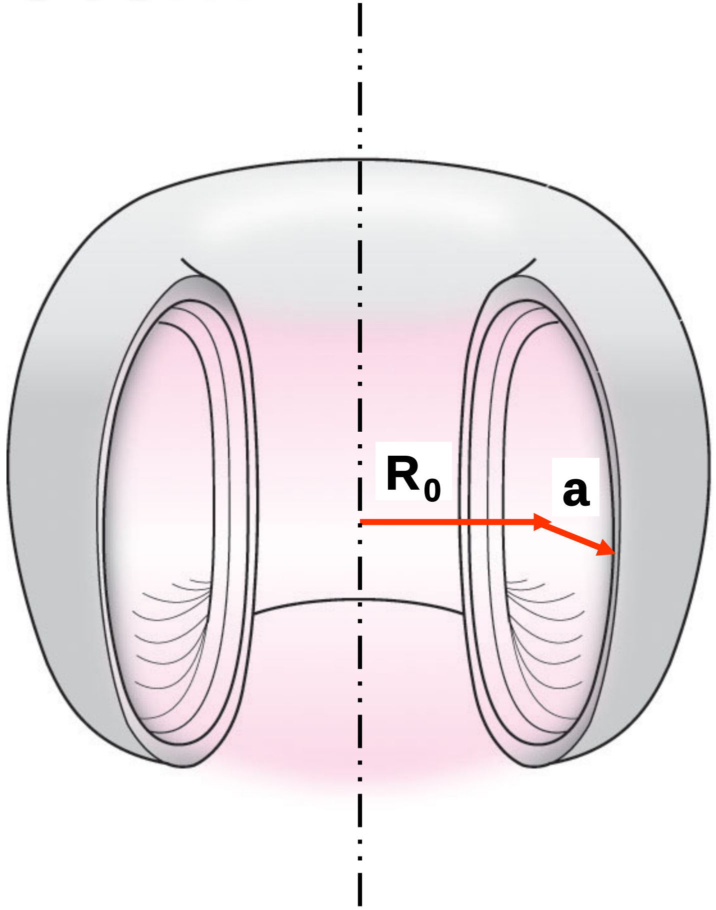


Plasma needs to be confined

R_0 = 1.8 m, a = 0.5 m in KSTAR

Closed Magnetic System

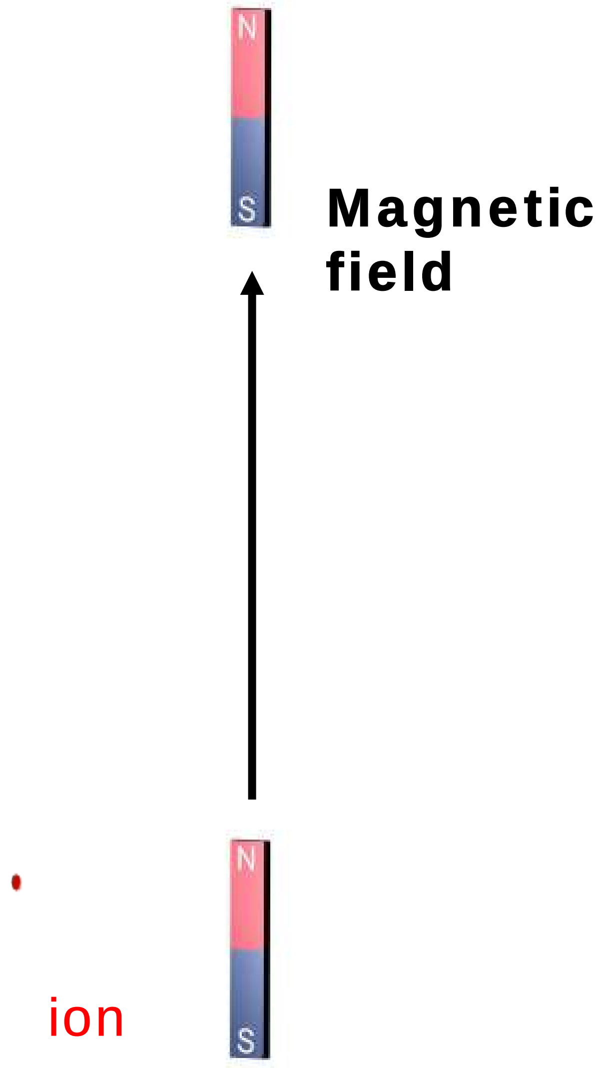


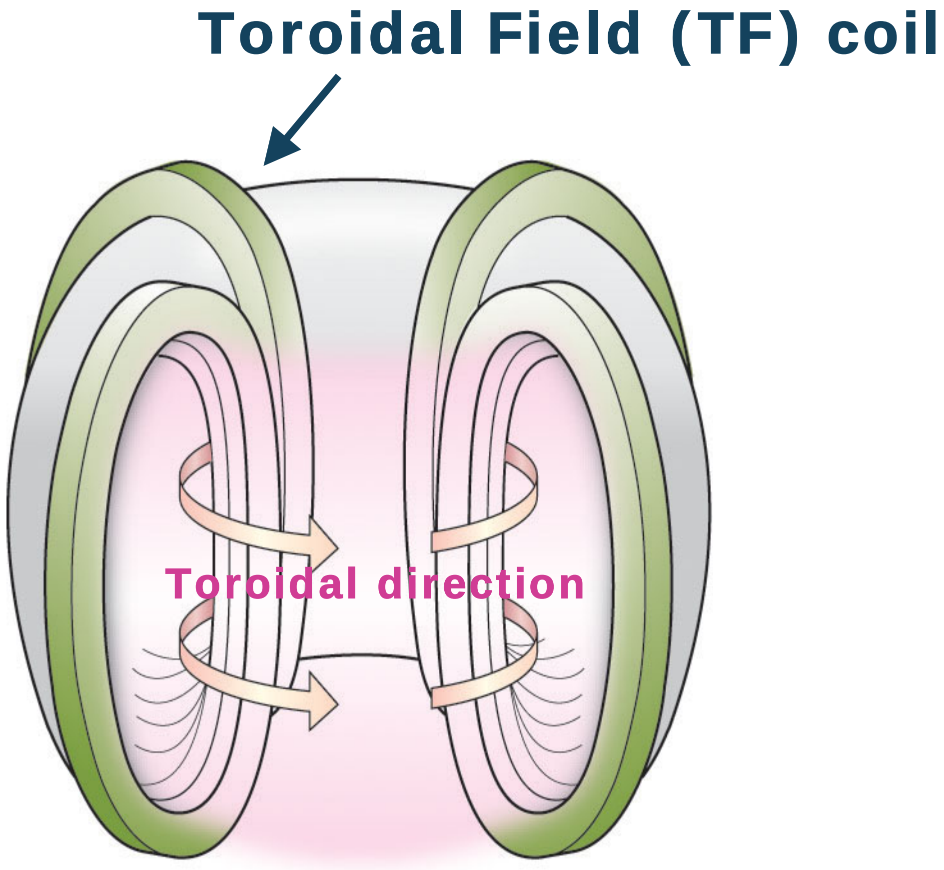


Plasma needs to be confined

R_0 = 6.2 m, a = 2.0 m in ITER

Closed Magnetic System

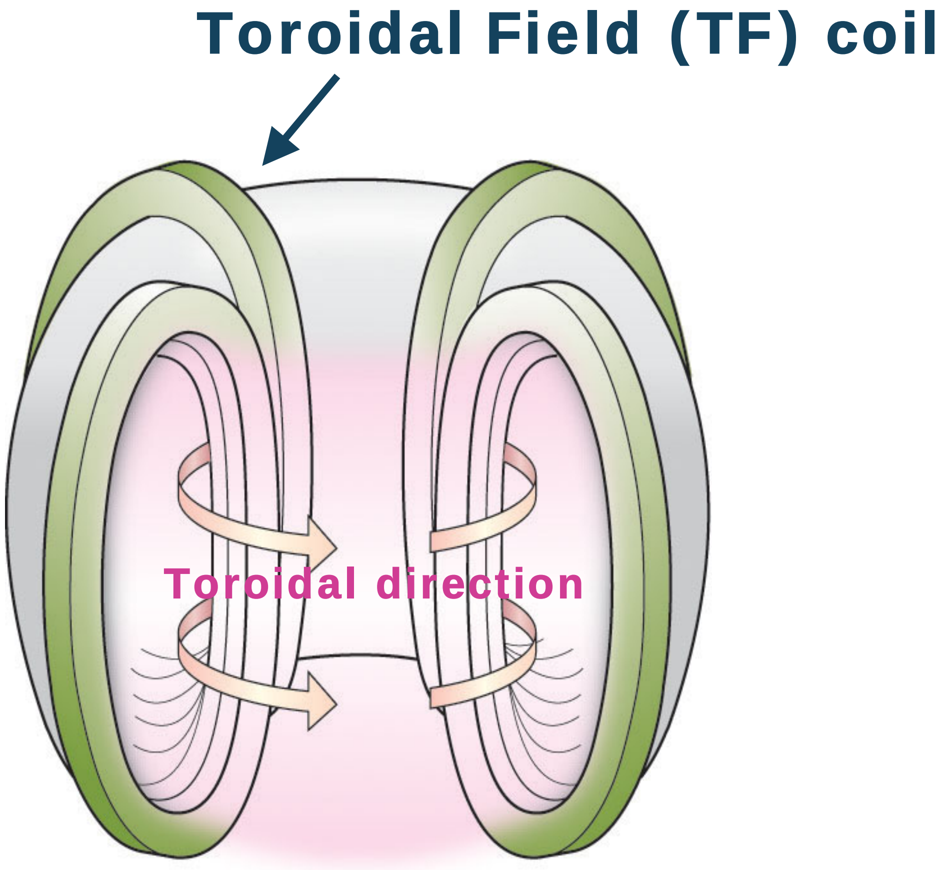




Applying toroidal magnetic field
3.5 T in KSTAR, 5.3 T in ITER

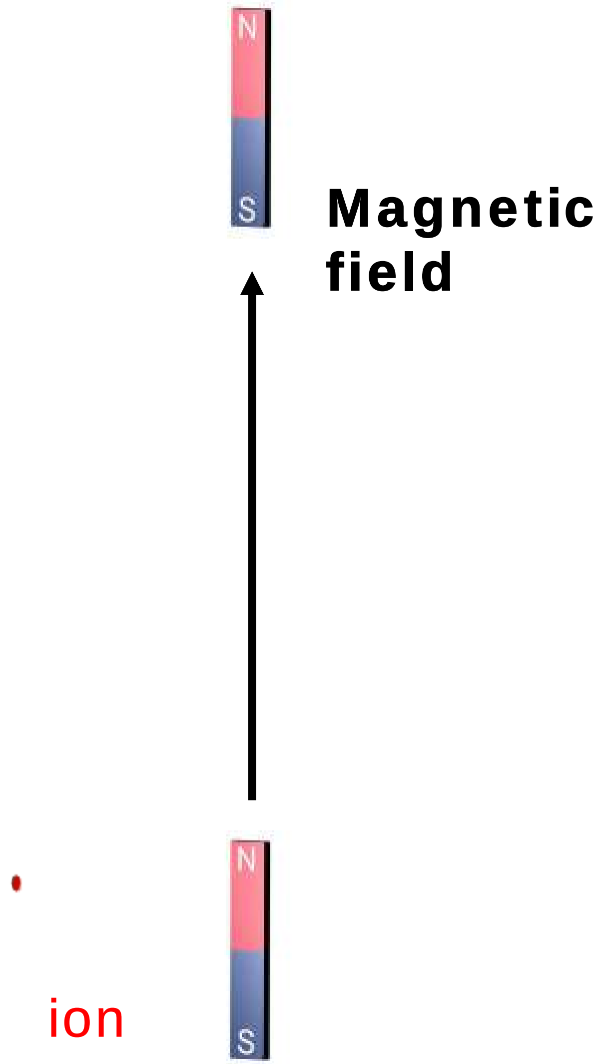
Closed Magnetic System

KSTAR

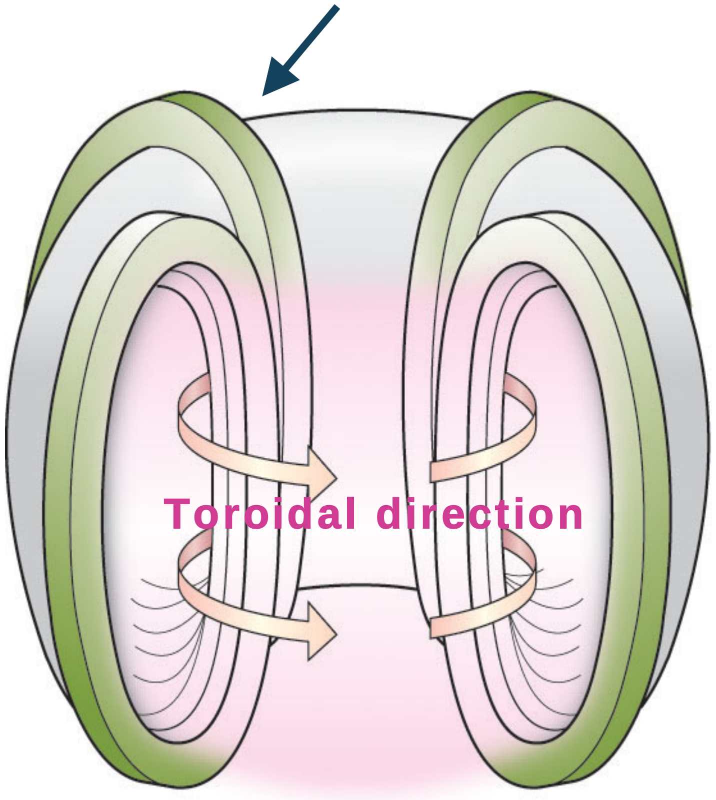


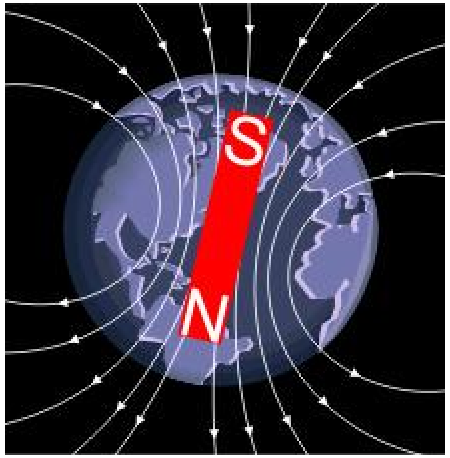
Applying toroidal magnetic field
3.5 T in KSTAR, 5.3 T in ITER

Closed Magnetic System



Toroidal Field (TF) coil



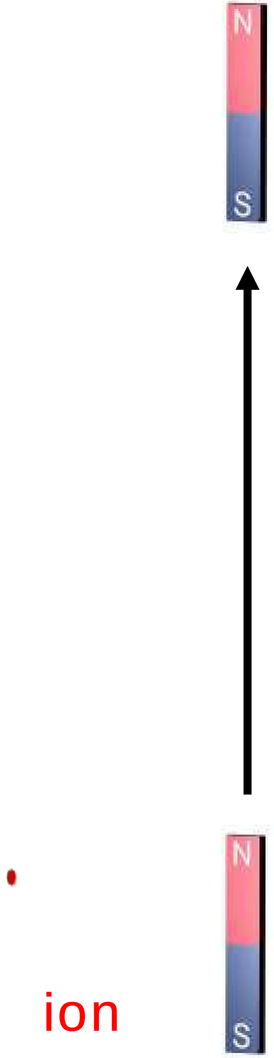


Magnetic field of earth?

0.5 Gauss = 0.00005 T

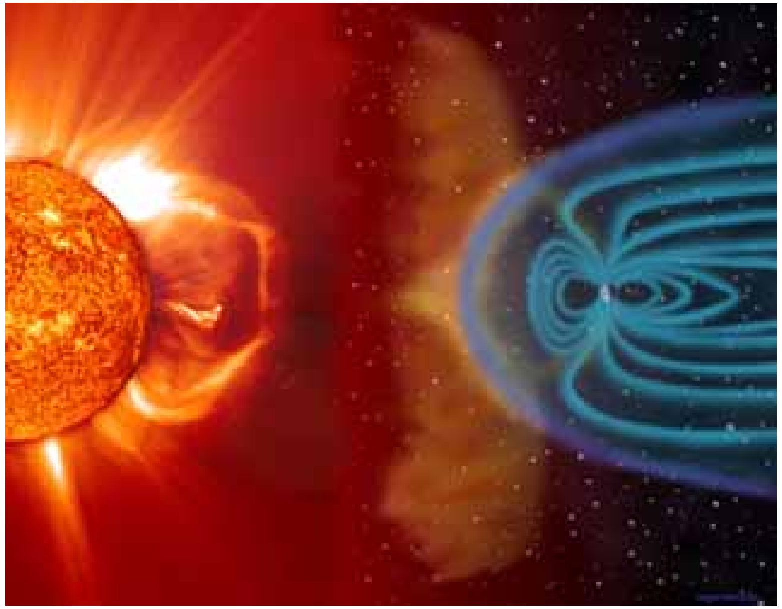
http://www.crystalinks.com/earthsmagneticfield.html

Closed Magnetic System



Magnetic field

ion

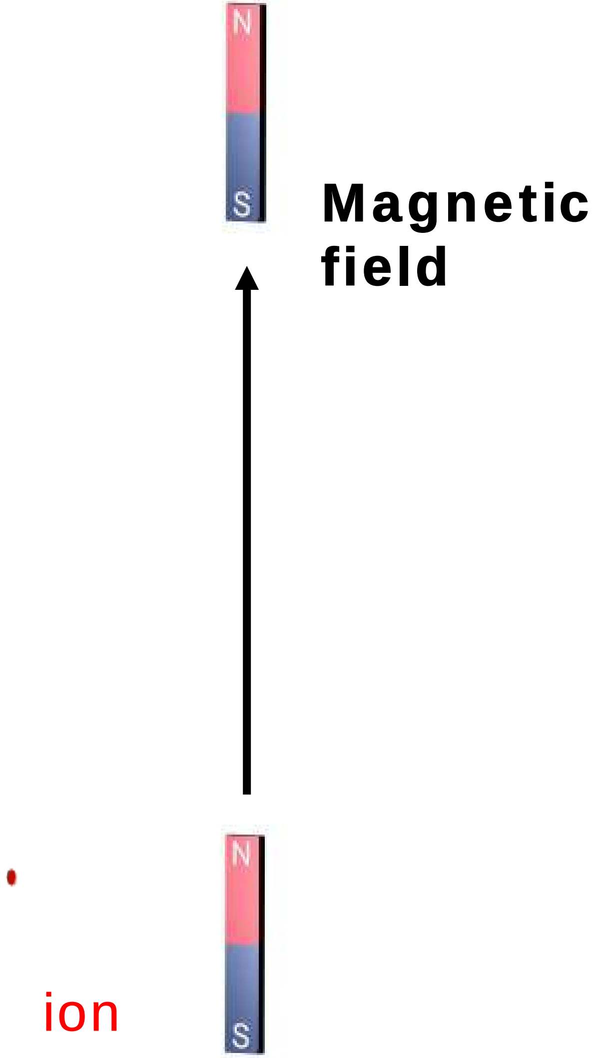


Magnetic field of earth?

0.5 Gauss = 0.00005 T

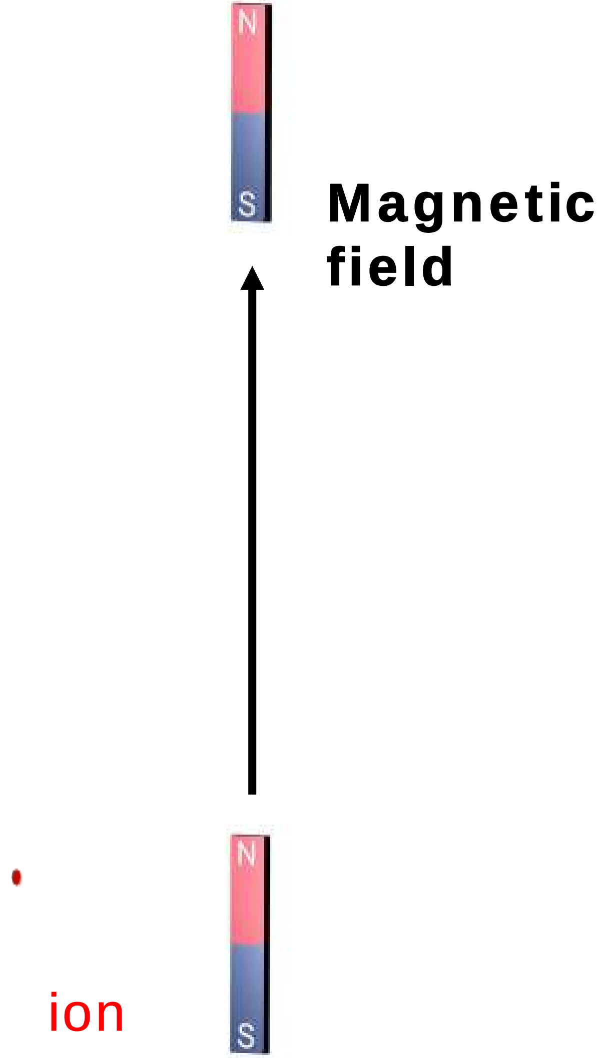
http://www.transformacionconciencia.com/archives/2384

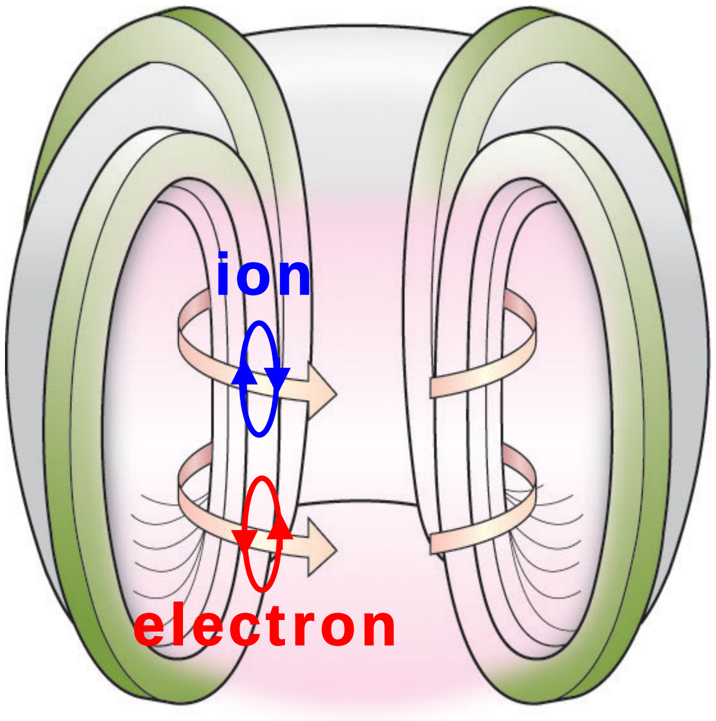
Closed Magnetic System



Lesch, Astrophysics, IPP Summer School (2008)

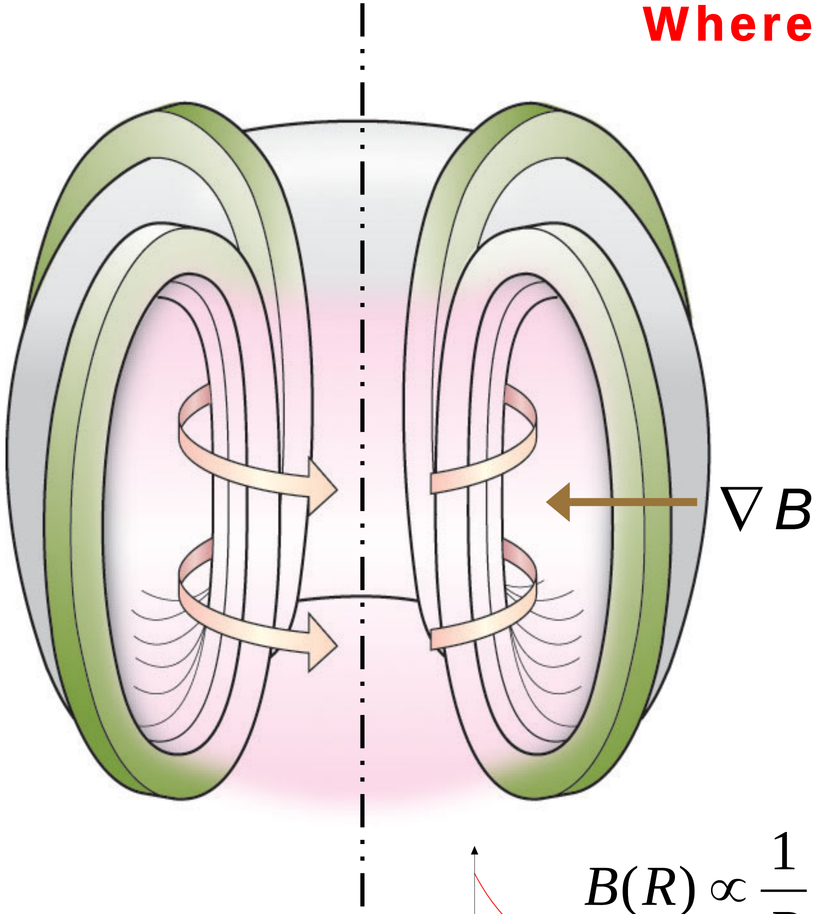
Closed Magnetic System



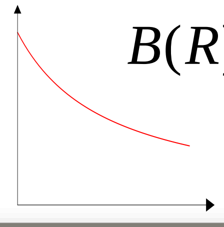


What kind of drift motions?

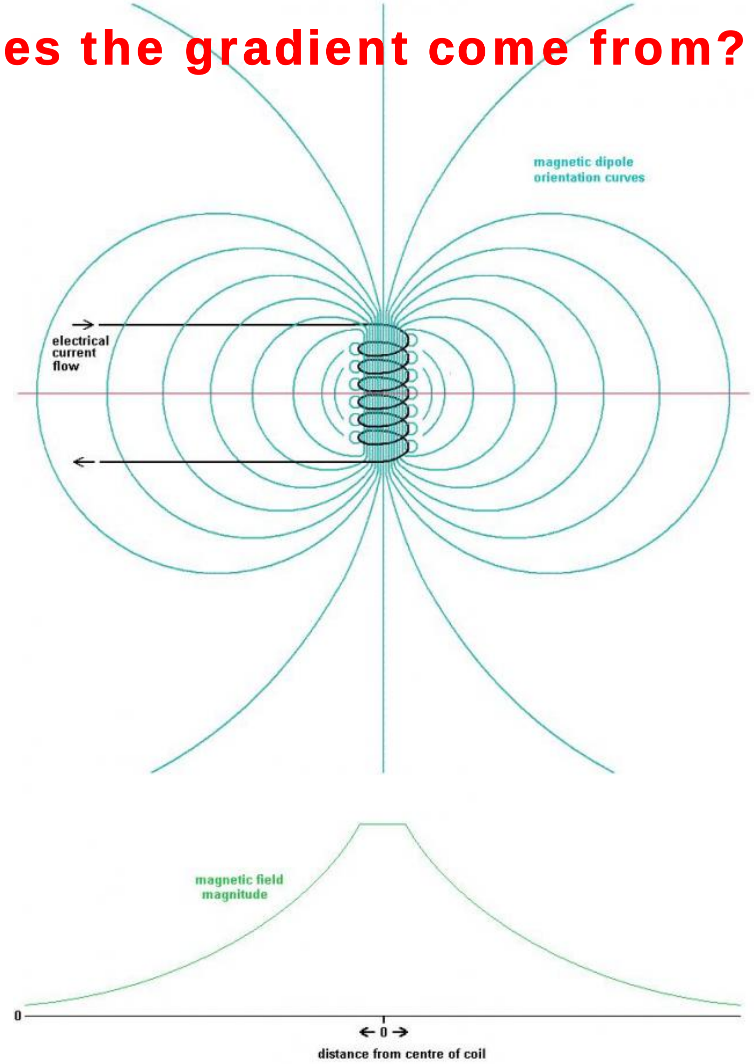
Closed Magnetic System



$$B(R) \propto \frac{1}{R}$$



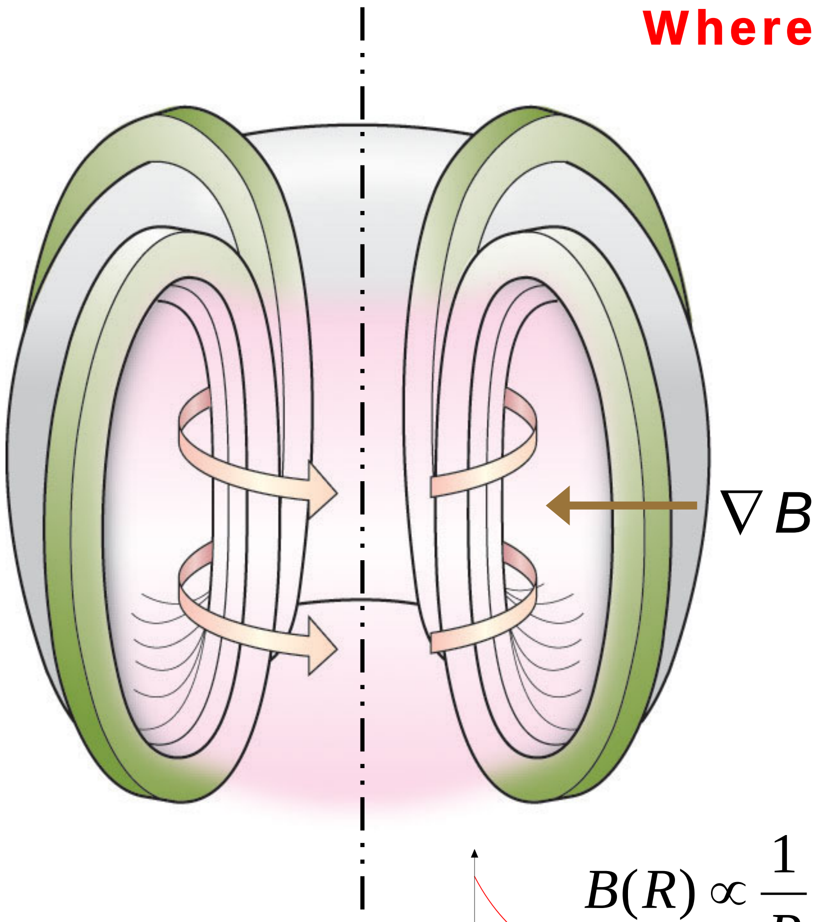
Where does the gradient come from?

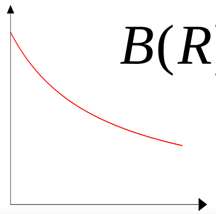


http://magnetoglide.wordpress.com/

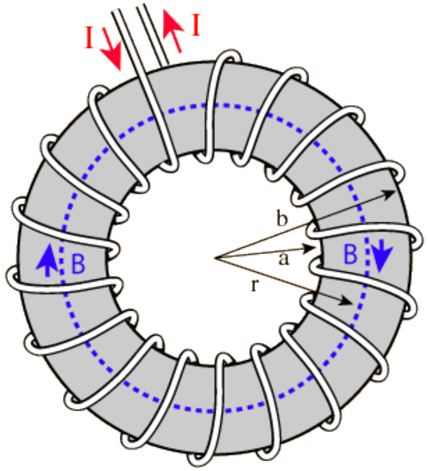
Closed Magnetic System

Where does the gradient come from?





$$B(R) \propto \frac{1}{R}$$



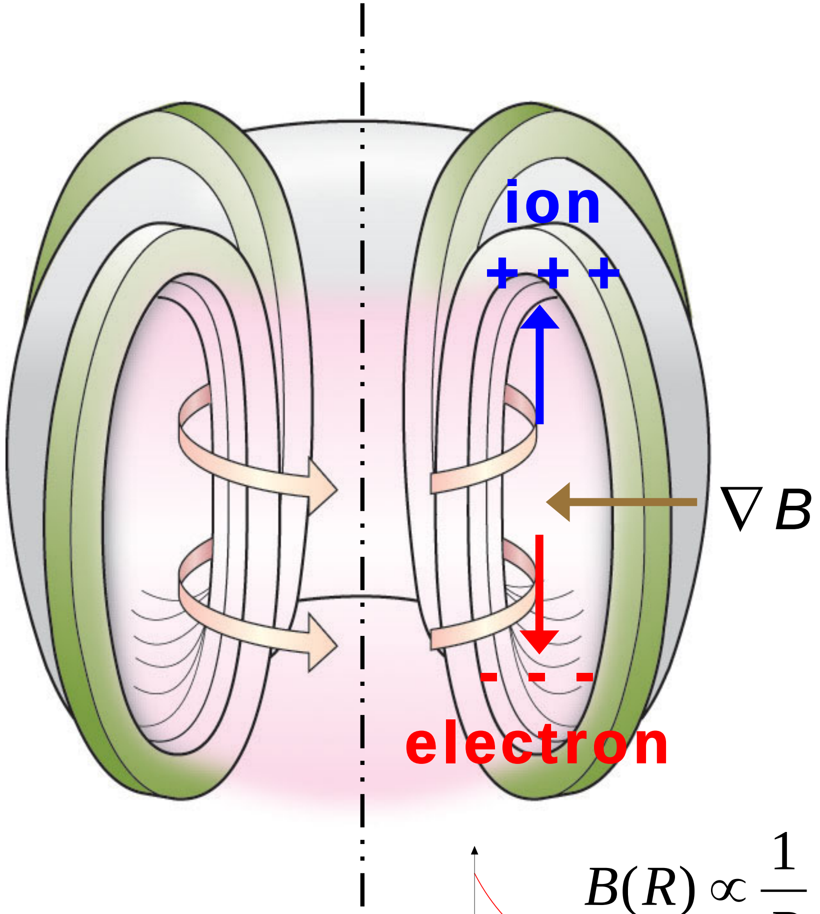
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\oint \mathbf{B}_\phi \cdot d\mathbf{l} = \mu_0 N I_c$$

$$B_\phi(R) = \frac{\mu_0 N I_c}{2\pi R}$$

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/toroid.html

Closed Magnetic System



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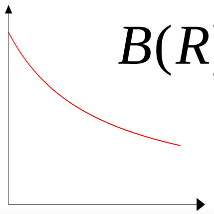


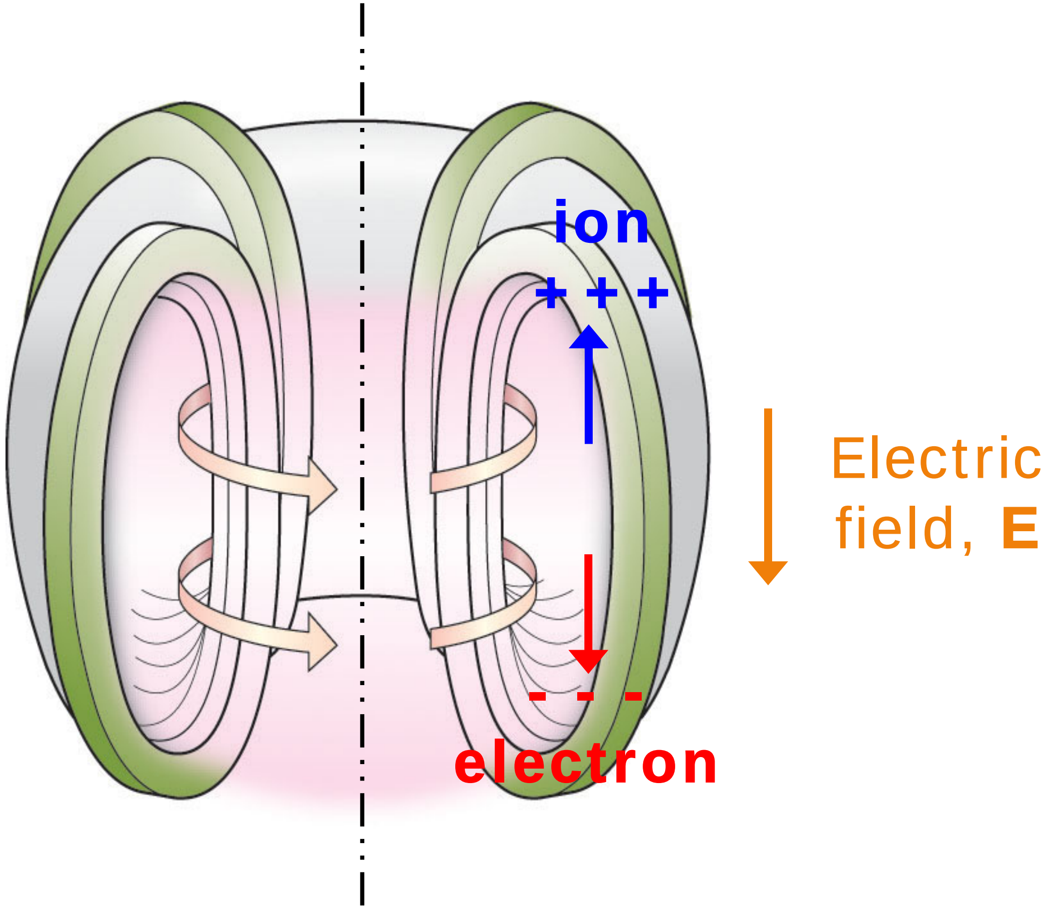
$$\mathbf{v}_{D,R} = \frac{m v_\parallel^2}{q B_0^2} \frac{\mathbf{R}_0 \times \mathbf{B}_0}{R^2}$$

$$\mathbf{v}_{D,\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\mathbf{B} \times \nabla B}{B^2}$$

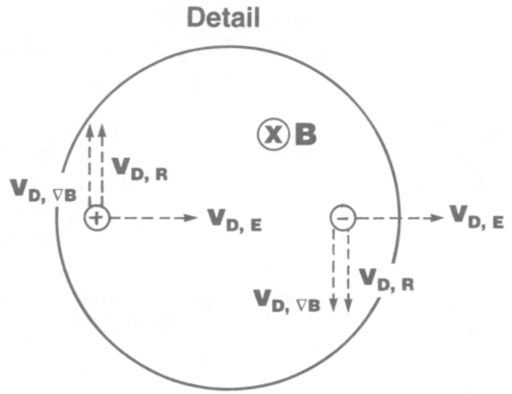
$$= \frac{m v_\perp^2}{2qB} \frac{\mathbf{B} \times \nabla B}{B^2}$$

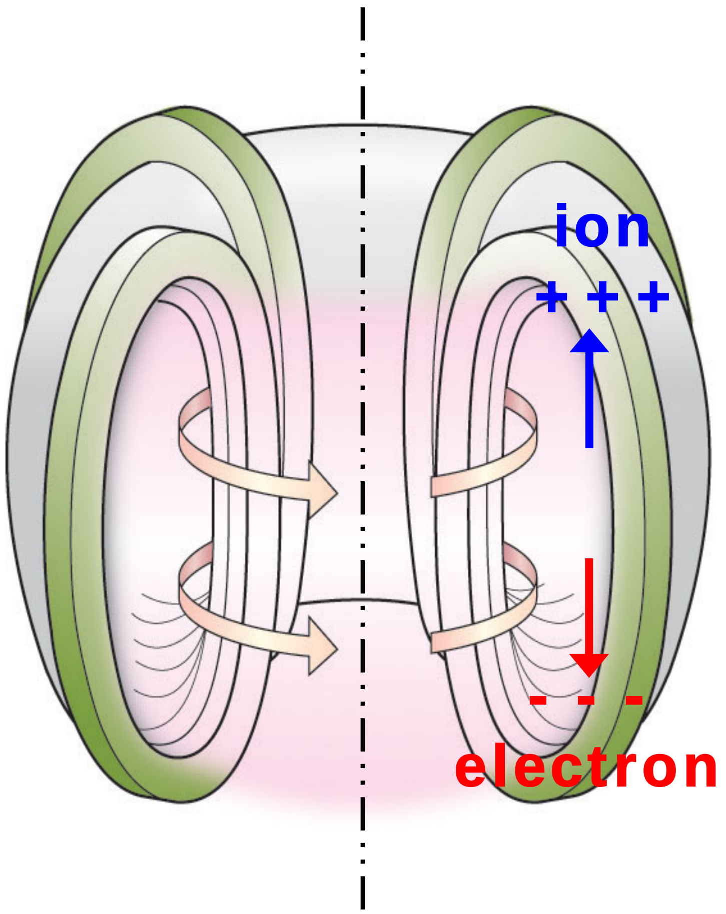
$$\mathbf{v}_D = \frac{m}{q} \frac{1}{R_0 B_\phi(R_0)} \left[v_\parallel^2 + \frac{v_\perp^2}{2} \right] \mathbf{e}_Z$$

Closed Magnetic System



Closed Magnetic System





ExB drift

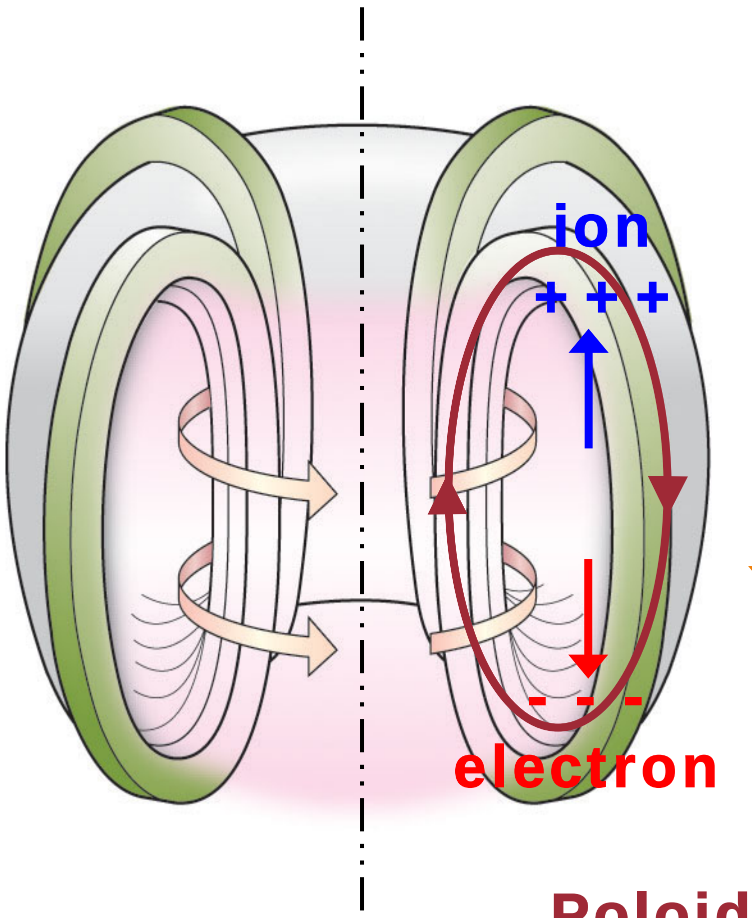
Electric field, **E**

$$\mathbf{v}_{D,E} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{E}{B_\phi(R_0)}\cdot\frac{\mathbf{R}}{R_0}$$

ExB drift

A. A. Harms et al, “Principles of Fusion Energy”, World Scientific (2000)

Closed Magnetic System



ExB drift

Electric field, **E**

ExB drift

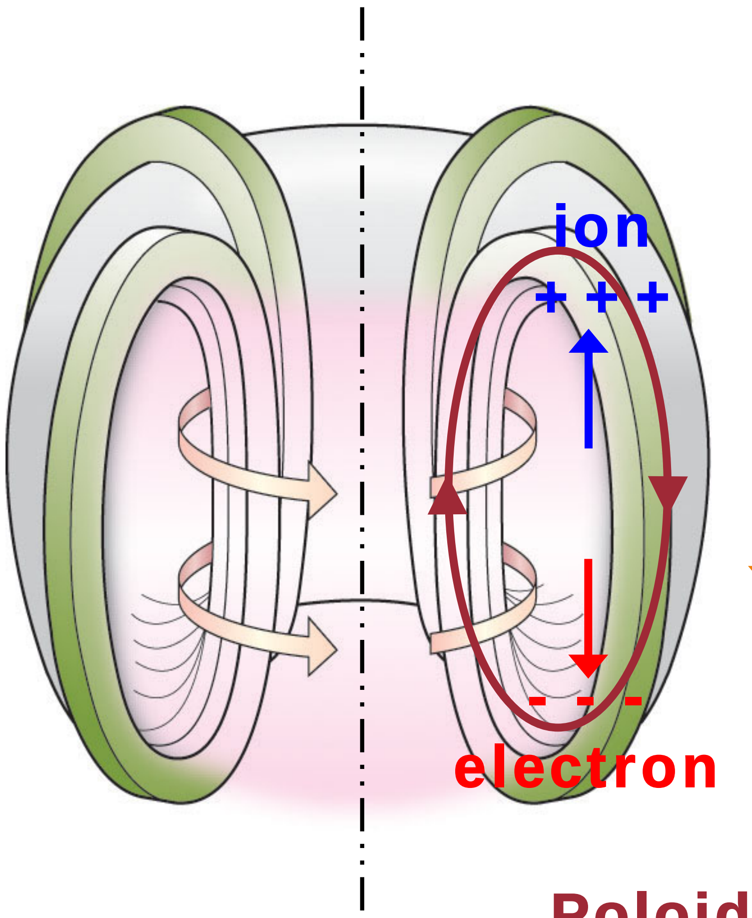
Poloidal magnetic field required

Tokamak .VS. Stellarator

What is a tokamak?

Closed Magnetic System

Is the plasma current driven if toroidal magnetic filed is applied?



Ex**B** drift

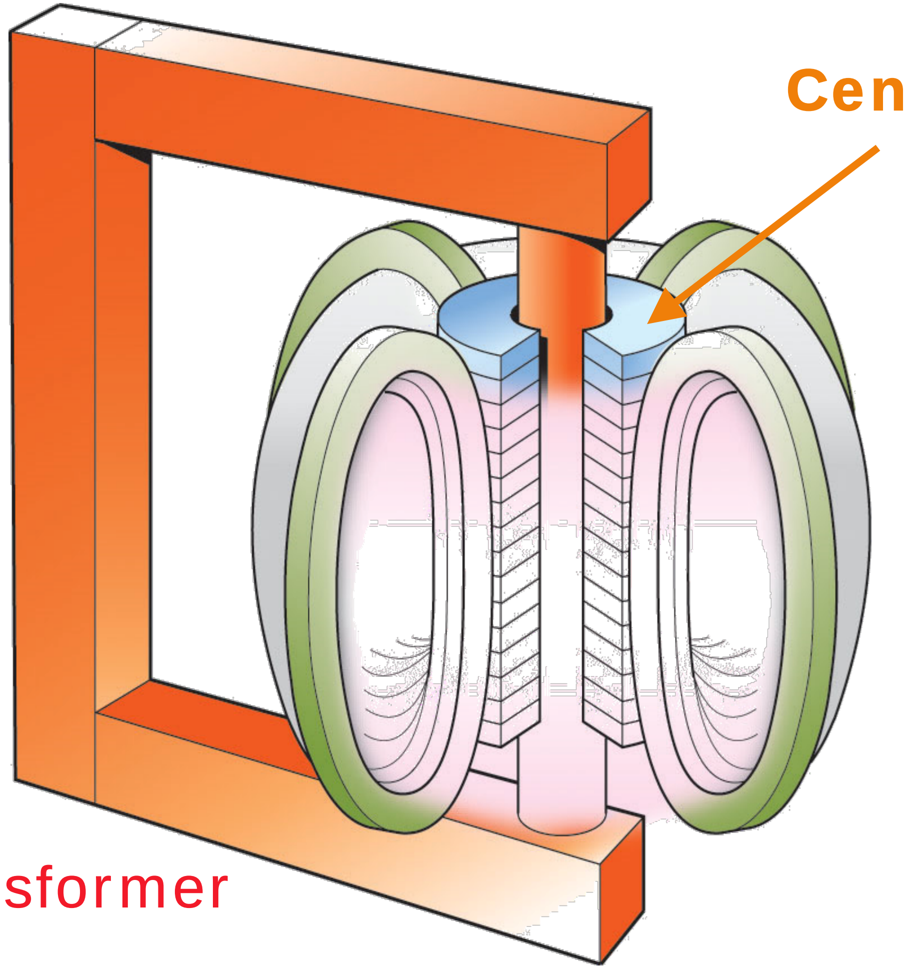
Electric field, **E**

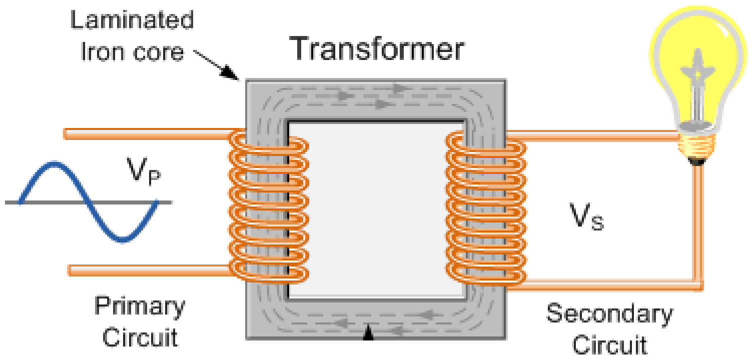
Ex**B** drift

Poloidal magnetic field required

How to drive plasma current?

Tokamak



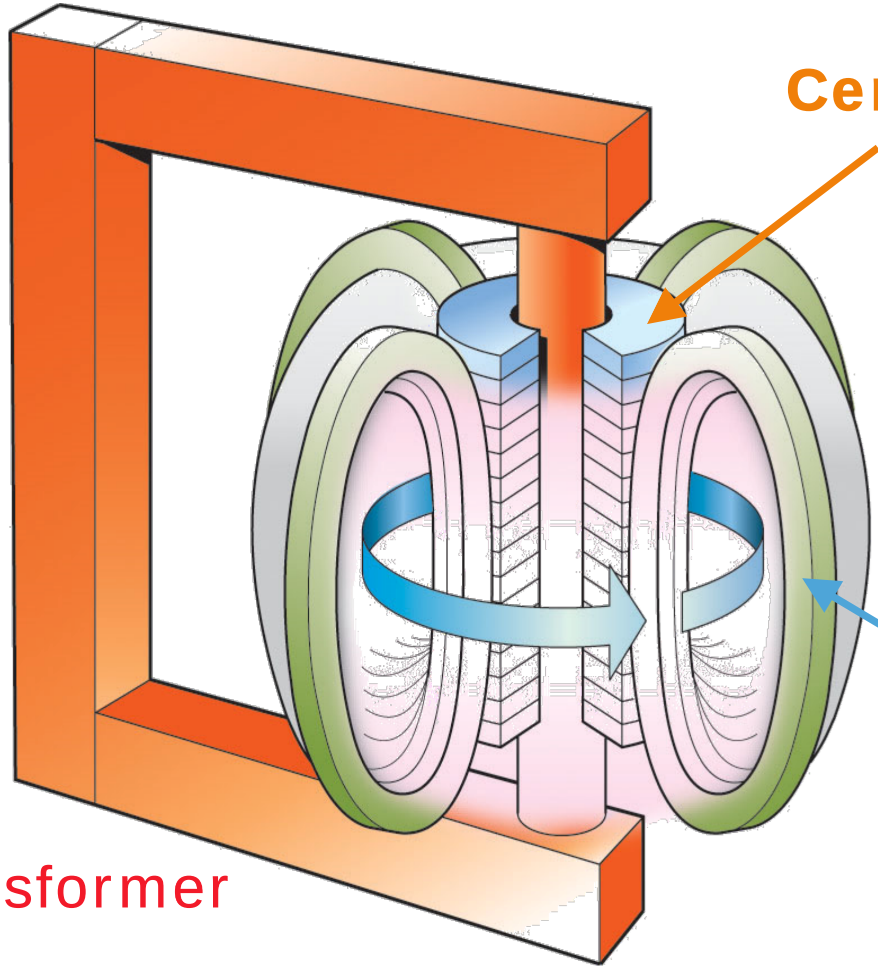


Faraday's law

$$v = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$$

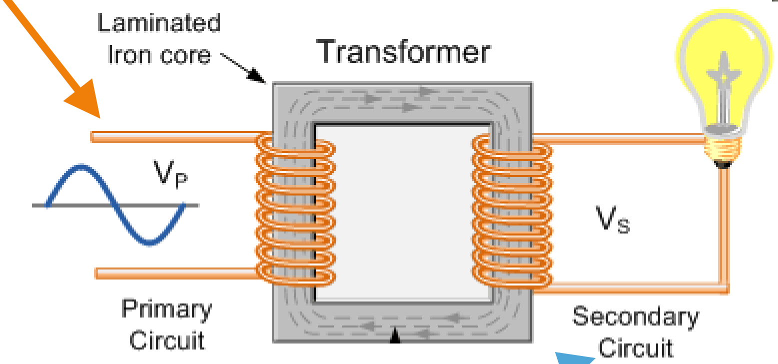
http://www.electronics-tutorials.ws/electromagnetism/electromagnetic-induction.html

Tokamak

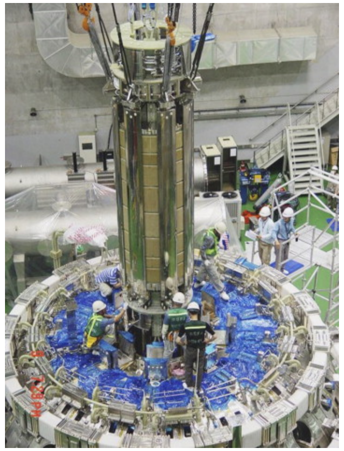


Transformer

Central Solenoid (CS)



Plasma

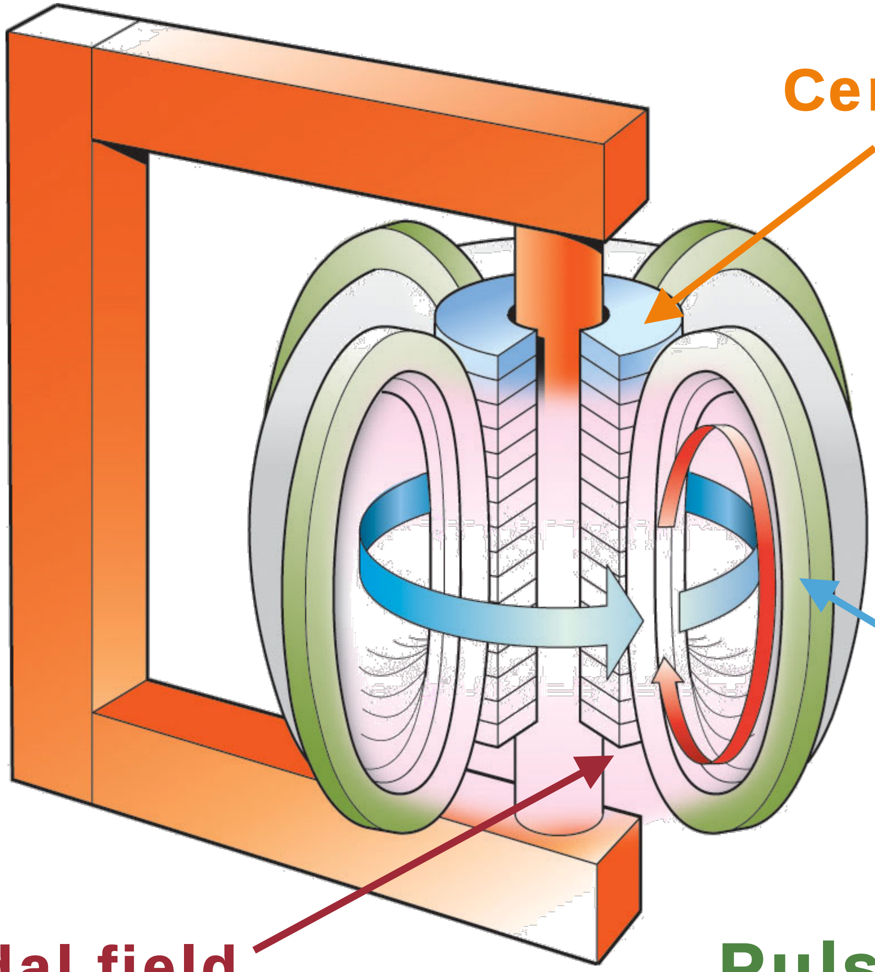


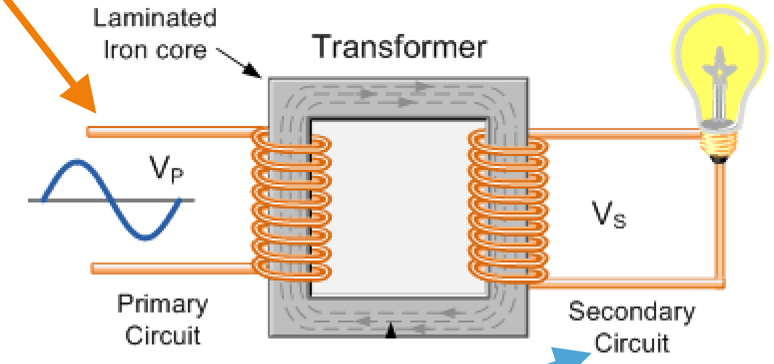
KSTAR

http://www.electronics-tutorials.ws/electromagnetism/electromagnetic-induction.html

Tokamak

Central Solenoid (CS)





Plasma

Poloidal field

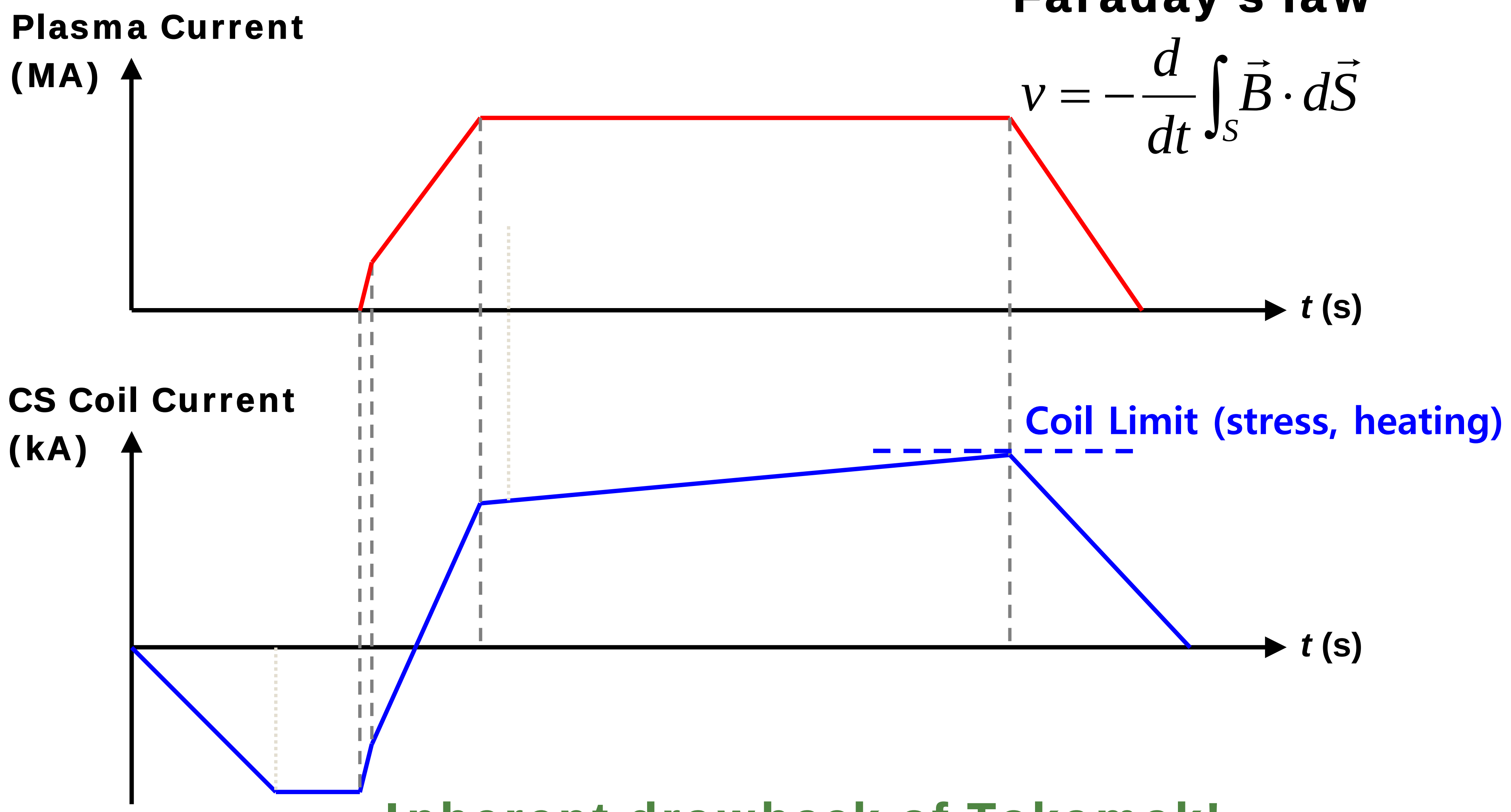
Pulsed Operation!

http://www.electronics-tutorials.ws/electromagnetism/electromagnetic-induction.html

Pulsed Operation

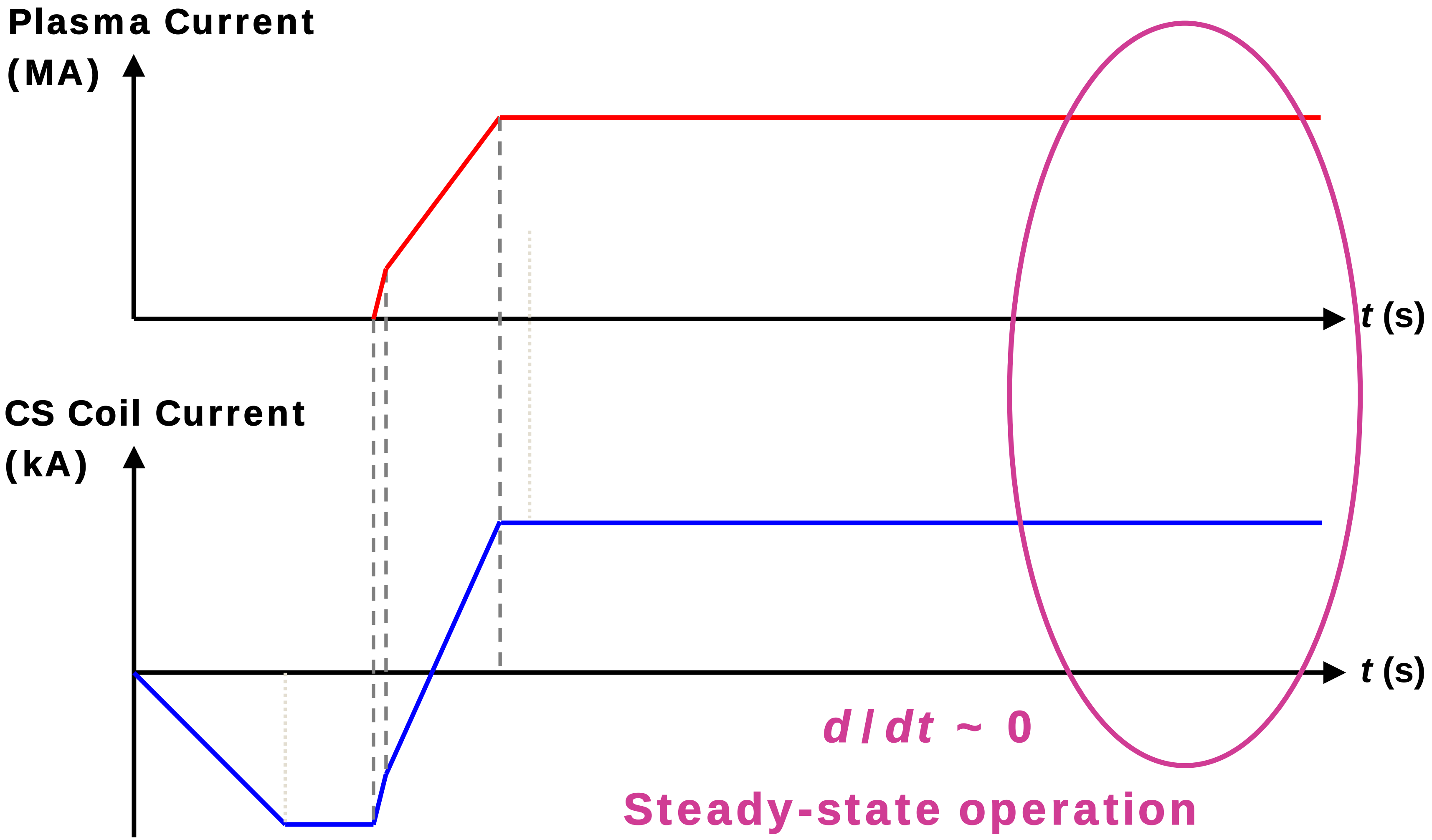
Faraday's law

$$v = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}$$



Inherent drawback of Tokamak!

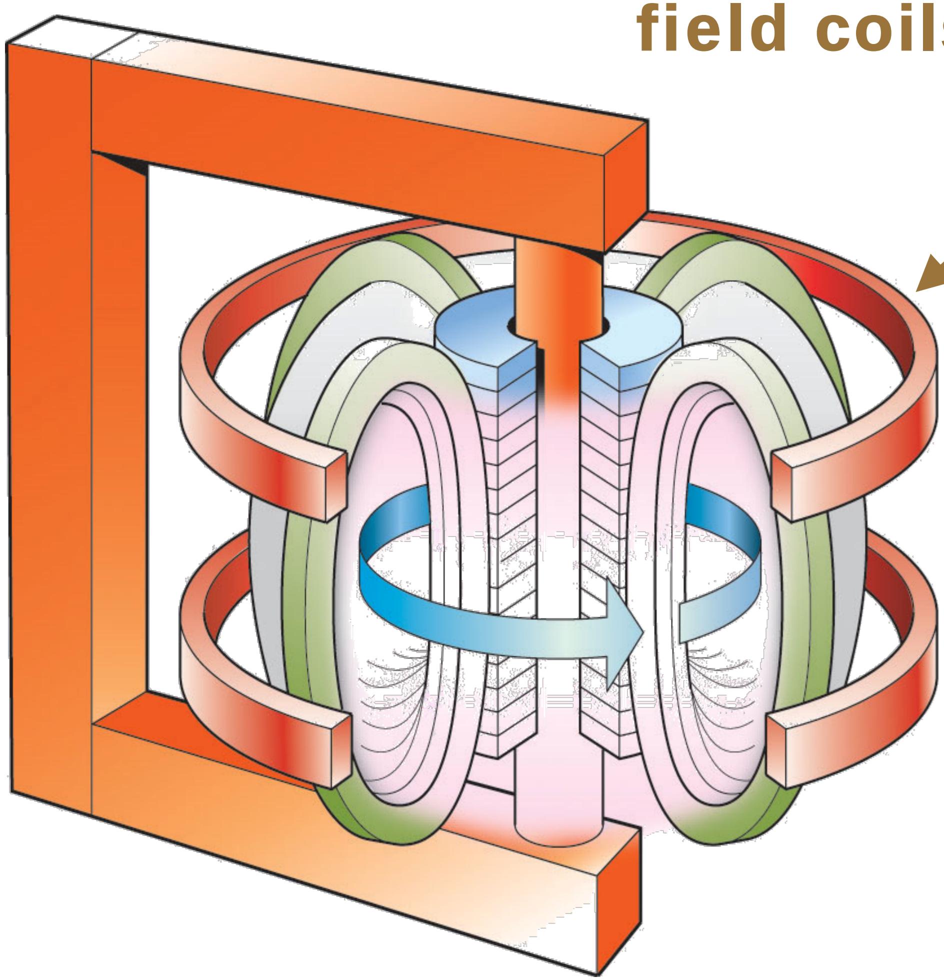
Steady-State Operation

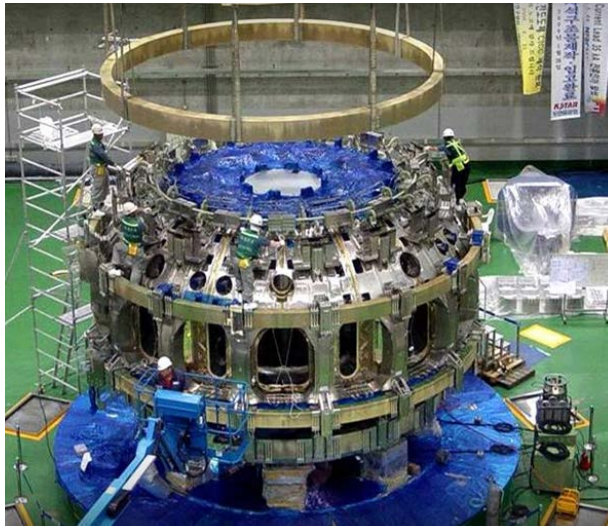


Steady-state operation
by self-generated and externally driven current

Tokamak

Adding vertical (equilibrium) field coils (PF: Poloidal Field)

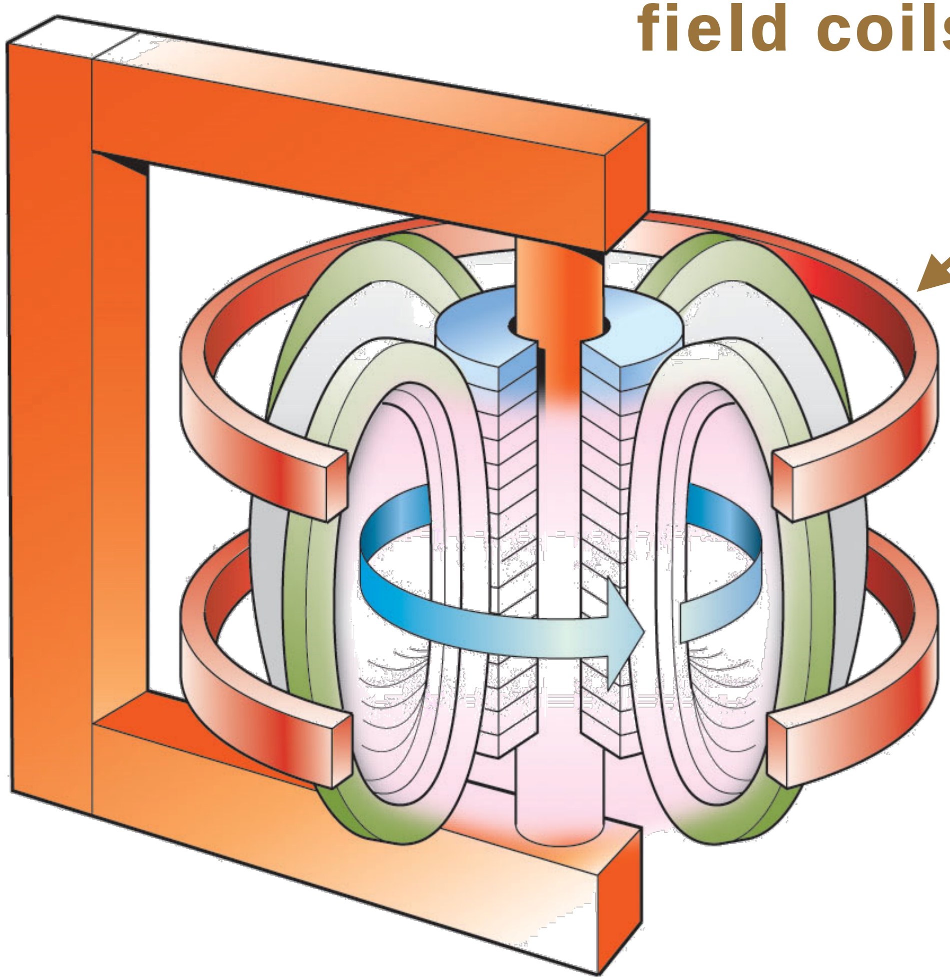




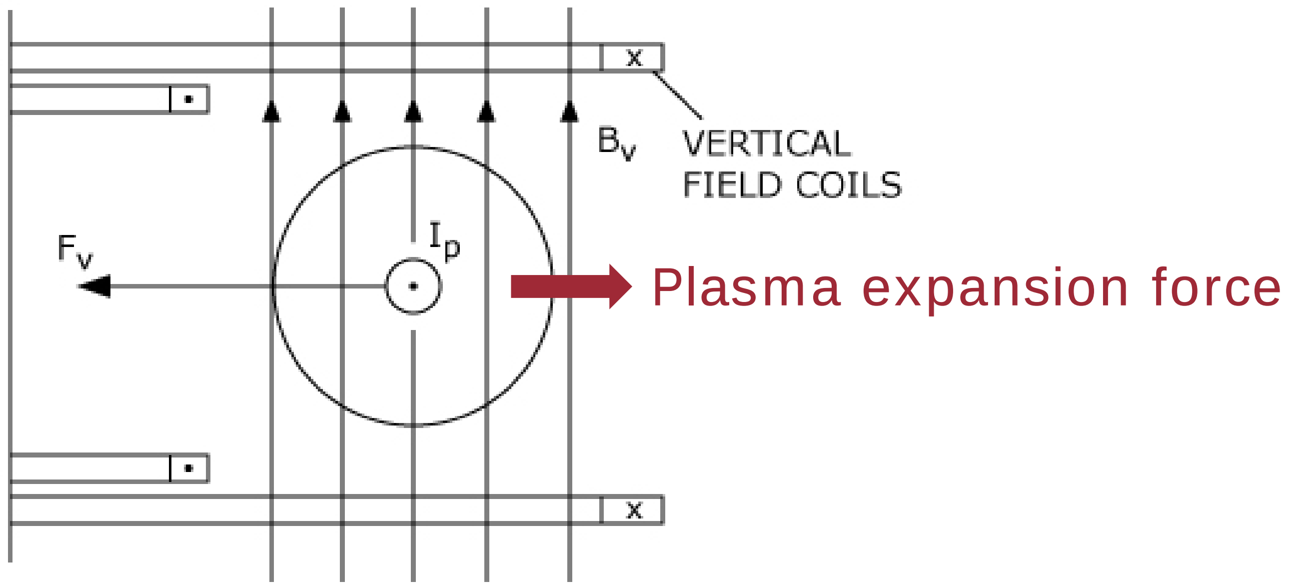
KSTAR

Tokamak

Adding vertical (equilibrium) field coils (PF: Poloidal Field)



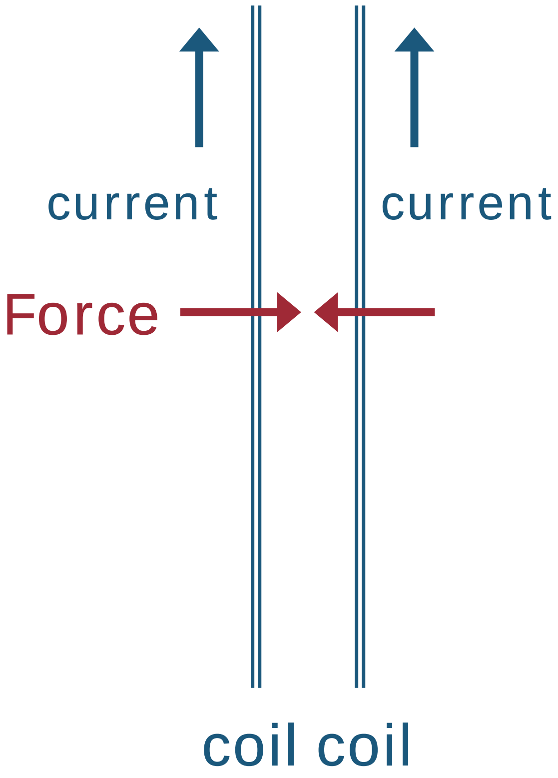
Tokamak

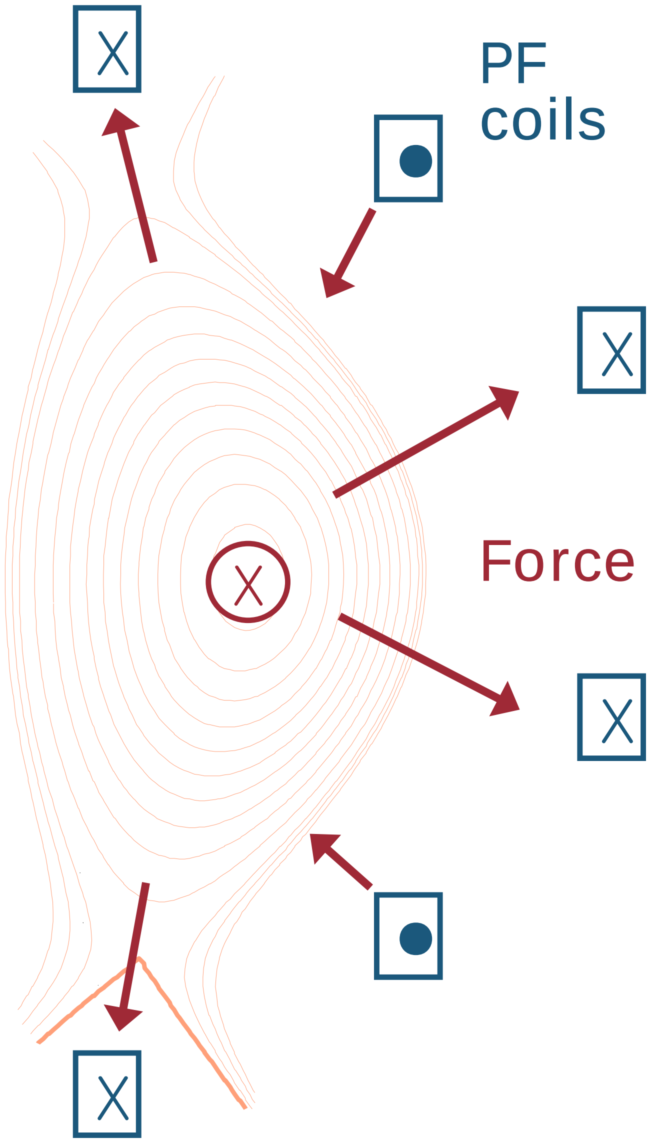


Force balance by vertical field coils:
Plasma positioning

J.P. Freidberg, "Ideal Magneto-Hydro-Dynamics", lecture note

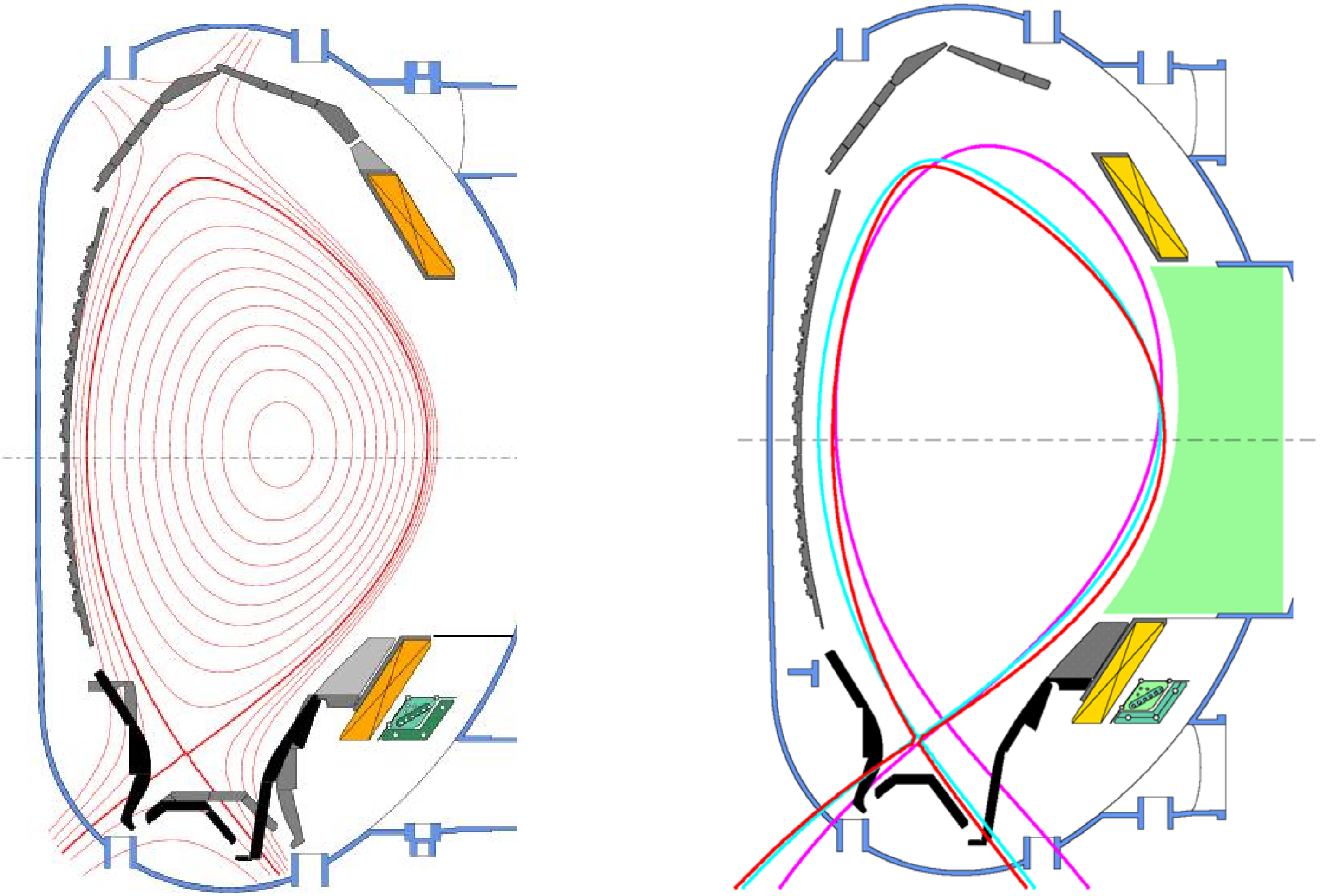
Tokamak





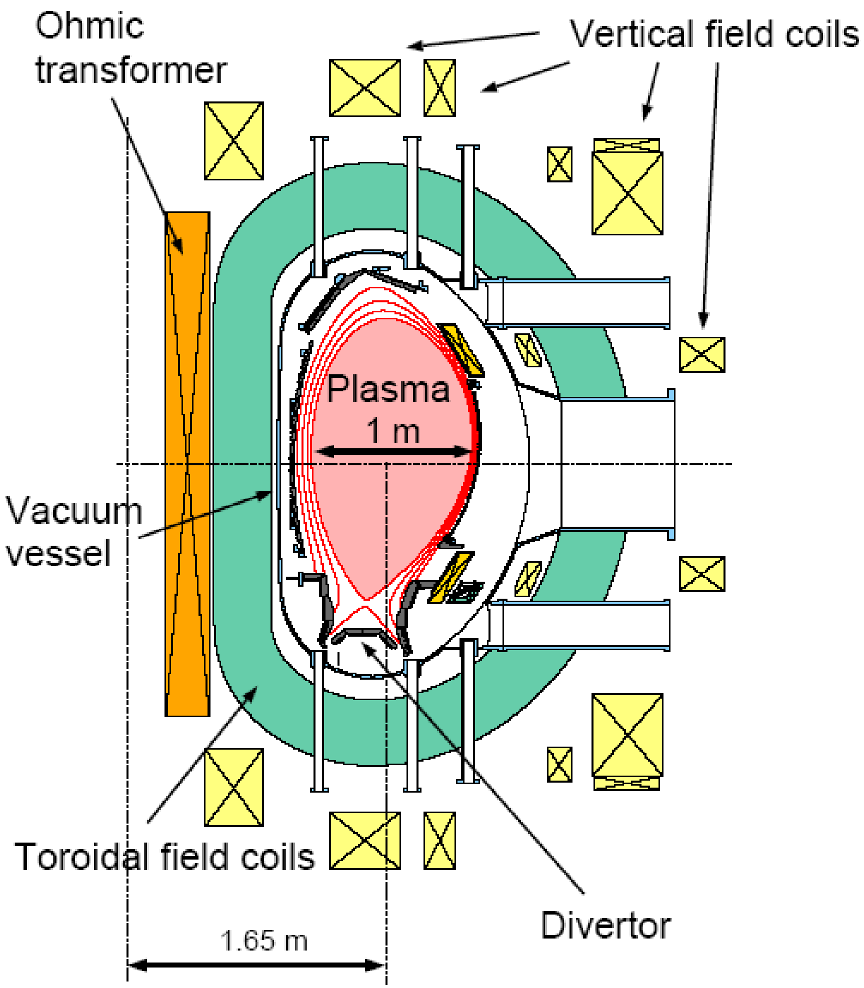
Plasma shaping by PF coils

Tokamak

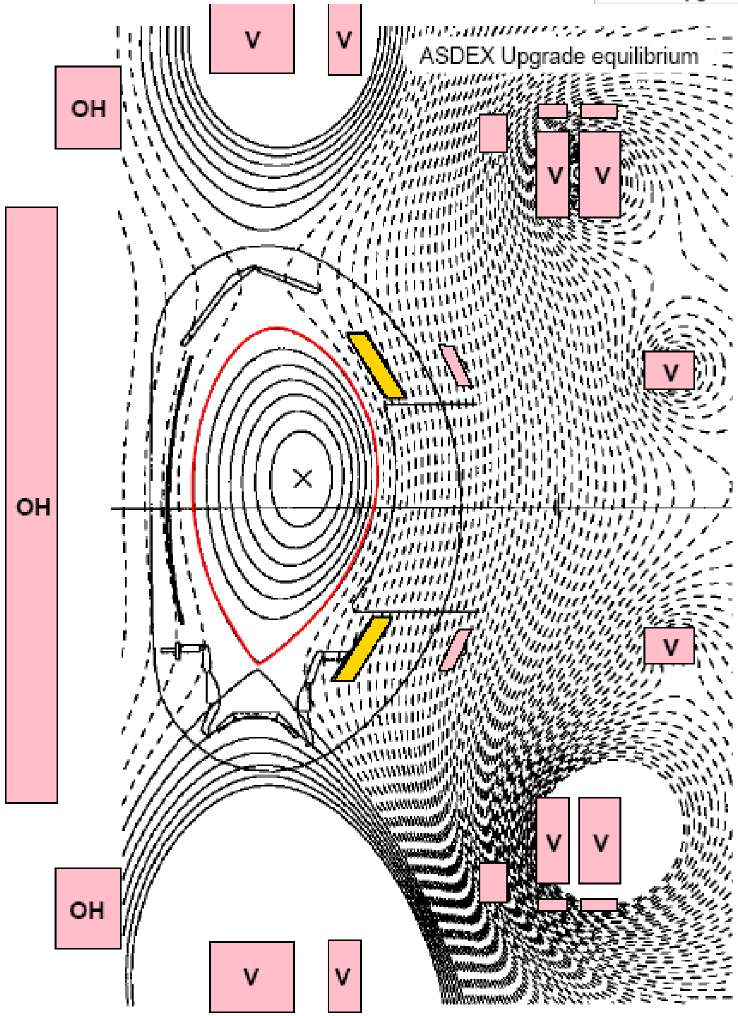


The plasma shape can be modified by PF coil currents.

Tokamak



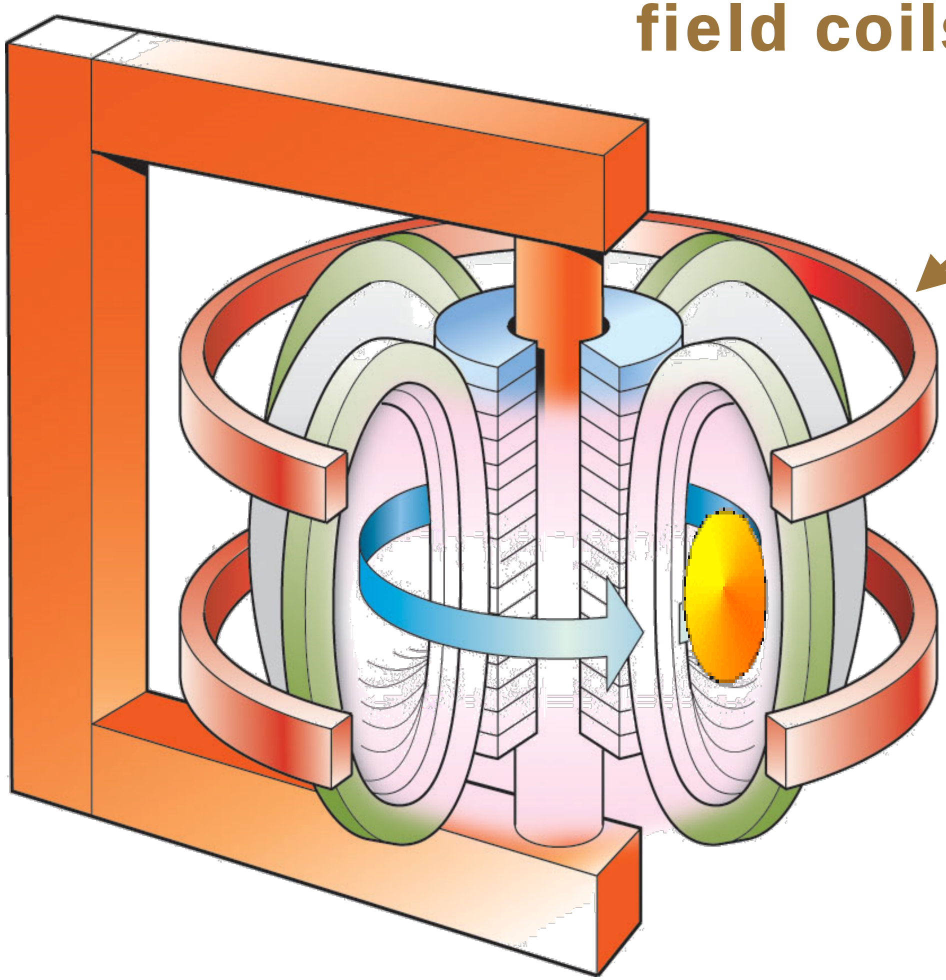
Ohmic transformer
Vertical field coils
Plasma
1 m
Vacuum vessel
Toroidal field coils
Divertor
1.65 m



ASDEX Upgrade equilibrium
OH
V

Tokamak

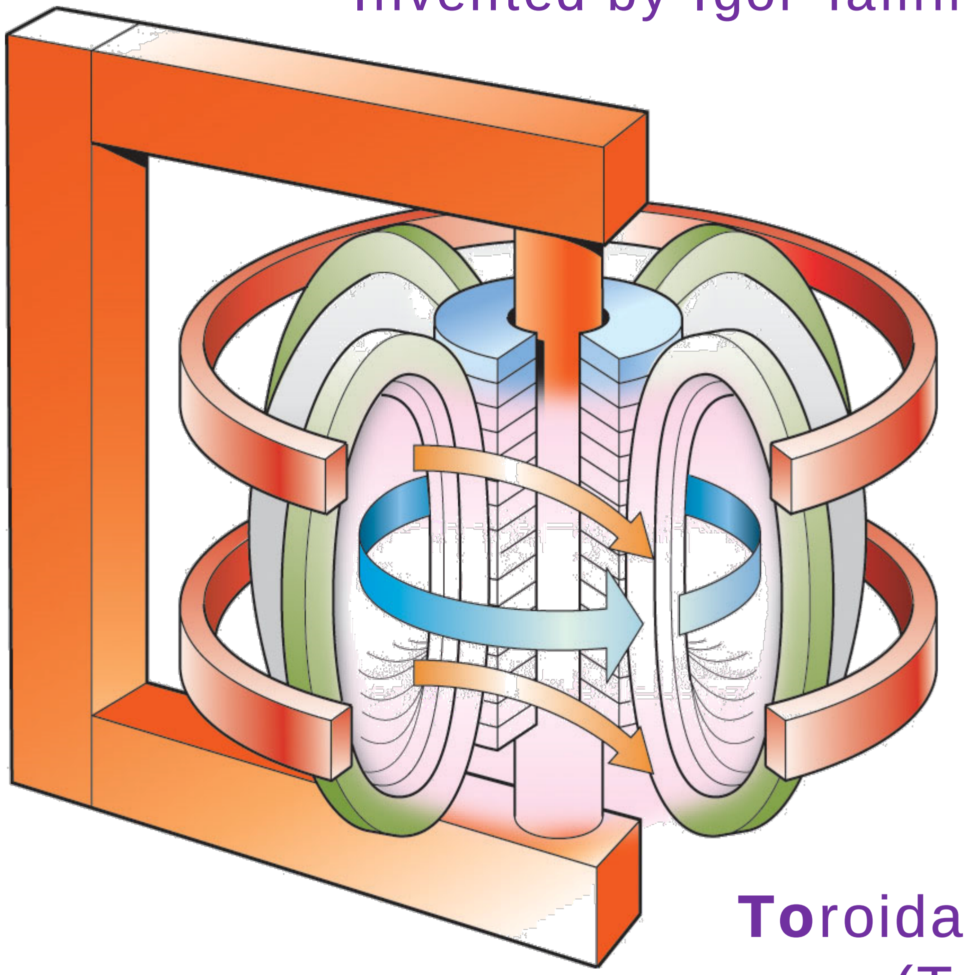
Adding vertical (equilibrium) field coils (PF: Poloidal Field)

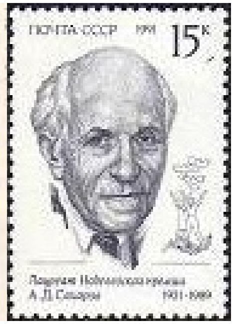


Plasma positioning & shaping by PF coils

Tokamak

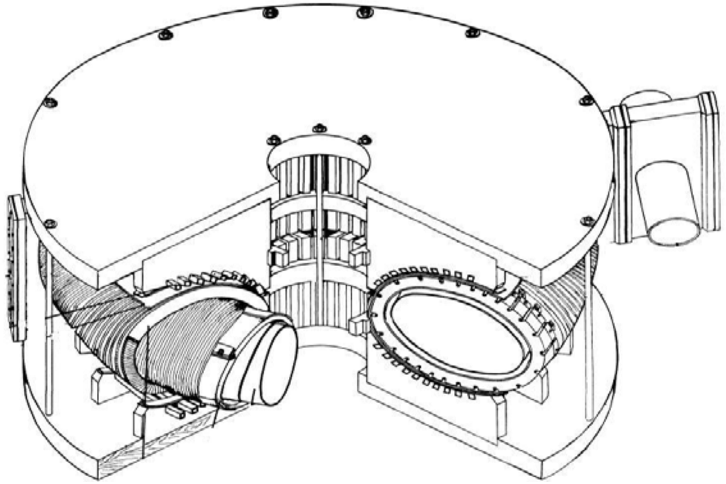
Invented by Igor Tamm and Andrei Sakharov in 1952





Toroidalnaja **ka**mera **ma**gnitnaja **k**atushka
(Toroidal chamber magnetic coil)

Tokamak

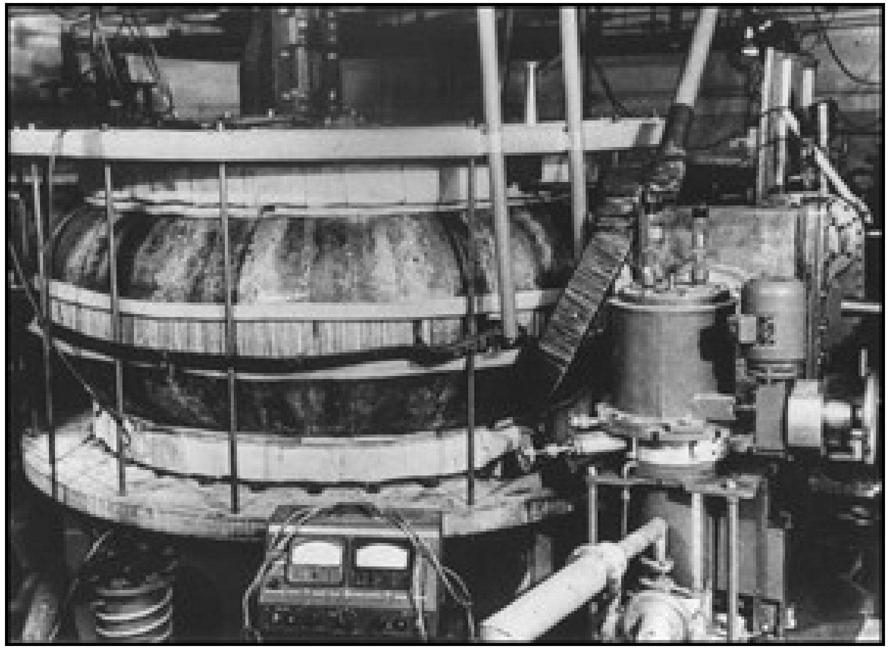


Cutaway of the Toroidal Chamber in Artsimovitch's Paper *Research on Controlled Nuclear Fusion in the USSR*

WIKIPEDIA

Toroidalnaja **ka**mera **ma**gnitnaja **k**atushka
(Toroidal chamber magnetic coil)

1958 IAEA FEC, Geneva, Switzerland



T1: The world's first tokamak,
Kurchatov Institute, Moscow Russia

Tokamak

JET (Joint European Torus): R_0 = 3 m, a = 0.9 m, 1983-today

http://imgarcade.com/1/spherical-tokamak/

Tokamak

JET (Joint European Torus): R_0 = 3 m, a = 0.9 m, 1983-today

Tokamak

JET (Joint European Torus): R_0 = 3 m, a = 0.9 m, 1983-today

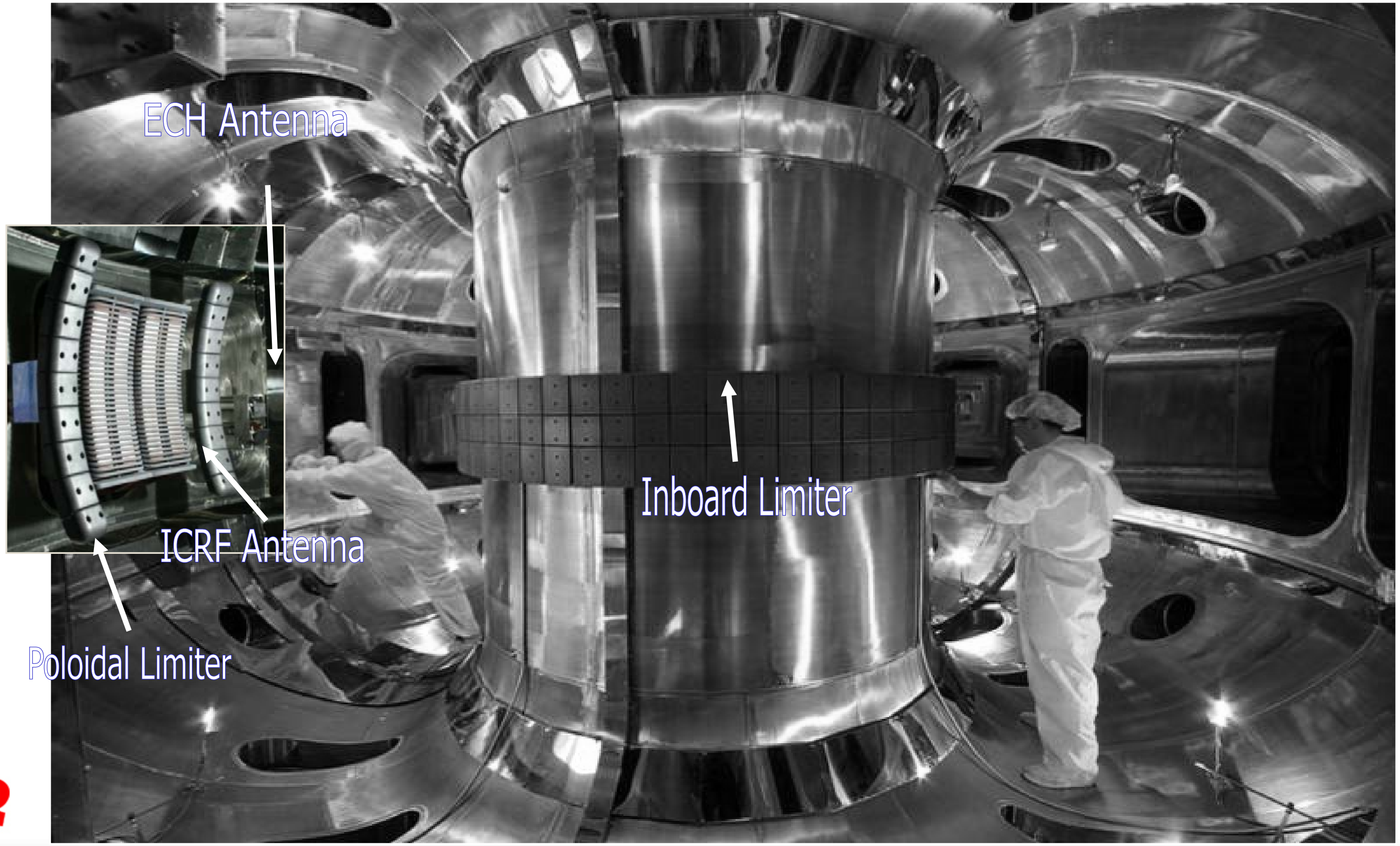
Tokamak

http://cafe.naver.com/gamebox26.cafe?iframe_url=/ArticleRead.nhn%3Farticleid=16140

Tokamak

KSTAR (Korea Superconducting Tokamak Advanced Research):

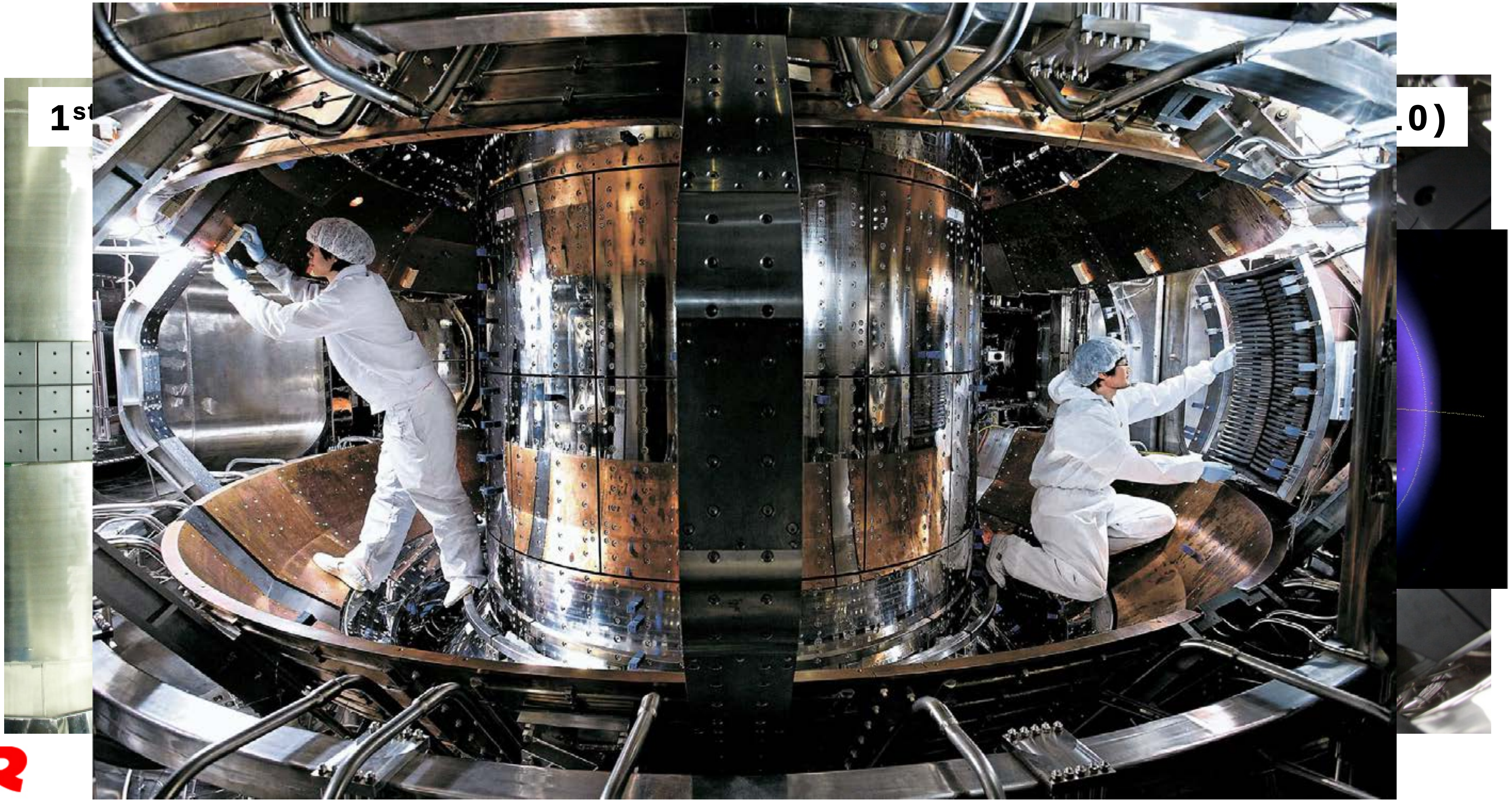
R_0 = 1.8 m, a = 0.5 m, 2007-today



KSTAR

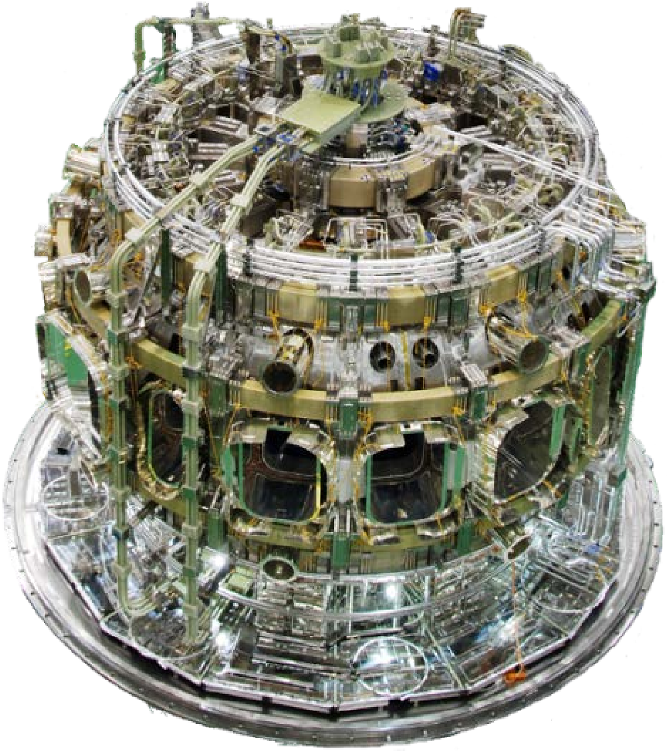
Tokamak

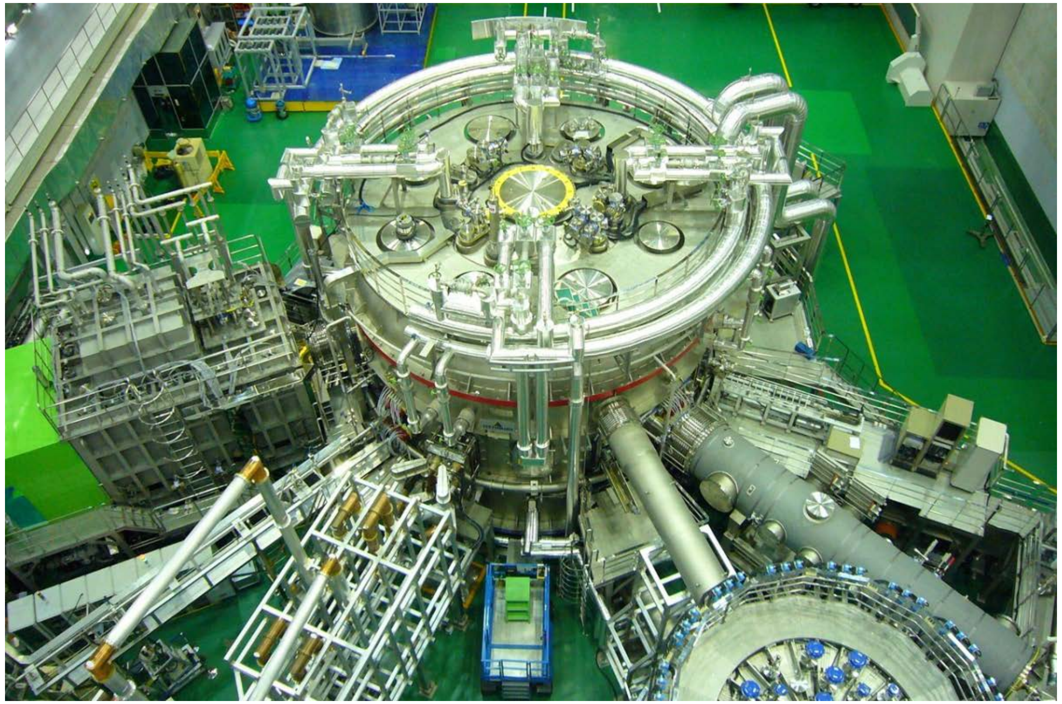
KSTAR (Korea Superconducting Tokamak Advanced Research):
R_0 = 1.8 m, a = 0.5 m, 2007-today



Tokamak

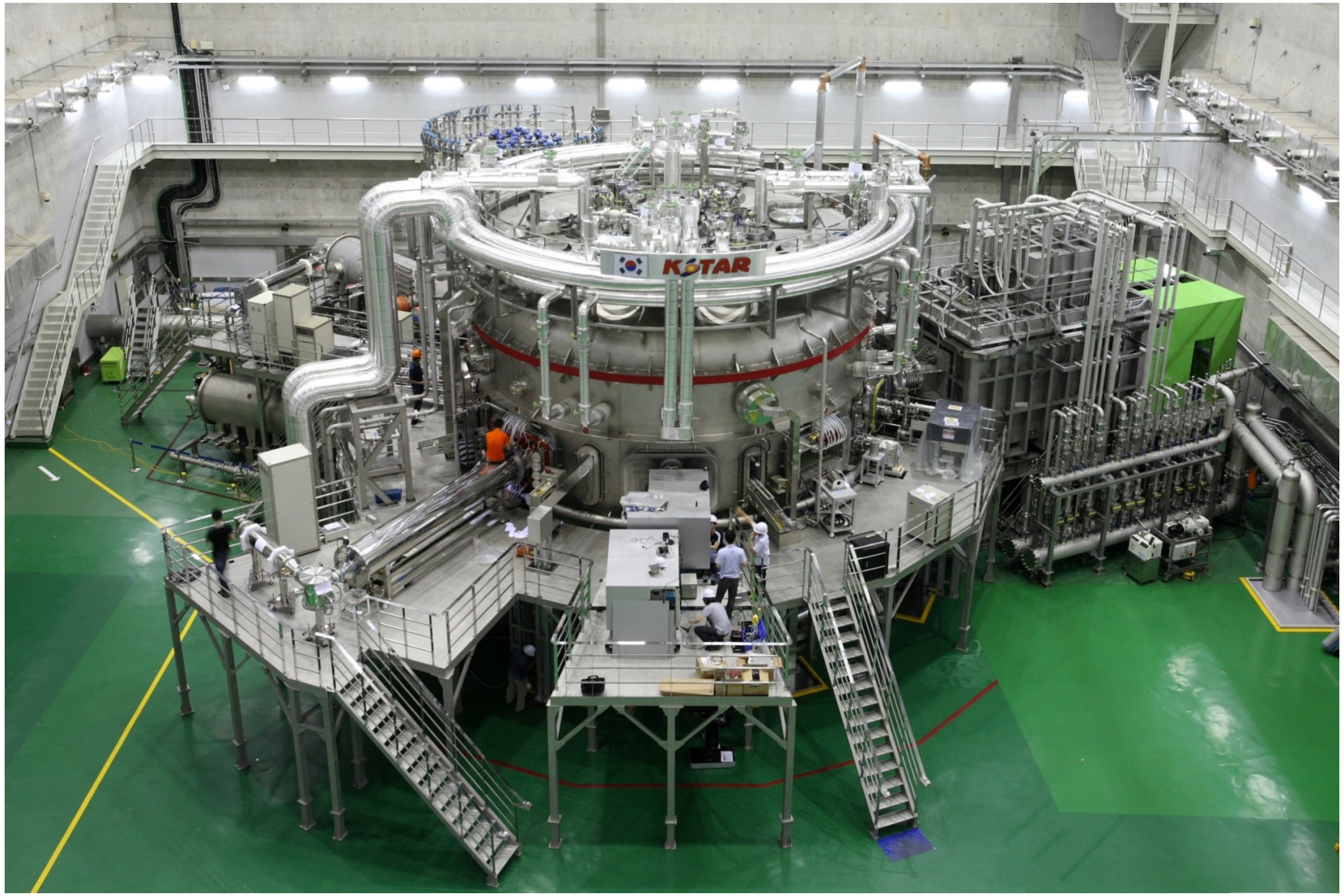
KSTAR (Korea Superconducting Tokamak Advanced Research):
R_0 = 1.8 m, a = 0.5 m, 2007-today





Tokamak

KSTAR (Korea Superconducting Tokamak Advanced Research):
R_0 = 1.8 m, a = 0.5 m, 2007-today



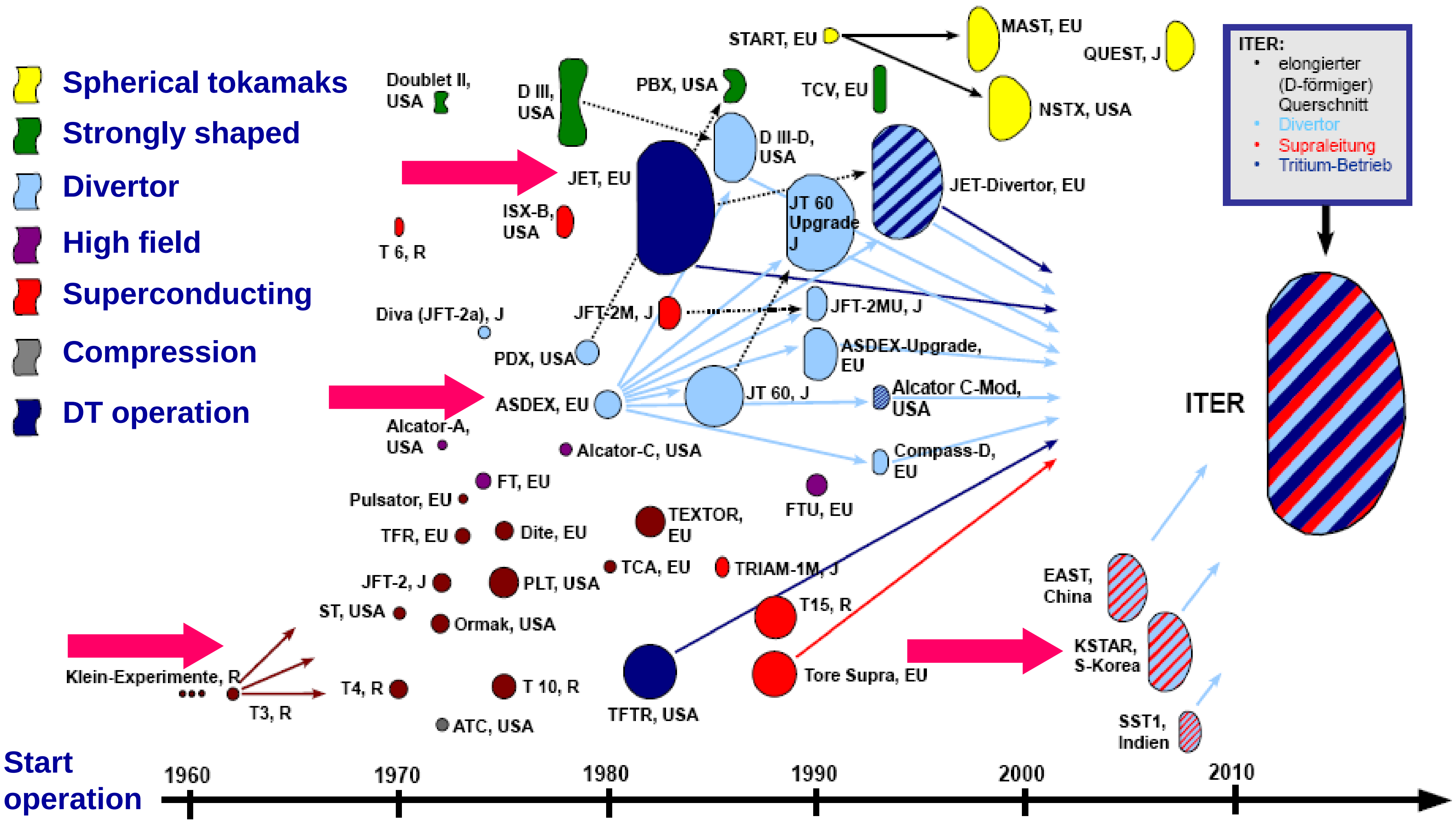
Tokamak

KSTAR (Korea Superconducting Tokamak Advanced Research):
R_0 = 1.8 m, a = 0.5 m, 2007-today

KSTAR 1st plasma

Analyse the KSTAR 1st plasma

Tokamak

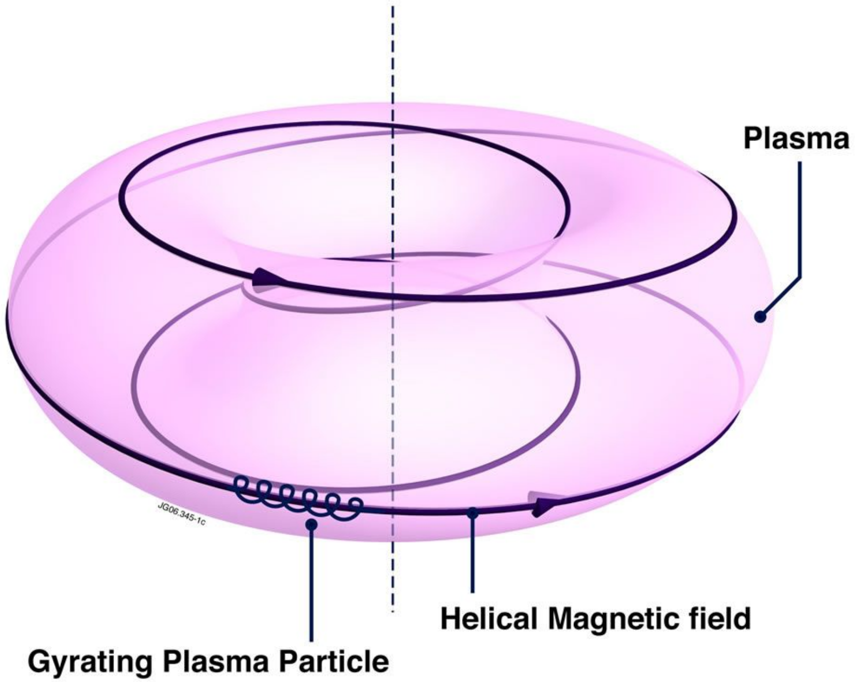


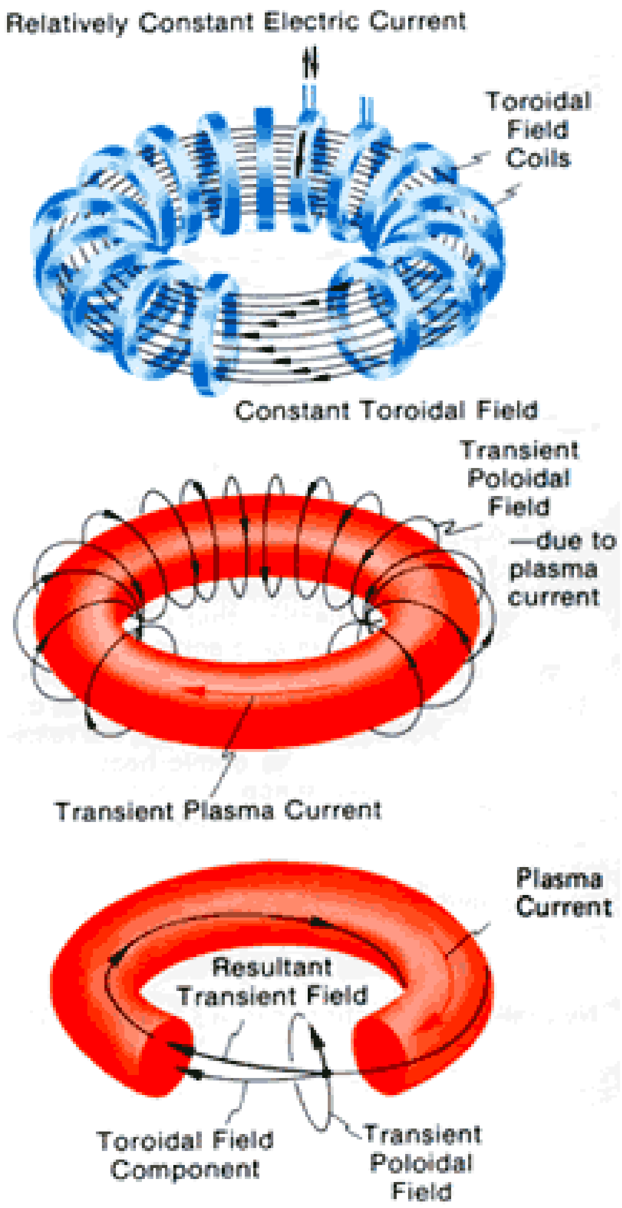
Spherical tokamaks
Strongly shaped
Divertor
High field
Superconducting
Compression
DT operation
START, EU
MAST, EU
QUEST, J
NSTX, USA
Doublet II, USA
D III, USA
PBX, USA
TCV, EU
D III-D, USA
JET, EU
JET-Divertor, EU
ISX-B, USA
T 6, R
JT 60 Upgrade J
Diva (JFT-2a), J
JFT-2M, J
JFT-2MU, J
PDX, USA
ASDEX-Upgrade, EU
ASDEX, EU
JT 60, J
Alcator C-Mod, USA
Alcator-A, USA
Alcator-C, USA
Compass-D, EU
FT, EU
Pulsator, EU
FTU, EU
TEXTOR, EU
TFR, EU
Dite, EU
TCA, EU
TRIAM-1M, J
JFT-2, J
PLT, USA
ST, USA
Ormak, USA
T15, R
Klein-Experimente, R
T4, R
T 10, R
TFTR, USA
Tore Supra, EU
T3, R
ATC, USA
EAST, China
KSTAR, S-Korea
SST1, Indien
ITER
ITER:
elongierter (D-förmiger) Querschnitt
Divertor
Supraleitung
Tritium-Betrieb
Start operation
1960
1970
1980
1990
2000
2010

Basic tokamak variables

How to characterize a tokamak?

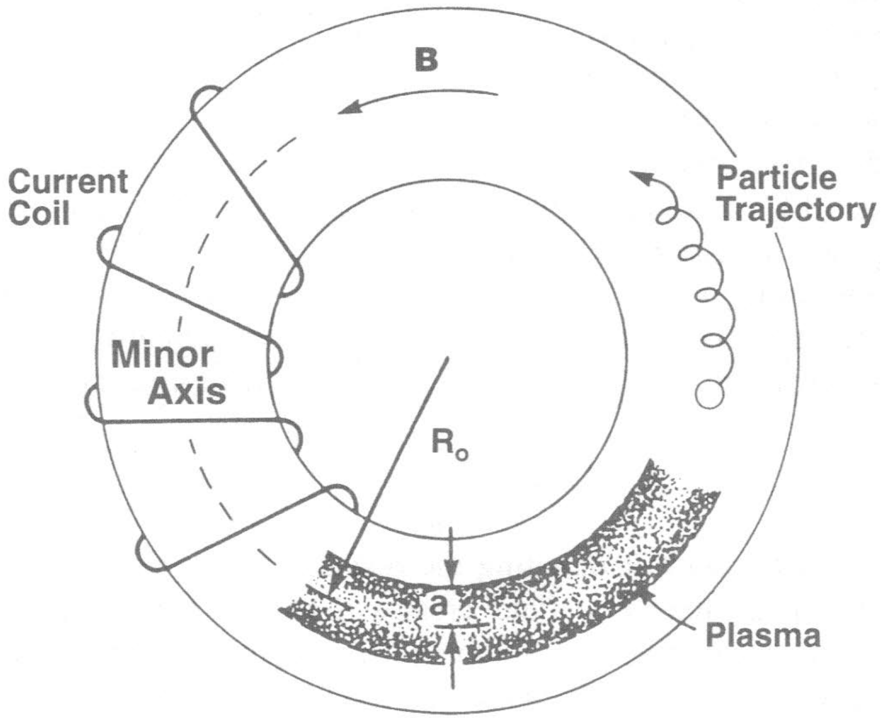
Tokamak

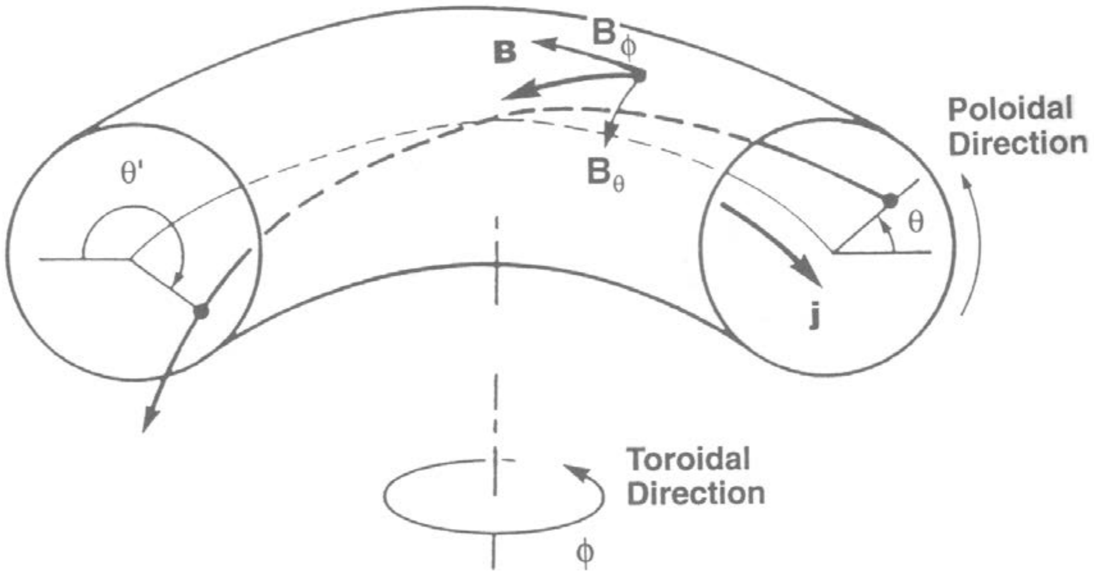

Plasma
Helical Magnetic field
Gyrating Plasma Particle


Relatively Constant Electric Current
Toroidal Field Coils
Constant Toroidal Field
Transient Poloidal Field
—due to plasma current
Transient Plasma Current
Plasma Current
Resultant Transient Field
Toroidal Field Component
Transient Poloidal Field

Basic Tokamak Variables

- **Cylindrical and local coordinates for a tokamak**



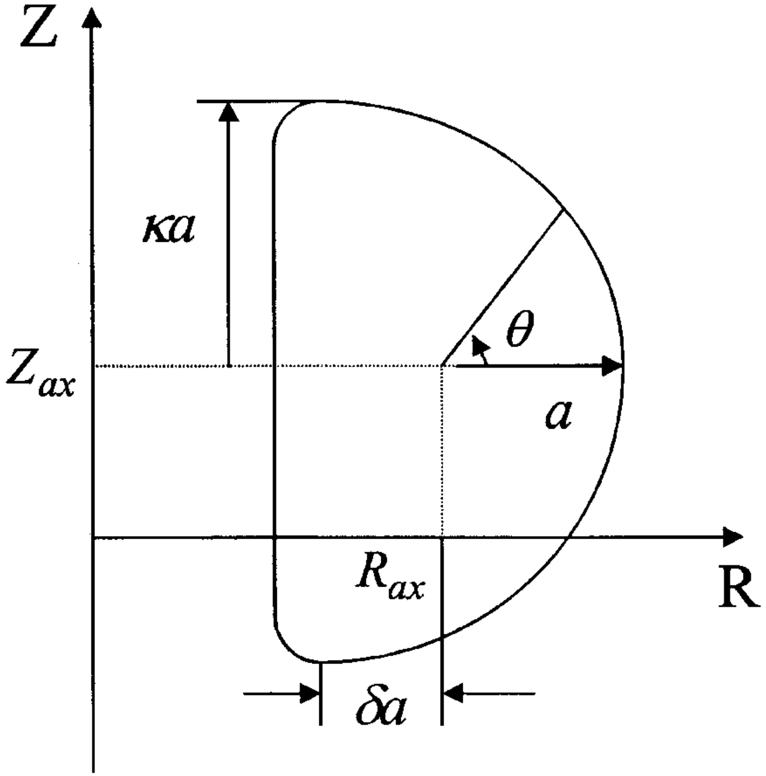


- Major radius: R_0, minor radius: a
- Aspect ratio: R_0/a ~ 3-5
  ex) KSTAR: 3.6, ITER: 3.1

A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Basic Tokamak Variables

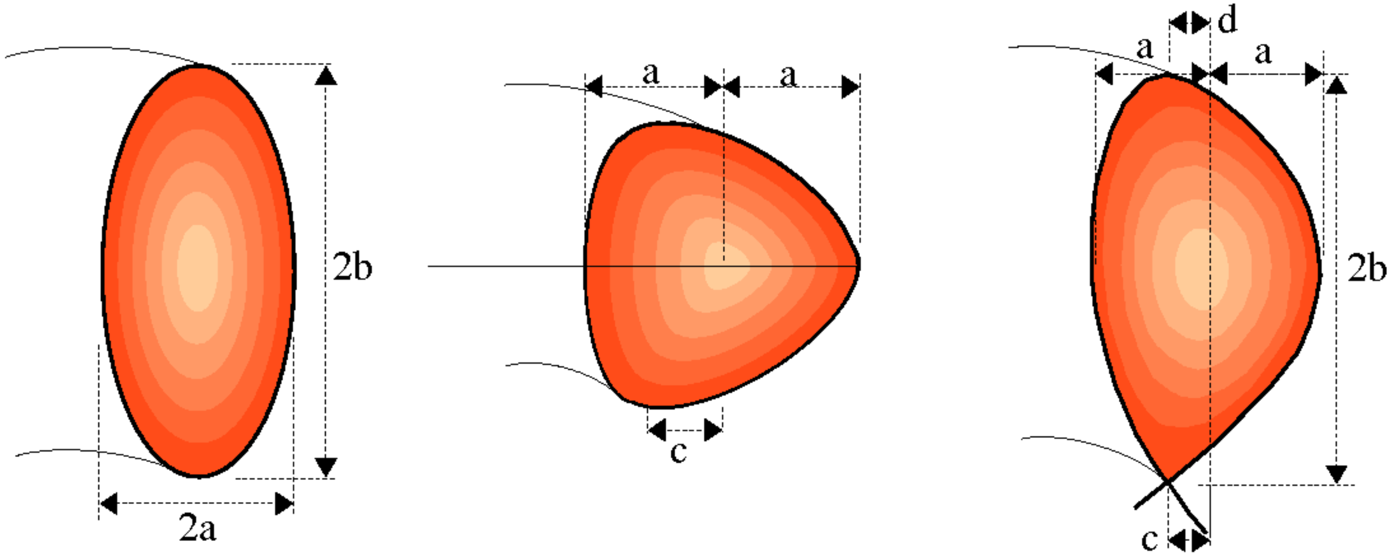
- **Plasma equilibrium parameters**



- Elongation: κ
- Triangularity: δ

Basic Tokamak Variables

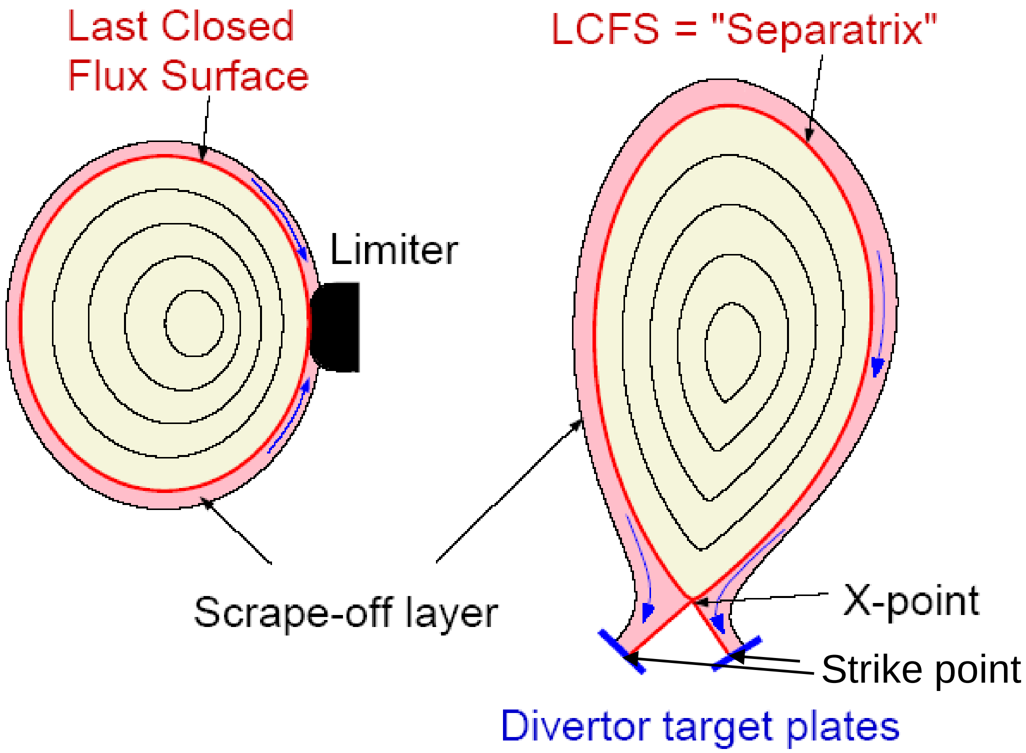
- **Plasma equilibrium parameters**

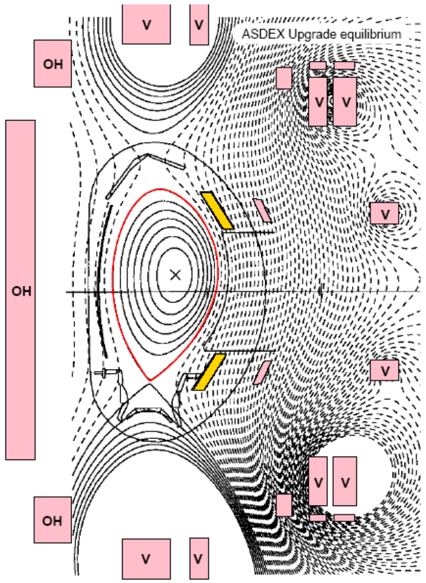


$$\kappa = \frac{b}{a} \qquad \delta = \frac{c+d}{2a}$$

Basic Tokamak Variables

- **Separation of plasma from wall by a limiter and a divertor**



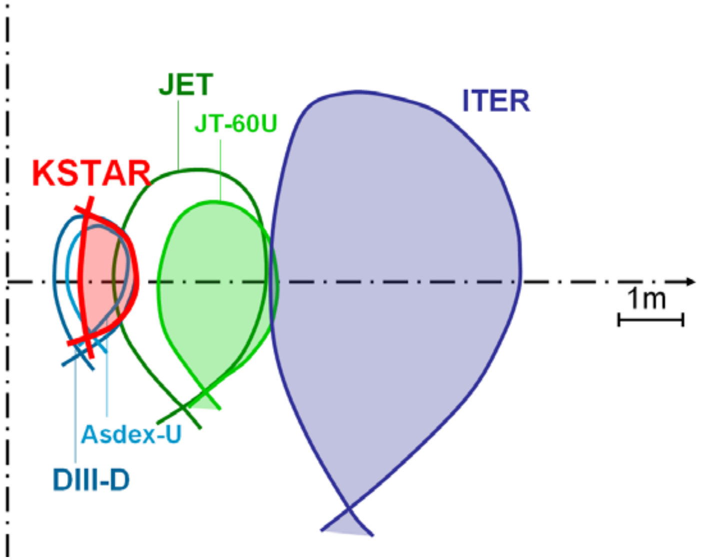


Basic Tokamak Variables

• **Plasma equilibrium parameters**

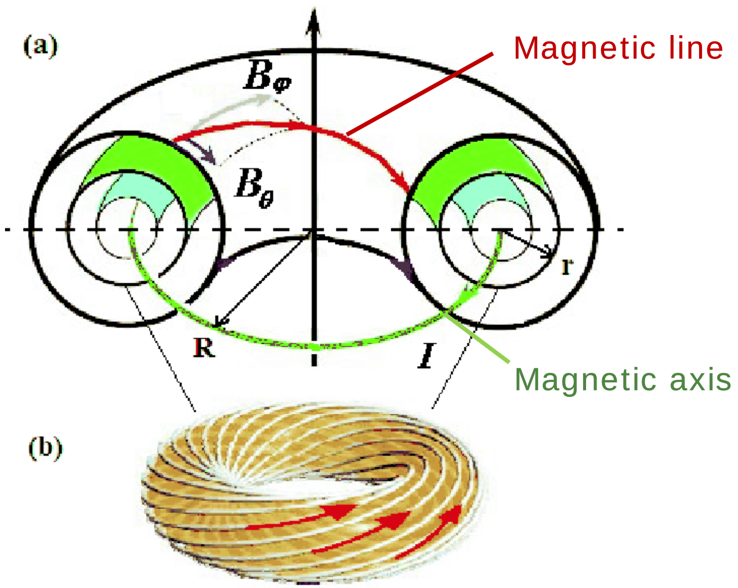
Parameters	KSTAR	ITER
Major Radius, R_0	1.8 m	6.2 m
Minor Radius, a	0.5 m	2.0 m
Plasma Current, I_P	2.0 MA	15 MA
Elongation, κ_x	2.0	1.85
Triangularity, δ_x	0.8	0.5
Toroidal Field, B_0	3.5 T	5.3 T
Pulse Length	300 s	500 s
Fuel	H, D	D, T

- Plasma shape



Magnetic Flux Surface

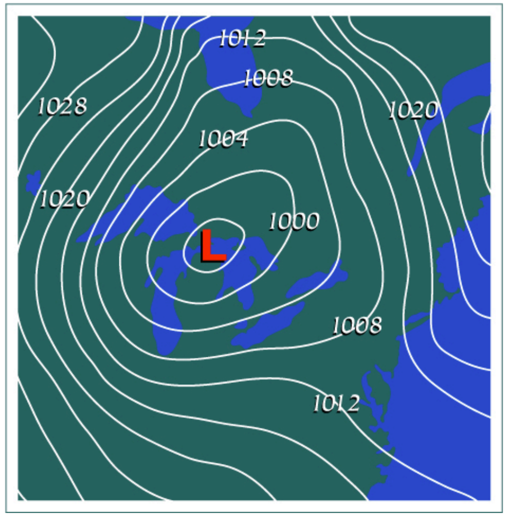
- In a tokamak configurations with confined plasmas the magnetic lines lie on a set of nested toroidal surfaces called flux surfaces.
- Pressure is constant along a magnetic field line.
- Magnetic lines lie in surfaces of constant pressure.
- Flux surfaces are surfaces of constant pressure.
- The current lines lie on surfaces of constant pressure.



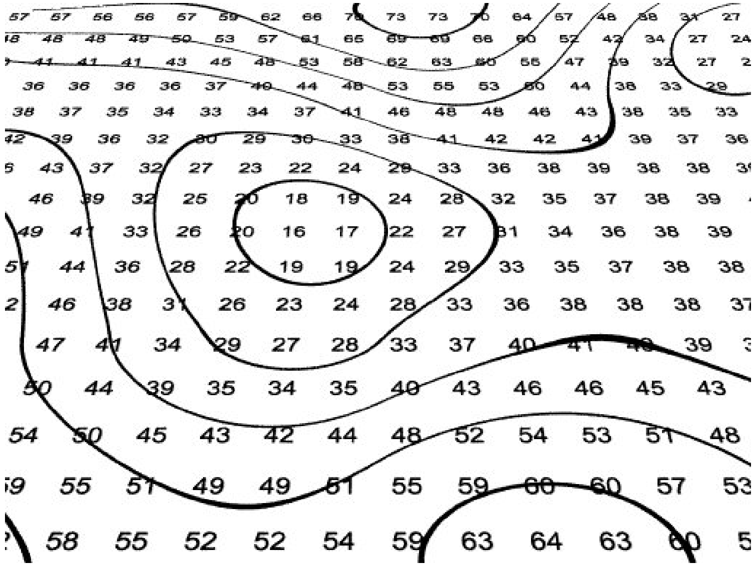
$$\vec{J} \times \vec{B} = \nabla p$$

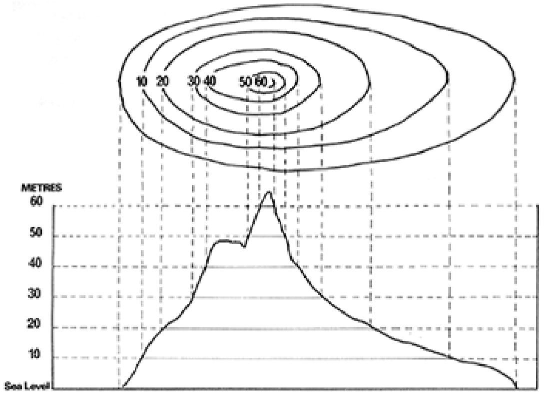
$$\vec{B} \cdot \nabla p = 0 \quad \vec{J} \cdot \nabla p = 0$$

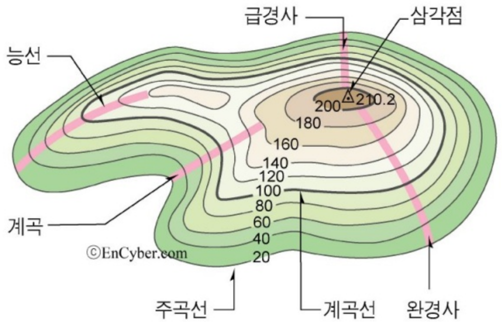
Magnetic Flux Surface



Isobar: p = const.







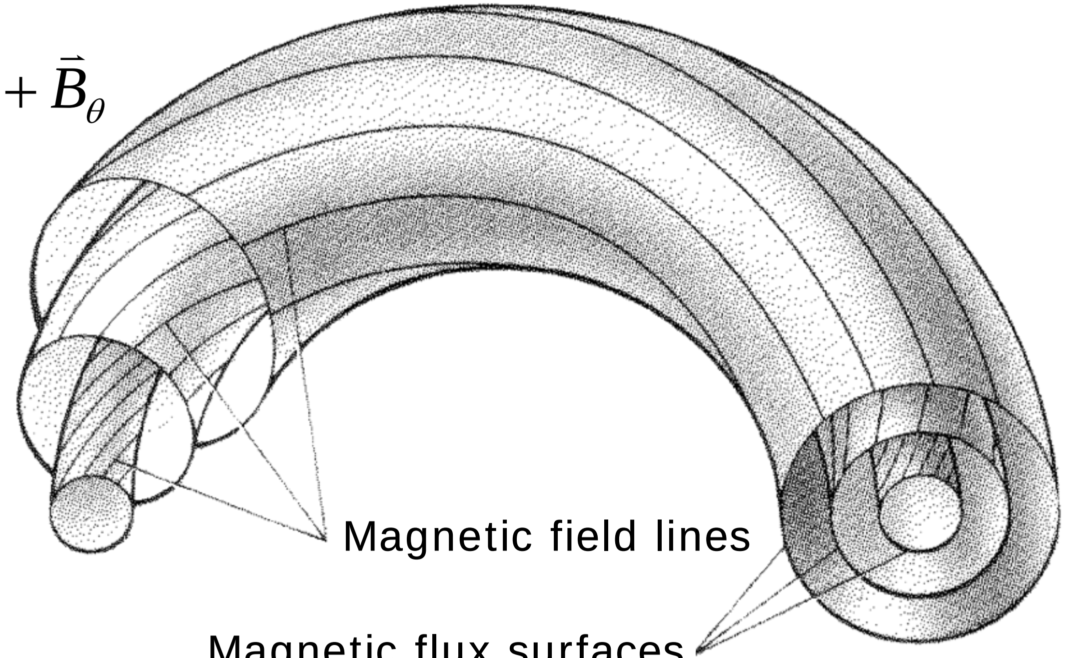
http://www.freewebs.com/weatherexplorer/apps/blog/show/286262
http://www.econym.demon.co.uk/isotut/isobars.htm

http://blog.daum.net/eco205sky/12854349

Basic Tokamak Variables

- **Safety factor *q* = number of toroidal orbits per poloidal orbit**

$$\vec{B} = \vec{B}_\phi + \vec{B}_\theta$$



- Rotational transform:
  Δθ? when *ΔΦ* = 2π

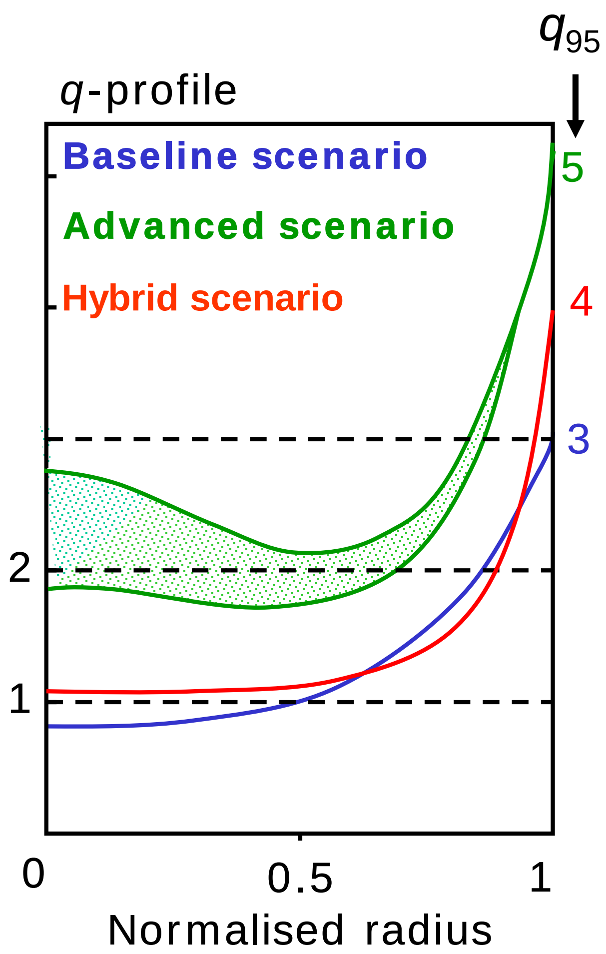
$$\iota = \frac{\Delta\theta}{\frac{\Delta\phi}{2\pi}} = \frac{\frac{B_\theta}{r}}{\frac{B_\phi}{2\pi R}} = \frac{2\pi B_\theta}{r B_\phi}$$

$$\leftarrow \quad \frac{R d\phi}{B_\phi} = \frac{r d\theta}{B_\theta}$$

$$q = \frac{\text{number of toroidal windings}}{\text{number of poloidal windings}} = \frac{2\pi}{\iota} = \frac{r}{R}\frac{B_\phi}{B_\theta}$$

Basic Tokamak Variables

- **Safety factor q = number of toroidal orbits per poloidal orbit**



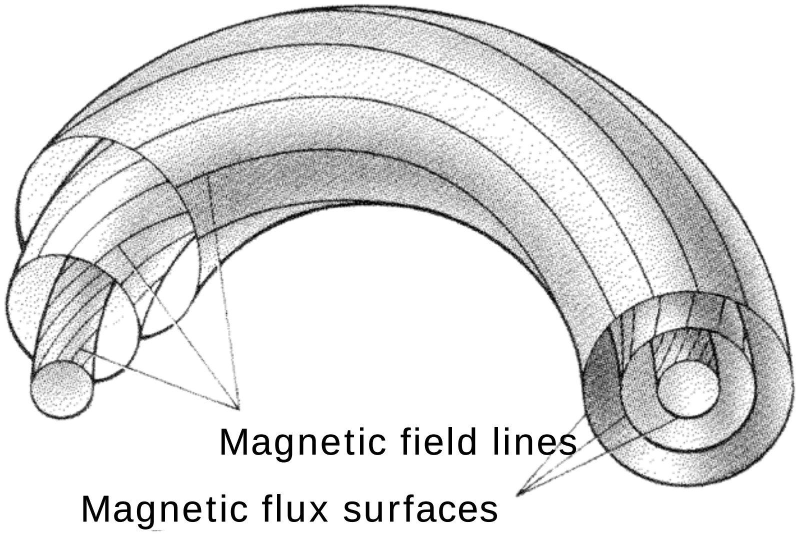
Basic Tokamak Variables

- **Magnetic Shear**

  - Measuring the change in pitch angle of a magnetic field line from one flux surface to the next
  - Playing an important role in stabilizing MHD instabilities, particularly those driven by the pressure gradient:

$$s(r) \equiv \frac{r}{q}\frac{dq}{dr}$$

A perturbation aligned with $\mathbf{B}(r)$ will, at a point with increased minor radial distance $r+dr$, encounter field lines at a different angle which again will vary as the perturbation grows to another distance $r+dr'$. Any helically resonant instabilities are thus radially localised.



Tokamak equilibrium

Tokamak Equilibrium

$$\nabla p = \vec{J} \times \vec{B}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

- Consider the axisymmetric torus, the simplest, multi-dimensional configuration
- We shall derive the Grad–Shafranov equation for axisymmetric equilibria.
- This provides a complete description of toroidal equilibrium:

  radial pressure balance, toroidal force balance,

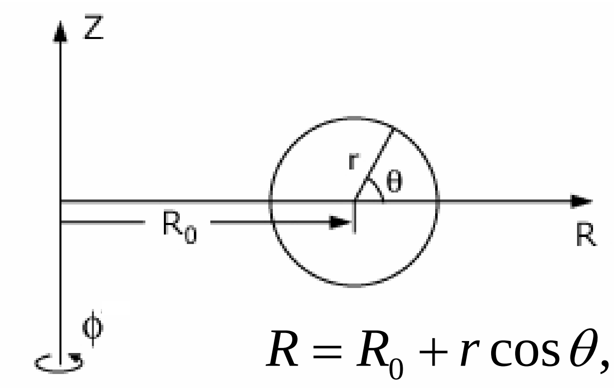
  β limits, q-profiles, etc.

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

  - Nonlinear
  - Partial differential equation
  - Grad and Rubin (1958), Shafranov (1960)
  - Toroidal axisymmetric $\partial/\partial\Phi = 0$



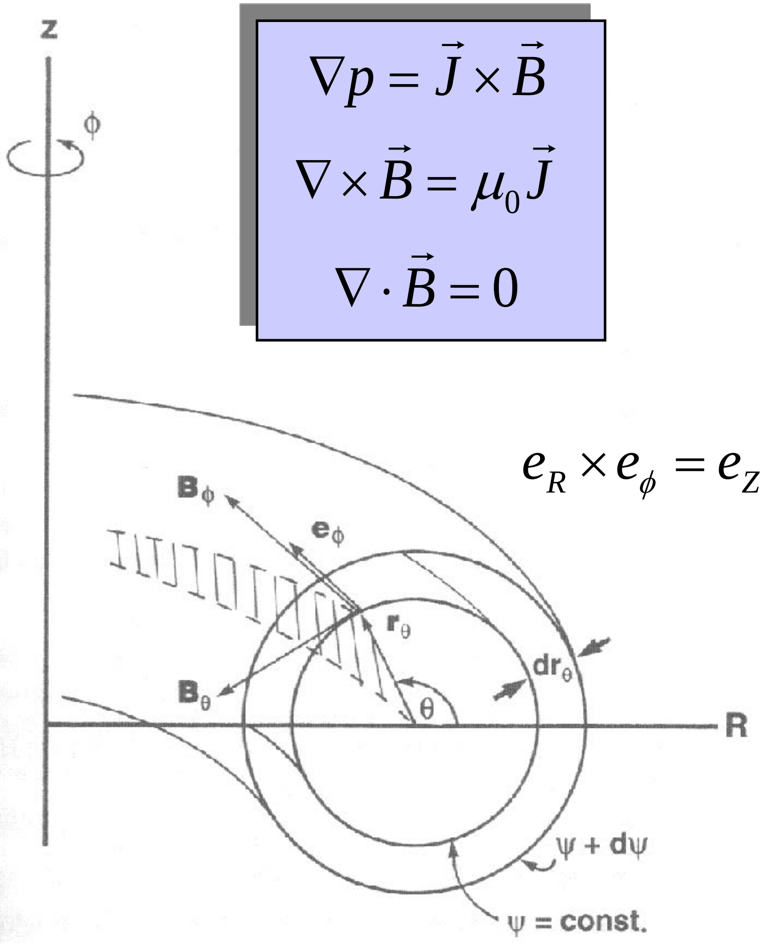
$$R = R_0 + r\cos\theta, \quad Z = r\sin\theta$$

$$\nabla p = \vec{J}\times\vec{B}$$

$$\nabla\times\vec{B} = \mu_0\vec{J}$$

$$\nabla\cdot\vec{B} = 0$$

$$e_R \times e_\phi = e_Z$$



J.P. Freidberg, "Ideal Magneto-Hydro-Dynamics", lecture note

A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

Sequence of solution of the MHD equilibrium equations

1. The $\nabla\cdot\mathbf{B} = 0$
2. Ampere's law: $\mu_0\mathbf{J} = \nabla\times\mathbf{B}$
3. The momentum equation: $\mathbf{J}\times\mathbf{B} = \nabla p$

$$\boxed{\begin{aligned}\nabla p &= \vec{J}\times\vec{B}\\ \nabla\times\vec{B} &= \mu_0\vec{J}\\ \nabla\cdot\vec{B} &= 0\end{aligned}}$$

$$\vec{J}\times\vec{B} = \nabla p$$

$$\nabla p = \left(\vec{J}_\phi + \vec{J}_p\right)\times\left(\vec{B}_\phi + \vec{B}_p\right)$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Momentum equation $\vec{J} \times \vec{B} = \nabla p$

$$\nabla p = \left(\vec{J}_\phi + \vec{J}_p\right) \times \left(\vec{B}_\phi + \vec{B}_p\right)$$

- aim: to express each term with ψ and F $\qquad \psi = \frac{\psi_p}{2\pi}, \quad F = \frac{\mu_0 I_p}{2\pi}$

$$\nabla \cdot \vec{B} = 0 \longrightarrow \vec{B}_p = -\frac{1}{R}\vec{e}_\phi \times \nabla \psi$$

$$\mu_0 \vec{J}_p = \nabla \times \vec{B}_\phi \longrightarrow \vec{J}_p = -\frac{1}{\mu_0 R}\vec{e}_\phi \times \nabla F, \quad B_\phi = \frac{F}{R}$$

$$\mu_0 \vec{J}_\phi = \left(\nabla \times \vec{B}\right)_\phi \longrightarrow J_\phi = -\frac{1}{\mu_0 R}\Delta^* \psi$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

  - The $\nabla\cdot$**B** Equation

In cylindrical coordinates for toroidal axisymmetric fields ($\partial/\partial\Phi=0$)

$$\nabla\cdot\vec{B}=0 \qquad \frac{1}{R}\frac{\partial(RB_R)}{\partial R}+\cancel{\frac{1}{R}\frac{\partial B_\phi}{\partial\phi}}+\frac{\partial B_Z}{\partial Z}=0$$

$$\vec{B}\cdot\nabla\psi=0 \qquad B_R\frac{\partial\psi}{\partial R}+B_Z\frac{\partial\psi}{\partial Z}=0$$

Magnetic flux surface

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial Z},\quad B_Z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

Ψ: Stream function for the poloidal magnetic field

Tokamak Equilibrium

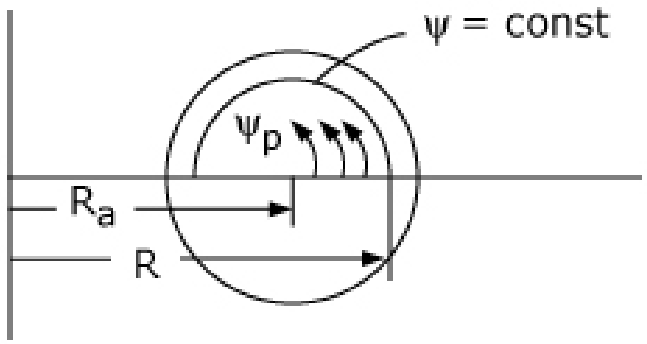
- **The Grad-Shafranov Equation**
  - The stream function ψ is closely related to the poloidal flux in the plasma.

$$\psi_p = \int \vec{B}_p \; d\vec{A}$$

$$= \int_0^{2\pi} d\phi \int_{R_a}^{R} R dR B_Z(R, Z=0)$$

$$= \int_{R_a}^{R} 2\pi R \frac{1}{R}\frac{\partial \psi}{\partial R} dR \quad \longleftarrow \quad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R}$$

$$= 2\pi\left[\psi(R,0) - \psi(R_a,0)\right] = 2\pi\psi$$



Poloidal flux on axis is zero

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - The $\nabla\cdot\mathbf{B}$ Equation

$$\nabla\cdot\vec{B}=0 \qquad \frac{1}{R}\frac{\partial(RB_R)}{\partial R}+\frac{1}{R}\frac{\partial B_\phi}{\partial \phi}+\frac{\partial B_Z}{\partial Z}=0$$

$$\vec{B}\cdot\nabla\psi=0 \qquad B_R\frac{\partial\psi}{\partial R}+B_Z\frac{\partial\psi}{\partial Z}=0$$

Magnetic flux surface

Ψ: Stream function for the poloidal magnetic field

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial Z},\quad B_Z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

$$\vec{B}=B_\phi\vec{e}_\phi+\vec{B}_p$$

$$2\pi d\psi=2\pi(\psi+d\psi)-2\pi\psi=\int_{A+\delta A}\vec{B}_p\cdot d\vec{A}-\int_A\vec{B}_p\cdot d\vec{A}=\int_{\delta A}\vec{B}_p\cdot d\vec{A}\approx B_p 2\pi R dr$$

$$RB_p=\frac{d\psi}{dr}=\left|\nabla\psi\right| \qquad \longleftarrow \quad 2\pi\psi=\psi_p=\int\vec{B}_p\ d\vec{A}$$

$$\vec{B}_p=-\frac{1}{R}\vec{e}_\phi\times\nabla\psi$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Ampere's law

$$\mu_0 \vec{J}_p = \nabla \times \vec{B}_\phi$$

$$\int_{\delta A} \nabla \times \vec{B}_\phi \cdot d\vec{A} = \oint \vec{B}_\phi \cdot d\vec{l} = 2\pi d(RB_\phi) = 2\pi d\psi \frac{\partial (RB_\phi)}{\partial \psi}$$

$$\int_{\delta A} \mu_0 \vec{J}_p \cdot d\vec{A} = \mu_0 J_p 2\pi R dr$$

$$\Rightarrow \quad \frac{\partial (RB_\phi)}{\partial \psi} = \frac{\mu_0 J_p R}{(d\psi / dr)}$$

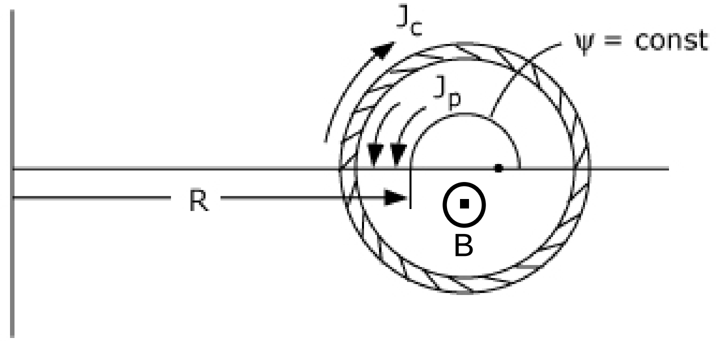
$$\Rightarrow \quad \mu_0 J_p R = \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial r} = \frac{\partial F}{\partial \psi}\left|\nabla \psi\right| = \frac{\partial F}{\partial r} = \left|\nabla F(\psi)\right| \quad \leftarrow \quad F(\psi) \equiv RB_\phi$$

$$\vec{J}_p = -\frac{1}{\mu_0 R}\vec{e}_\phi \times \nabla F, \quad B_\phi = \frac{F}{R}$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Interpretation of $F(\psi)$

$$
\begin{aligned}
I_p &= \int \vec{J}_p \cdot d\vec{A} \\
&= \int_0^{2\pi} d\phi \int_0^R RdRJ_Z(R, Z=0) \\
&= \int_0^{2\pi} d\phi \int_0^R RdR \frac{1}{\mu_0 R}\frac{\partial}{\partial R}(RB_\phi) \\
&= \int_0^R 2\pi R \frac{1}{\mu_0 R}\frac{\partial F}{\partial R} dR \\
&= 2\pi\left[F(R,0) - F(0,0)\right]/\mu_0 \\
&= 2\pi F(\psi)/\mu_0
\end{aligned}
$$



$$\leftarrow \quad J_Z = \frac{1}{\mu_0}\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right), \quad F(\psi) \equiv RB_\phi$$

$$R = 0 \rightarrow F(\psi) = 0$$

$I_p(\psi)$ is the total poloidal current and the toroidal field coils passing through the circle $\psi(R, 0)$ = const.

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Ampere's law

$$\mu_0 \vec{J} = \nabla \times \vec{B}$$

$$J_\phi = \frac{1}{\mu_0}\left(\frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R}\right) = -\frac{1}{\mu_0}\left(\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{1}{R}\frac{\partial^2 \psi}{\partial Z^2}\right)$$

$$\equiv -\frac{1}{\mu_0 R}\Delta^* \psi \qquad \leftarrow \quad B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z}, \quad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R}$$

$$\Delta^* \psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{\partial^2 \psi}{\partial Z^2}$$ elliptic operator

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Momentum equation

$$\vec{J}\times\vec{B}=\nabla p$$

Symmetry

$$\vec{J}_p=-\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F$$

$$\vec{B}_p=-\frac{1}{R}\vec{e}_\phi\times\nabla\psi$$

$$B_\phi=\frac{F(\psi)}{R}$$

$$J_\phi=-\frac{1}{\mu_0 R}\Delta^*\psi$$

$$\begin{aligned}\nabla p&=\left(\vec{J}_\phi+\vec{J}_p\right)\times\left(\vec{B}_\phi+\vec{B}_p\right)\\&=\left(\vec{J}_\phi-\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F\right)\times\left(\vec{B}_\phi-\frac{1}{R}\vec{e}_\phi\times\nabla\psi\right)\\&=-\vec{J}_\phi\times\left(\frac{1}{R}\vec{e}_\phi\times\nabla\psi\right)-\left(\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F\right)\times\vec{B}_\phi\\&=\frac{\vec{J}_\phi}{R}\nabla\psi-\frac{\vec{B}_\phi}{\mu_0 R}\nabla F\end{aligned}$$

$$\longleftarrow\quad A\times(B\times C)=B(A\cdot C)-C(A\cdot B)$$

$$J_\phi=R\frac{dp}{d\psi}+\frac{B_\phi}{\mu_0}\frac{dF}{d\psi}=R\frac{dp}{d\psi}+\frac{F(\psi)}{\mu_0 R}\frac{dF}{d\psi}=-\frac{1}{\mu_0 R}\Delta^*\psi$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

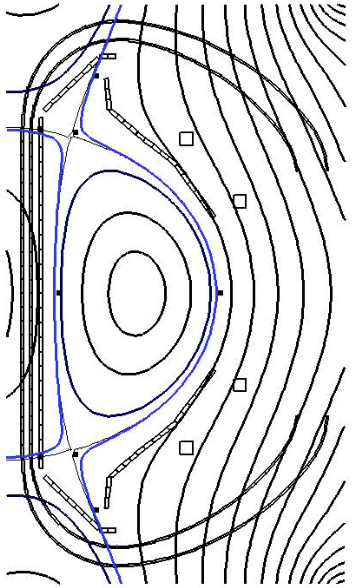
$$\Delta^{*}\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

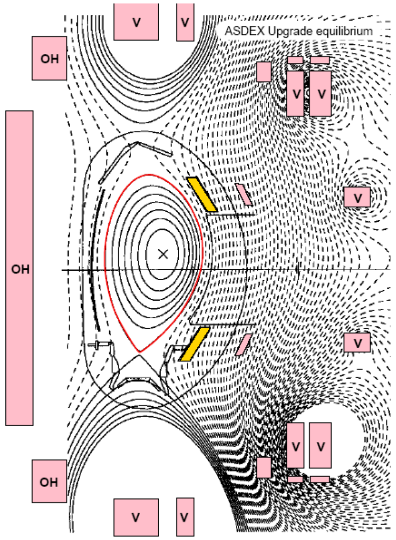
$\uparrow$ **J_ΦxB_p**: strength of **B_p**

$\uparrow$ ∇p: plasma load on a magnetic flux surface

$\uparrow$ **J_pxB_Φ**: strength of **B_Φ**

- BCs: provided by the transformer-induced poloidal magnetic field outside the plasma
- In practice, the G-S equation is solved numerically to find the geometrical location of the magnetic surfaces and the radial distribution of the axial current density in a way that is consistent with the experimentally measured pressure profiles (p) and the externally applied field (F).





References

Lesch, Astrophysics, IPP Summer School (2008)