Beams

i) Assume shape functions $\begin{bmatrix} u_o \\ v_o \\ w_o \end{bmatrix} = \psi(\vec{x})r(t)$

ii) Obtain station equations

$\delta\varepsilon_o \quad \delta\kappa_z \quad \delta\kappa_y$ are arbitrary

$[?] = 0$ at all x

$$\bar{C}\begin{bmatrix} \varepsilon_o \\ \kappa_z \\ \kappa_y \end{bmatrix} = \begin{bmatrix} P^m + P^E \\ -M_z^m - M_z^E \\ -M_y^m - M_y^E \end{bmatrix}$$

$\bar{C}$ has form

$$\bar{C} = \int_A \begin{bmatrix} c_{11} & yc_{11} & zc_{11} \\ yc_{11} & y^2c_{11} & yzc_{11} \\ zc_{11} & yzc_{11} & z^2c_{11} \end{bmatrix} dA$$

→ These are dependent on position

Beams

Sometimes

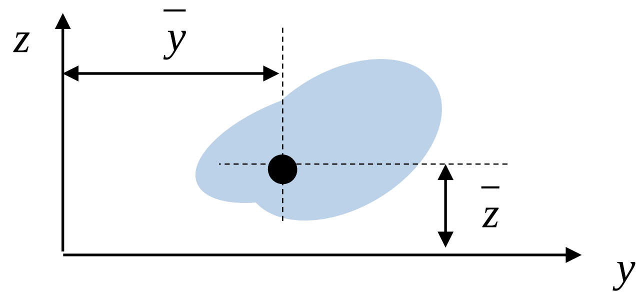
$$\bar{C} = E_R \begin{bmatrix} \bar{A} & \bar{y}\bar{A} & \bar{z}\bar{A} \\ \bar{y}\bar{A} & \bar{I}_{zz} & \bar{I}_{yz} \\ \bar{z}\bar{A} & \bar{I}_{yz} & \bar{I}_{yy} \end{bmatrix}$$

Modulus weighted properties

$\bar{A}$ modulus weighted area $\int \frac{c_{11}}{E_r} dA$

$\bar{y}$ location of modulus weighted centroid

$$\bar{y} = \frac{1}{\bar{A}} \int y \frac{c_{11}}{E_r} dA \qquad \bar{z} = \frac{1}{\bar{A}} \int z \frac{c_{11}}{E_r} dA$$



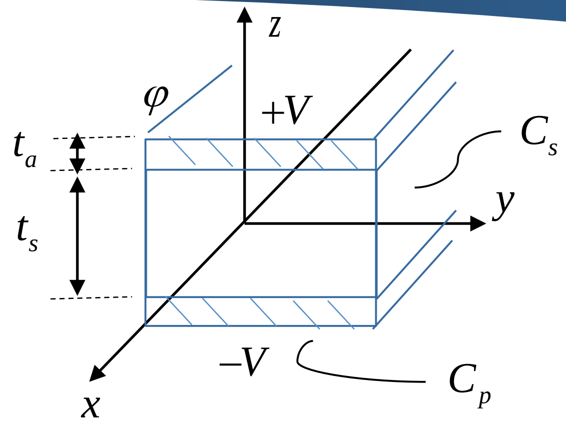
if you set axis location such that $\bar{y} = \bar{z} = 0$ ("principal axis")

$$\varepsilon_o = \frac{1}{E_r \bar{A}} \left(P^m + P^E \right)$$

$$\begin{bmatrix} \kappa_z \\ \kappa_y \end{bmatrix} = \frac{1/E_r}{\bar{I}_{zz}\bar{I}_{yy} - \bar{I}_{zz}^{\;2}} \begin{bmatrix} \bar{I}_{zz} & -\bar{I}_{yz} \\ -\bar{I}_{yz} & \bar{I}_{zz} \end{bmatrix} \begin{bmatrix} -M_z^m - M_z^E \\ -M_y^m - M_y^E \end{bmatrix}$$

Beams

❖ Simple Example



Notes : - we are already at modulus weighted centroid

- $\bar{I}_{yz} = 0$

- $M_z = 0$

➢ Material Properties

Structure $c_{11} = c_s$

Piezo $$\begin{bmatrix} T_1 \\ D \end{bmatrix} = \begin{bmatrix} c_{11}^E & -e_{31} \\ e_{31} & \varepsilon_{33}^S \end{bmatrix} \begin{Bmatrix} S_1 \\ E_3 \end{Bmatrix}$$

Piezo in z direction

transverse act

Beams

Electric field assumption

Case A : $E_3 = +E_o \quad (z > 0) \qquad E_3 = -E_o \quad (z < 0)$

Case B : $E_3 = E_o$ all z

Due to symmetry 2 equations (no mechanical loading)

$$\varepsilon_o = \frac{1}{c_s \bar{A}} P^E \qquad P^E = \int_A \vec{e}_t \bullet \vec{E} dA$$

$$\kappa_y = -M_y^E / c_s \bar{I}_{yy} \qquad -M_y^E = \int_A z\vec{e}_r \bullet \vec{E} dA$$

$$\vec{e}_r \bullet \vec{E} = e_{31} E_o$$

Case A :

$$P^E = 0$$

$$-M_y^E = 2b \int_{\frac{t_s}{2}}^{\frac{t_s}{2}+t_a} z e_{31} E_o dz = b e_{31} E_o \left[\left(\frac{t_s}{2} + t_a \right)^2 + \left(\frac{t_s}{2} \right)^2 \right]$$

Case B :

$$P^E = 2 b e_{31} E_o t_a$$

$$-M_y^E = 0$$

Beams

Furthermore $\bar{A} = \int_A \frac{c_{11}(z)}{c_s} dA = bt_s + 2bt_a \left(\frac{c_p}{c_s} \right)$

$$\bar{I}_{yz} = b\int_A z^2 \frac{c_{11}(z)}{c_s} dA = \frac{2}{3} b \left(\frac{t_s}{2} \right)^3 + \frac{2}{3} b \frac{c_p}{c_s} \left[\left(\frac{t_s}{2} + t_a \right)^3 + \left(\frac{t_s}{2} \right)^3 \right]$$

Case B:

$$\varepsilon_o = \frac{2be_{31}E_o t_a}{c_s bt_s + 2bt_a c_p} = \underbrace{\left(\frac{e_{31}E_o}{c_p} \right)}_{\underset{\sim}{\Lambda}} \frac{1}{1+\psi} \qquad \kappa_y = 0$$

$$\psi = \frac{c_s t_s}{c_p t_a}$$

Case A:

$$\varepsilon_o = 0 \qquad \kappa_y = -\frac{M_y^E}{c_s \bar{I}_{yy}} = \frac{3 \overbrace{\left(\frac{e_{31}E_o}{c_p} \right)}^{\underset{\sim}{\Lambda}} \left[\left(\frac{t_s}{2} + t_a \right)^2 + \left(\frac{t_s}{2} \right)^2 \right]}{2 \left[\frac{c_s}{c_p} \left(\frac{t_s}{2} \right)^3 + \left(\frac{t_s}{2} + t_a \right)^3 - \left(\frac{t_s}{2} \right)^3 \right]}$$

Beams

Limiting case

$$t_s = 0 \qquad \frac{3\left(\frac{e_{31}E_o}{c_p}\right)t_a^2}{2t_a^3} = \frac{3}{2}\frac{1}{t_a}\left(\frac{e_{31}E_o}{c_p}\right)$$

Some comments on the Electrical side

→ Known E

$$E = \psi(x, y, z)v(t)$$

→ $$\begin{bmatrix} P^E \\ -M_z^E \\ -M_y^E \end{bmatrix} = v(t)\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Beams

❖ Torsion

- very different from bending
- 2-D distribution of stresses on cross section
- 2 ways to approach

a) assumed stress function approach

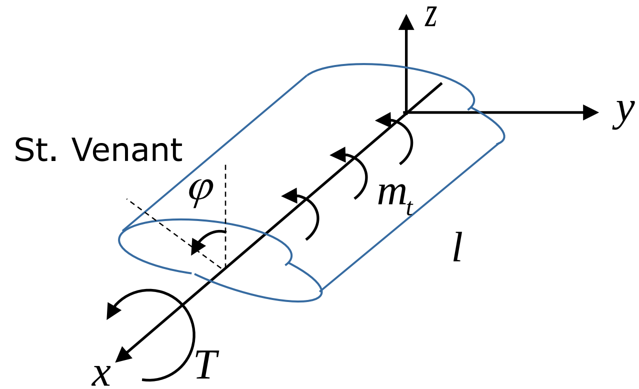
-membrane analogy $T = \frac{d\varphi}{dy} \quad T_6 = \frac{d\varphi}{dz}$

$\nabla^2 \varphi = 2G\theta$ θ : rate of twist

b) assumed displacement function

" warping " function

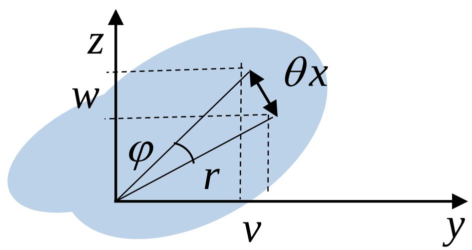
} St. Venant's torsion



$\varphi' = \theta = \text{constant}$

Beams

- Assumption
  - Each section rotates as a rigid body
  - Rate of twist θ = constant ("uniform torsion")
  - Cross section is free to warp but, same in all cross sections



$$v = -xz\theta$$
$$w = xy\theta$$
$$u = \theta f(y,z)$$

strains $\quad S_1, S_2, S_3, S_4 = 0$

$$S_5 = \frac{\partial u}{\partial z} + \frac{dw}{dx} = \theta\left(\frac{\partial f}{\partial z} + y\right) \qquad S_6 = \frac{\partial u}{\partial y} + \frac{dv}{dx} = \theta\left(\frac{\partial f}{\partial y} - z\right)$$

consider the warping function

equilibrium → only shear present

$$\boxed{\frac{\partial T_6}{\partial y} + \frac{\partial T_5}{\partial z} = 0} \qquad \frac{\partial T_6}{\partial x} = 0 \qquad \frac{\partial T_5}{\partial x} = 0$$

Beams

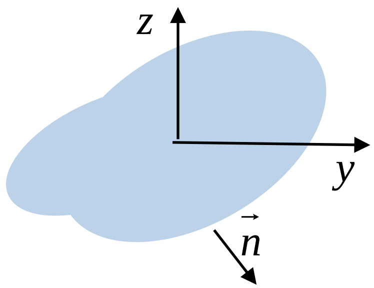
assume $\begin{bmatrix} T_5 \\ T_6 \end{bmatrix} = G \begin{Bmatrix} s_5 \\ s_6 \end{Bmatrix}$ → $\nabla^2 f = 0$ Laplace's equation

$$\nabla^2 \varphi = 2G\theta$$

→ Stress function

BC's

- along contour



→ from stress BC's

$$T_6 m + T_5 n = 0$$

where, m: direction cosine between and y axis

n : direction cosine between z axis

This gives

$$\left(\frac{\partial f}{\partial y} - z\right) m + \left(\frac{\partial f}{\partial z} - y\right) n = 0$$

on stress-free contour

f must be continuous and differentiable

Beams

- Constitutive relations

$$T = c^E S - e_t E$$

$$\begin{bmatrix} T_5^E \\ T_6^E \end{bmatrix}$$

Only read shear

$$\begin{bmatrix} T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{55} & c_{56} \\ c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} S_5 \\ S_6 \end{Bmatrix} + \begin{bmatrix} & e_t^{(5,6)} & \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$T_5 = c_5 S_5 - T_5^E$$

$$T_6 = c_6 S_6 - T_6^E$$

- Variational principle

$$\int_{t_1}^{t_2} \int_V \left[\delta T - \delta U_1^m + \delta W_1^m \right] dVdt = 0 \quad (\delta T = 0)$$

$$\delta U_1^m = \int_V T_5 \delta S_5 + T_6 \delta S_6 dV$$

$$= \int_x \int_A \left\{ \left[c_{55}\theta \left(\frac{\partial f}{\partial z} + y \right) - T_5^E \right] \delta\theta \left(\frac{\partial f}{\partial z} + y \right) + \left[c_{66}\theta \left(\frac{\partial f}{\partial y} - z \right) - T_6^E \right] \delta\theta \left(\frac{\partial f}{\partial y} - z \right) \right\} dAdx$$

$$= \int_x \left[\theta \bar{K} \delta\theta - M_t^E d\theta \right] dx$$

$$\bar{K} = \int_A \left[c_{55} \left(\frac{\partial f}{\partial z} + y \right)^2 - c_{66} \left(\frac{\partial f}{\partial y} - z \right)^2 \right] dx \qquad M_t^E = \int_A \left[T_5^E \left(\frac{\partial f}{\partial z} + y \right) - T_6^E \left(\frac{\partial f}{\partial y} - z \right) \right] dA$$

Beams

- Work terms
  - assume there is a distributed torque
  - end tip torque

$$\delta\varphi = \delta\theta$$

$$\delta W_1^m = T\delta\varphi_{x=L} + \int_x m_t \delta\varphi dx$$

$$= T\delta\varphi_{x=L} + \left(\int m_t dx\right)\delta\varphi\Big|_0^L - \int_x \left(\int m_t dx\right)\delta\theta dx$$

BC's

=0

$$\delta W_1^m = \left(T + \int_0^L m_t dx\right)\delta\varphi_{x=L} - \left(\int m_t dx\right)\delta\varphi_{x=0} - \int_x \left(\int m_t dx\right)\delta\theta dx$$

$$M_t^m$$

$$M_t^{m'} = -m_t \qquad M_t = -\int_0^L m_t dx = -T$$

→ $\int_{t_1}^{t_2}\int_A \left[\theta\bar{\kappa}\delta\theta - M_t^E\delta\theta - M_t^m\delta\theta\right]dxdt = 0$

$\delta\theta$: arbitrary @ each section

$$\theta\bar{\kappa} = M_t^E + M_t^m$$

$$\bar{\kappa} = G_r\bar{J} = G_r\int_A\left[\frac{c_{55}}{G_r}\left(\frac{\partial f}{\partial z} + y\right)^2 - \frac{c_{66}}{G_r}\left(\frac{\partial f}{\partial y} - z\right)^2\right]dA$$

$$M_t^E = \int_A\left[T_5^E\left(\frac{\partial f}{\partial z} + y\right) - T_6^E\left(\frac{\partial f}{\partial y} - z\right)\right]dA$$