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.13 Suspended Load

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Mass Conservation of Suspended Sediment

- Suspended sediment differs from bed load sediment in t hat it may be diffused throughout the vertical column of fl uid via turbulence.
- Suspended particles are transported solely by convectiv e fluxed.
  - By mean flow motion
  - Turbulence of the flow
- Here, only the main elements of the theory of equilibrium suspensions need to estimate suspended load are discu ssed

Mass Conservation of Suspended Sediment

- In the earlier lecture, the Lagrangian particle velocity was denoted as u_{pi}. Here the corresponding Eulerian particle velocity field is denoted as $v_i(x_j,t)$. The approximation for fine particles takes the Eulerian form

$$v_i = u_i - v_s \delta_{i3}$$

- The above velocities are instantaneous values. In the ab ove form of the relation, $x_3=z$ is constrained to be upward vertical. The form for an arbitrary Cartesian system is

$$v_i = u_i - v_s \mathbf{k_i}$$

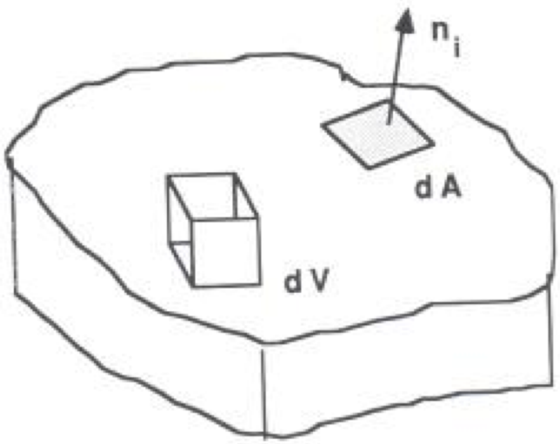
- As long as suspended sediment is sufficiently coarse not to undergo Brownian motion, molecular effects can be n eglected.

Mass Conservation of Suspended Sediment

- The volume convective flux F_{si} of suspended sediment is given by the following expression:

$$F_{si} = cv_i$$

- Consider the arbitrary volume, fixed in Eulerian space, illu strated below. It has volume V and surface area A. The l ocal unit outward vector to the surface is denoted as n_i.
- The equation of mass balance o f suspended sediment can be w ritten in words as



$$\frac{\partial}{\partial t}\left[mass\ in\ volume\right] = \left[net\ mass\ \inf low\ rate\right]$$

Mass Conservation of Suspended Sediment

- Noting that the mass inflow rate across an area dS is give n by $-\rho_s F_{si} n_i dS$, then

$$\frac{\partial}{\partial t}\left(\iiint_V \rho_s c\, dV\right) = -\rho_s \iint_S F_{si} n_i\, dS$$

- According to the divergence theorem,

$$\iint_S F_{si} n_i\, dS = \iiint_V \frac{\partial F_{si}}{\partial x_i} dV$$

- Then $\frac{\partial}{\partial t}\left(\iiint_V \rho_s c\, dV\right) = -\rho_s \iiint_V \frac{\partial F_{si}}{\partial x_i} dV$

$$\rho_s \iiint_V \left[\frac{\partial c}{\partial t} + \frac{\partial v_i c}{\partial x_i}\right] dV = 0$$

$$\Rightarrow \quad \frac{\partial c}{\partial t} + \frac{\partial v_i c}{\partial x_i} = 0$$

Mass Conservation of Suspended Sediment

- The final form of equation

$$\frac{\partial c}{\partial t}+\frac{\partial F_{si}}{\partial x_i}=0$$

$$F_{si}=\left(u_i-v_s k_i\right)c$$

- In other descriptions of the above two equations are

$$\frac{\partial c}{\partial t}+\frac{\partial uc}{\partial s}+\frac{\partial vc}{\partial n}+\frac{\partial}{\partial z}\left[\left(w-v_s\right)c\right]=0$$

Reynolds averaging : Closure

- The parameter u and c are not decomposed into average s over turbulence and fluctuations about the mean,

$$c = \bar{c} + c'; \qquad u_i = \bar{u}_i + u'$$

- Put this into the previous equation and using continuity eq uations, also doing average then

$$\frac{\partial c}{\partial t} + \frac{\partial \bar{F}_{si}}{\partial x_i} = 0$$

$$\bar{F}_{si} = \left(\bar{u}_i - v_s k_i\right)\bar{c} + \overline{u_i' c'}$$

- Now put into and expansion

$$\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial c}{\partial s} + \bar{v}\frac{\partial \bar{c}}{\partial n} + \left(\bar{w} - v_s\right)\frac{\partial \bar{c}}{\partial z} = -\frac{\partial \overline{u'c'}}{\partial s} - \frac{\partial \overline{v'c'}}{\partial n} - \frac{\partial \overline{w'c'}}{\partial z}$$

Reynolds averaging : Closure

- In the mean flux of suspended sediment (previous one)

$$\overline{F}_{si} = \left(\overline{u}_i - v_s k_i\right)\overline{c} + \overline{u_i' c'}$$

- Is composed of two components, i.e., a mean convective f lux and a Reynolds flux.
- Reynolds flux is clearly diffusive in nature, and the simple st closure assumption one could make for these terms is t he appropriate analogy

$$\overline{u_i' c'} = -D_d \frac{\partial \overline{c}}{\partial x_i}$$

- D_d is the kinematic eddy diffusivity is assumed to be a sca lar quantity. For the case of non isotropic case,

$$\overline{u_i' c'} = -D_{dij} \frac{\partial \overline{c}}{\partial x_{ij}}$$

Boundary conditions

- The previous flux expression is closed with a Fickian assumption and represents a convective-diffusive equation for suspended sediment.
- The condition of vanishing flux of suspended sediment across (normal to) the water surface defines the upper boundary condition.
- Let's denote a unit vector normal to the water surface. The boundary condition there takes the form

$$\bar{F}_{sz}\Big|_{z=H} = 0$$

$$\bar{F}_{sz} = -v_s \bar{c} + \overline{w'c'}$$

Boundary conditions

- The boundary condition at the bed differs from the one at the water surface, in that it must account for entrainment of sediment into the flow from the bed and deposition of s ediment from the flow onto the bed.
- For a flat (averaged over bedforms) bed, the mean deposi tional flux of suspended sediment onto the bed is given b y $-D_r$, which needs to be evaluated at a distance $z=b$ near the bed,

$$D_r = v_c \bar{c}_b$$

- Denotes the volume rate of deposition of suspended sedi ment per unit bed area. $\bar{c}_b$ denotes a near-bed value of m ean volumetric sediment concentration.

Boundary conditions

- The component of the Reynolds flux of suspended sediment near the bed may be termed the rate of erosion, or more accurately, entrainment of bed sediment. The entrainment rate E_r is

$$E_r = \overline{w'c'}$$

- It is seen from these equations that the net-upward, normal flux of suspended sediment at the bed is

$$\bar{F}_{sz}\Big|_{z=b} = E_r - D_r = v_s\left(E_s - \bar{c}_b\right)$$

- Where

$$E_s \equiv \frac{E_r}{v_S}$$

- Denotes a dimensionless rate of entrainment of bed sediment into suspension

Boundary conditions

- Typically a relation is assumed of the form

$$E_s = E_s\left(\tau_{bs},\ \text{other parameter}\right)$$

- If it is furthermore assumed that an equilibrium steady, uni form suspension has been achieved.
- No net deposition nor erosion from the bed.

$$\left.\overline{F}_{sz}\right|_{z=b} = 0$$

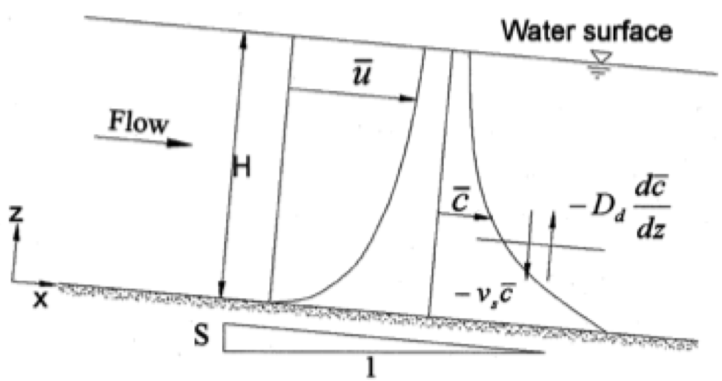
$$E_s = \overline{c}_b$$

- This relation simply states that the entrainment rate equal s the deposition rate at equilibrium thus there is no net no rmal flux of suspended sediment at the bed.

Equilibrium Suspension in a Wide Channel

- Consider normal flow in a wide, rectangular open channel
- The bed is assumed to be erodible and has no curvature when avera ged over bed forms such as ripples and dunes
- The z-coordinate is quasi-vertical, implying low channel slope S.
- The suspension is likewise assumed to be in equilibrium.
- The flow and suspensions are uniform in s and n and stead in time.

$$\frac{d}{dz}\left(\overline{w'c'} - v_s\bar{c}\right) = 0$$



- Also $\overline{w'c'} - v_s\bar{c} = \text{constant}$

- Over whole depth,

  and at the bottom flux is zero

then, constant should be zero over the whole depth

$$\overline{w'c'} - v_s\bar{c} = 0$$

Equilibrium Suspension in a Wide Channel

- Closing with the eddy diffusivity model, then

$$D_d \frac{d\overline{c}}{dz} + v_s \overline{c} = 0$$

- The equilibrium suspended sediment concentration decreases for inc reasing z, so that turbulence diffuses sediment from zones of high co ncentration to zones of low concentration.
- Thus the general boundary conditions are

$$-D_d \frac{d\overline{c}}{dz}\bigg|_{z=b} = v_s E_s$$

$$D_d \frac{d\overline{c}}{dz} + v_s \overline{c}\bigg|_{z=H} = 0$$

- The first specifies the near-bed rate of entrainment of sediment into s uspension and the second specifies the condition of vanishing upwar d normal sediment flux at the water surface.

Form of Eddy Diffusivity (Prandtl analogy)

- The simple approach taken here is that of Rouse (1957). It involves the use of the Prandtl analogy.
- Prandtl analogy
  - Fluid mass, heat, momentum, etc. should all diffuse at the same kinematic rate due to turbulence and thus have the s ame kinematic eddy diffusivity, because each is a property of the fluid particles, and it is the fluid particles that are bei ng transported by Reynolds fluxes.
- Actually sediment have lower diffusivity and have mean f all velocity.
- But, we will use here since simple

Form of Eddy Diffusivity (Prandtl analogy)

- The velocity profile is approximated as logarithmic throughout the depth. To account for the possible existence of bedforms, the turbulent rough law is employed.

$$\frac{\overline{u}(z)}{u_*} = \frac{1}{\kappa}\ln\left(30\frac{z}{k_c}\right)$$

- Here k_c is a composite roughness chosen to include the effect of bedforms. Furthermore,

$$\rho u_*^2 \equiv \tau_b \cong -\rho\overline{w'u'}\Big|_{z=b}$$

- Where b is chosen to be very close to the bed, i.e.,

$$\frac{b}{H} \ll 1$$

Form of Eddy Diffusivity (Prandtl analogy)

- Now the kinematic eddy viscosity D_d is defined such that

$$\tau = -\rho \overline{w'u'} = D_d \frac{d\overline{u}}{dz}$$

- In the old previous class,

$$\tau = \tau_b \left(1 - \frac{z}{H}\right)$$

- Then with the previous equations, we can find

$$D_d = \kappa u_* z \left(1 - \frac{z}{H}\right)$$

- The above relation is the Rousean relation for the vertica l kinematic eddy viscosity.

Form of Eddy Diffusivity (Prandtl analogy)

- The mean diffusivity over the whole depth

$$\overline{D}_d = \frac{1}{H}\int_0^H D_d\,dz = \frac{1}{H}\int_0^H \kappa u_* z\left(1-\frac{z}{H}\right)dz$$

$$\overline{D}_d = \frac{\kappa}{6}u_* H \cong 0.0667 u_* H$$

Rousean Distribution of Suspended Sediment

- The nominal "near bed" elevation in applying the bottom boundary condition is taken to be z=b, where b is a dista nce taken to be very close to the bed.
- In the Rousean analysis, this value cannot be taken as z =0, because the previous diffusivity profile has singular t here.
- Now put the diffusivity into

$$D_d \frac{d\bar{c}}{dz} + v_s \bar{c} = \kappa u_* z \left(1 - \frac{z}{H}\right) \frac{d\bar{c}}{dz} + v_s \bar{c} = 0$$

- Integrate it

$$\int_b^z \frac{d\bar{c}}{\bar{c}} = -Z \int_b^z \frac{H dz}{z(H-z)} = \ln\left[\left(\frac{H-z}{z}\right)^Z\right]_b^z \qquad Z = \frac{v_s}{\kappa u_*}$$

Rousean Distribution of Suspended Sediment

- Further reduction yields the prof ile $\bar{c} = \bar{c}_b \left[\frac{(H-z)/z}{(H-b)/b} \right]^Z$
- In equilibrium $\bar{c}_b$ is equal to the dimensionless sediment rate E.
- This provides an empirical mea ns to evaluate E as a function of bottom skin shear stress and ot her parameters.