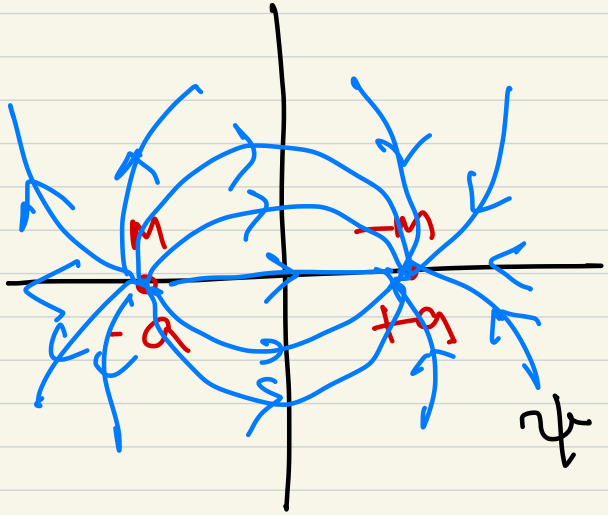
⑦ Doublet : source + sink pair w/ a vanishingly small distance a.



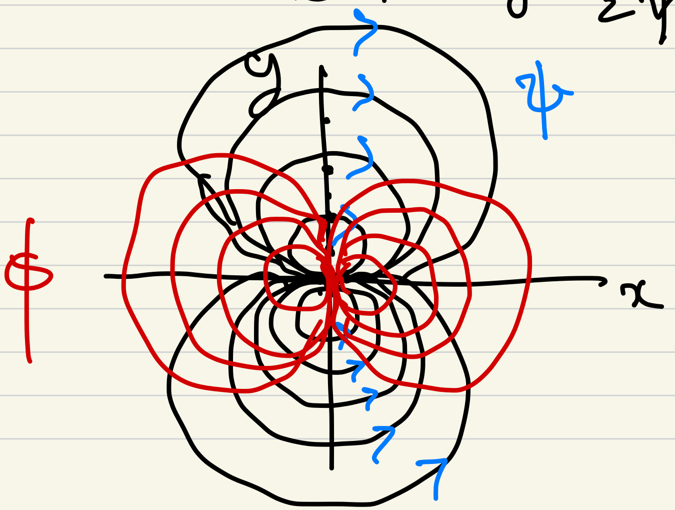
$a \to 0$

$2am = \text{const} = \lambda$: strength of doublet

$$\psi = \lim_{\substack{a\to 0 \\ 2am=\lambda}} \left(m \tan^{-1}\frac{y}{x+a} - m\tan^{-1}\frac{y}{x-a} \right)$$

$$= \cdots = -\frac{2amy}{x^2+y^2} = -\frac{\lambda y}{x^2+y^2} = -\frac{\lambda \sin\theta}{r}$$

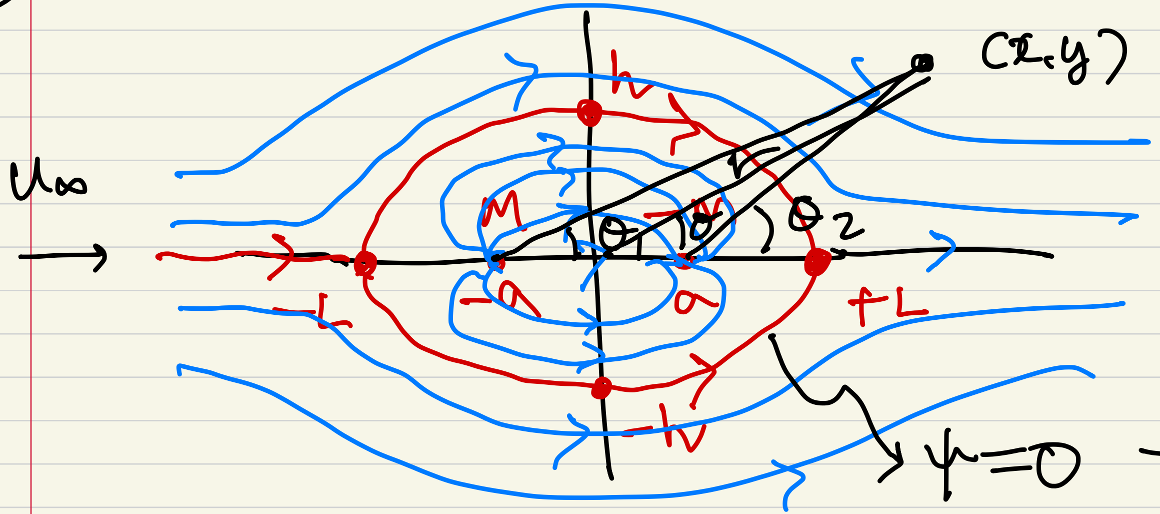
$\to \quad x^2 + \left(y + \frac{\lambda}{2\psi}\right)^2 = \left(\frac{\lambda}{2\psi}\right)^2$: streamlines $(\psi = c)$ are circles $\left(0, -\frac{\lambda}{2\psi}\right)$



similarly, $\phi = \frac{\lambda x}{x^2+y^2}$

$\to \left(x - \frac{\lambda}{2\phi}\right)^2 + y^2 = \left(\frac{\lambda}{2\phi}\right)^2$

⑧ Rankine oval : uniform stream + source + sink



$$\psi = u_\infty y - m \tan^{-1} \frac{2ay}{x^2+y^2-a^2}$$

$$= u_\infty r \sin\theta + m(\theta_1 - \theta_2)$$

$\psi = 0 \rightarrow \left. \begin{array}{l} u = \cdots \\ v = \cdots \end{array} \right\} \; u = v = 0$ (stag. pt.)

$$\downarrow$$

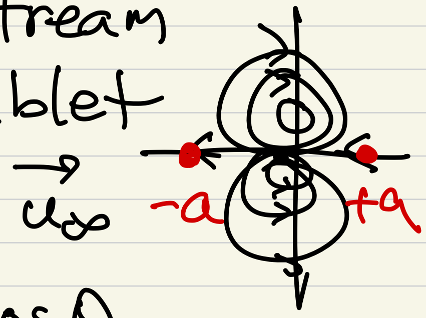
$$\frac{L}{a} = \left(1 + \frac{2m}{u_\infty a}\right)^{\frac{1}{2}}$$

@ $y = h,\ x = 0$: $\psi = u_\infty h - m \tan^{-1} \dfrac{2ay}{h^2 - a^2} = 0$

$$\rightarrow \frac{h}{a} = \cot \frac{h/a}{2m/u_\infty a}$$

$V^2 = u^2 + v^2$ on $\psi = 0 \rightarrow \dfrac{\partial V}{\partial \theta} = 0 \Rightarrow$ V_{max} @ $\theta = 90°$

⑨ Flow past a circular cylinder : uniform stream + doublet



$$\psi = u_\infty r \sin\theta - \frac{\lambda \sin\theta}{r} = \left(u_\infty r - \frac{\lambda}{r}\right)\sin\theta$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\left(u_\infty r - \frac{\lambda}{r}\right)\cos\theta = \frac{1}{r^2}\left(u_\infty r^2 - \lambda\right)\cos\theta$$

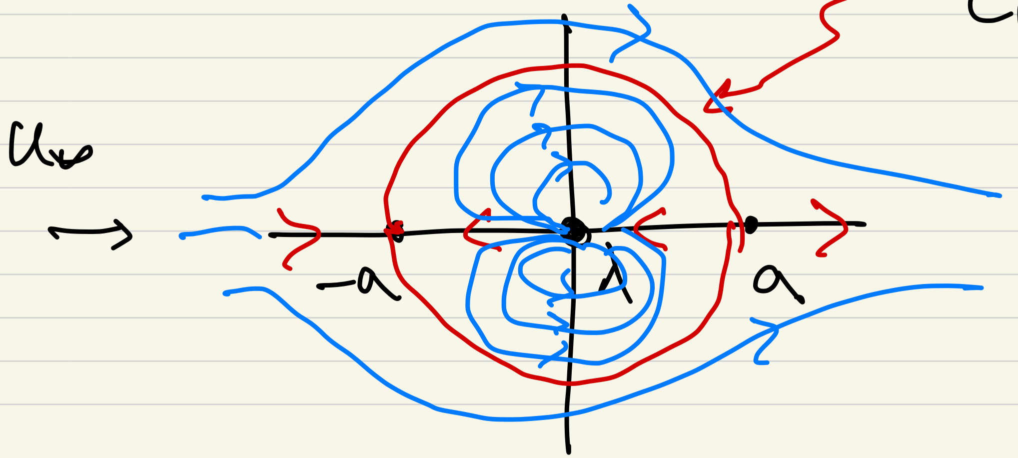
$$u_\theta = -\frac{\partial\psi}{\partial r} = -u_\infty \sin\theta - \frac{\lambda\sin\theta}{r^2} = -\frac{1}{r^2}\left(u_\infty r^2 + \lambda\right)\sin\theta$$

stag. pt. $u_r = u_\theta = 0 \rightarrow \theta = 0, \pi$

$\rightarrow u_\infty a^2 - \lambda = 0 \Rightarrow a^2 = \lambda / u_\infty \rightarrow \lambda = a^2 u_\infty$

$\psi = 0 : u_\infty r - \frac{\lambda}{r} = 0 \rightarrow r^2 = \frac{\lambda}{u_\infty} = a^2 \rightarrow r = a$

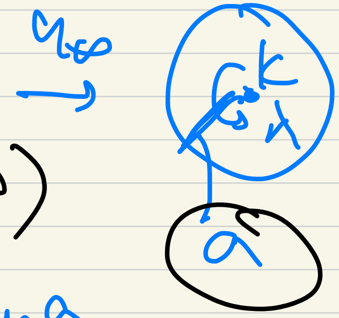
circle



$\nabla^2 \psi = 0$

(10) Flow past a circular cylinder with circulation

↳ uniform stream + doublet + vortex



$$\psi = U_\infty r \sin\theta - \frac{\lambda \sin\theta}{r} - k \ln r + k \ln a \quad (\lambda = a^2 u_\infty)$$

@ $r = a$, $$\psi = \underbrace{u_\infty a \sin\theta - \frac{\lambda \sin\theta}{a}}_{=0} - k \ln a + k \ln a$$

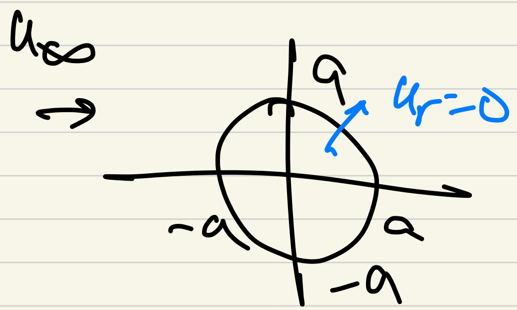
$$= -k \ln a + k \ln a$$

$$= 0$$

$$\Rightarrow \quad \psi = u_\infty \sin\theta \left(r - \frac{a^2}{r}\right) - k \ln \frac{r}{a}$$

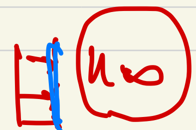
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_\infty \cos\theta \left(1 - \frac{a^2}{r^2}\right)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -u_\infty \sin\theta \left(1 + \frac{a^2}{r^2}\right) + \frac{k}{r}$$

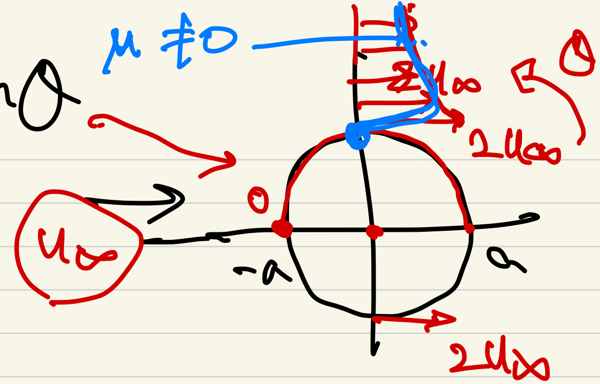


@ $r = a$ $\quad u_r(r=a) = 0$

$$u_\theta(r=a) = -2u_\infty \sin\theta + \frac{k}{a}$$



if $k=0$, $u_\theta(r=a) = -2u_\infty \sin\theta$



$(k \neq 0)$

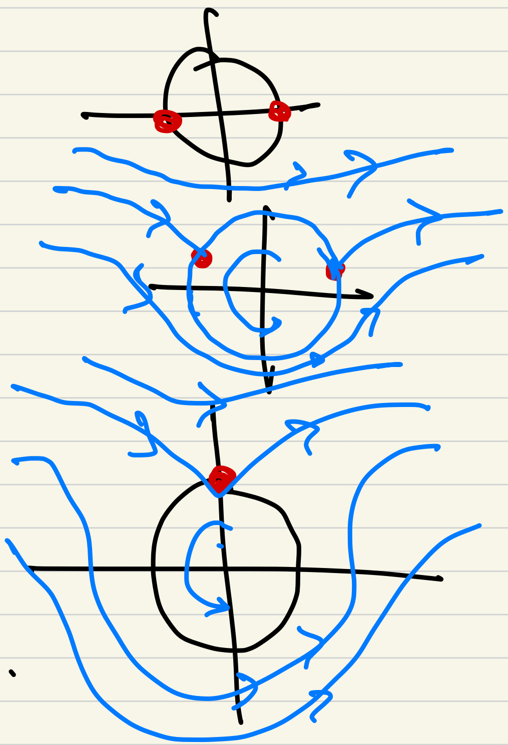
$\rightarrow u_\theta = 0 \Rightarrow \sin\theta = \dfrac{k/a}{2u_\infty} = \dfrac{k}{2u_\infty a}$

for small k, $\sin\theta_{stag} = \dfrac{k}{2u_\infty a} < 1$: two stag. pts.

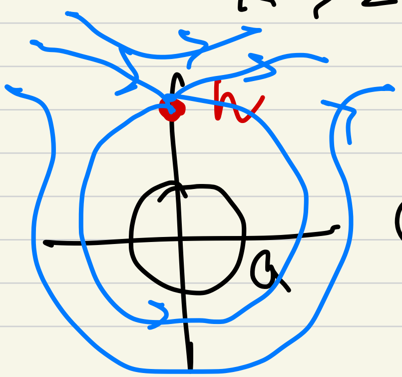
$k=0$: $\sin\theta_s = 0 \rightarrow \theta_s = 0, \pi$

$k = u_\infty a$: $\sin\theta_s = \frac{1}{2} \rightarrow \theta_s = \frac{\pi}{6}, \frac{5\pi}{6}$

$k = 2u_\infty a$: $\sin\theta_s = 1 \rightarrow \theta_s = \frac{\pi}{2}$



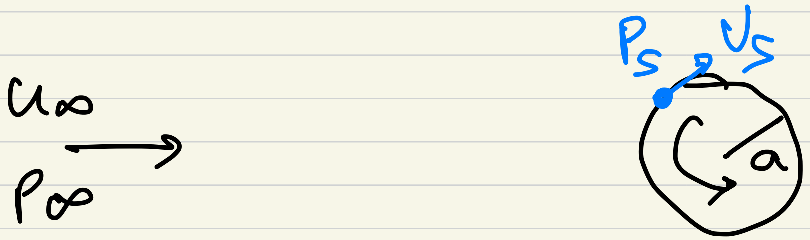
$k > 2u_\infty a$: stag. pt. does not exist on the cylinder surface.



@ $y=h$, $u_r = u_\theta = 0$

$\rightarrow \dfrac{h}{a} = \dfrac{1}{2}\left(\beta + \sqrt{\beta^2 - k}\right) \quad \left(\beta = \dfrac{k}{u_\infty a} > 2\right)$

⊛ Kutta-Joukowski theorem



$$p_\infty + \frac{1}{2}\rho u_\infty^2 = p_s + \frac{1}{2}\rho V_s^2 \qquad V_s : u_r = 0 \, , \; u_\theta \neq 0$$

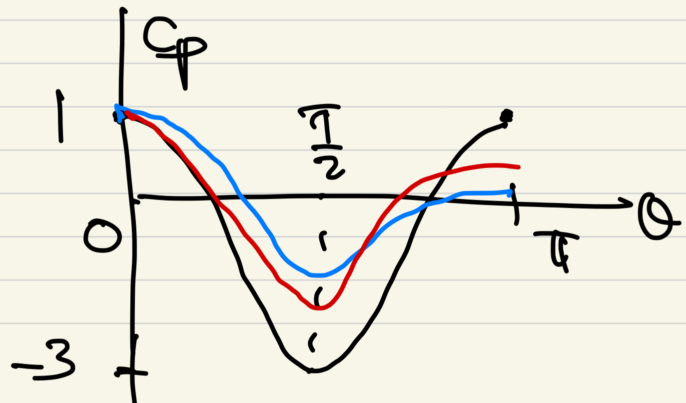
$$= p_s + \frac{1}{2}\rho\left(-2u_\infty \sin\theta + \frac{k}{a}\right)^2$$

$$\to \; p_s = p_\infty + \frac{1}{2}\rho u_\infty^2 \left(1 - 4\sin^2\theta + 4\beta\sin\theta - \beta^2\right) \qquad \left(\beta = \frac{k}{u_\infty a}\right)$$

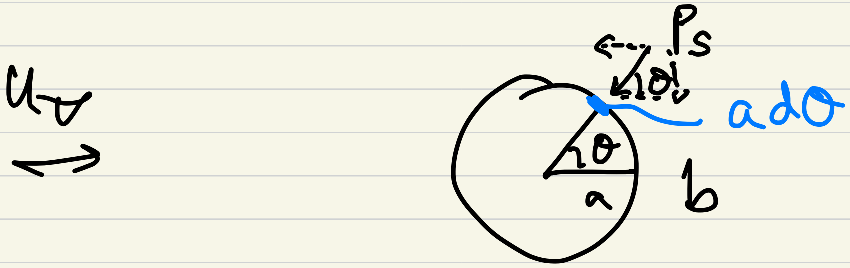
$$\to \; \boxed{C_p \equiv \frac{p_s - p_\infty}{\frac{1}{2}\rho u_\infty^2} = 1 - 4\sin^2\theta + 4\beta\sin\theta - \beta^2}$$

press. coeff

$$k = 0 : \; \beta = 0 \; \to \; \boxed{C_p = 1 - 4\sin^2\theta}$$



inviscid flow: $\mu = 0 \rightarrow \tau = 0$



Drag $D = \int_0^{2\pi} -p_s \cos\theta \, b \, a \, d\theta = \int_0^{2\pi} -(p_s - p_\infty)\cos\theta \, b \, a \, d\theta$

$= 0$!

" <u>d'Alembert's paradox</u> " (1752)

According to inviscid theory, the drag of any body of any shape immersed in a uniform stream is identically zero.

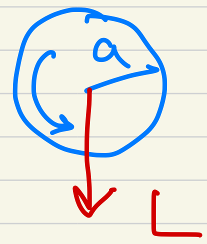
→ overcome by Prandtl (1904)

Lift $\quad L = \int_0^{2\pi} -p_s \sin\theta \cdot b a \, d\theta = \int_0^{2\pi} -(p_s - p_\infty) \sin\theta \cdot b a \, d\theta$

$$= \cdots = -\rho u_\infty \underbrace{2\pi k}_{=\Gamma} b$$

$$\rightarrow \boxed{L/b = -\rho u_\infty \Gamma}$$

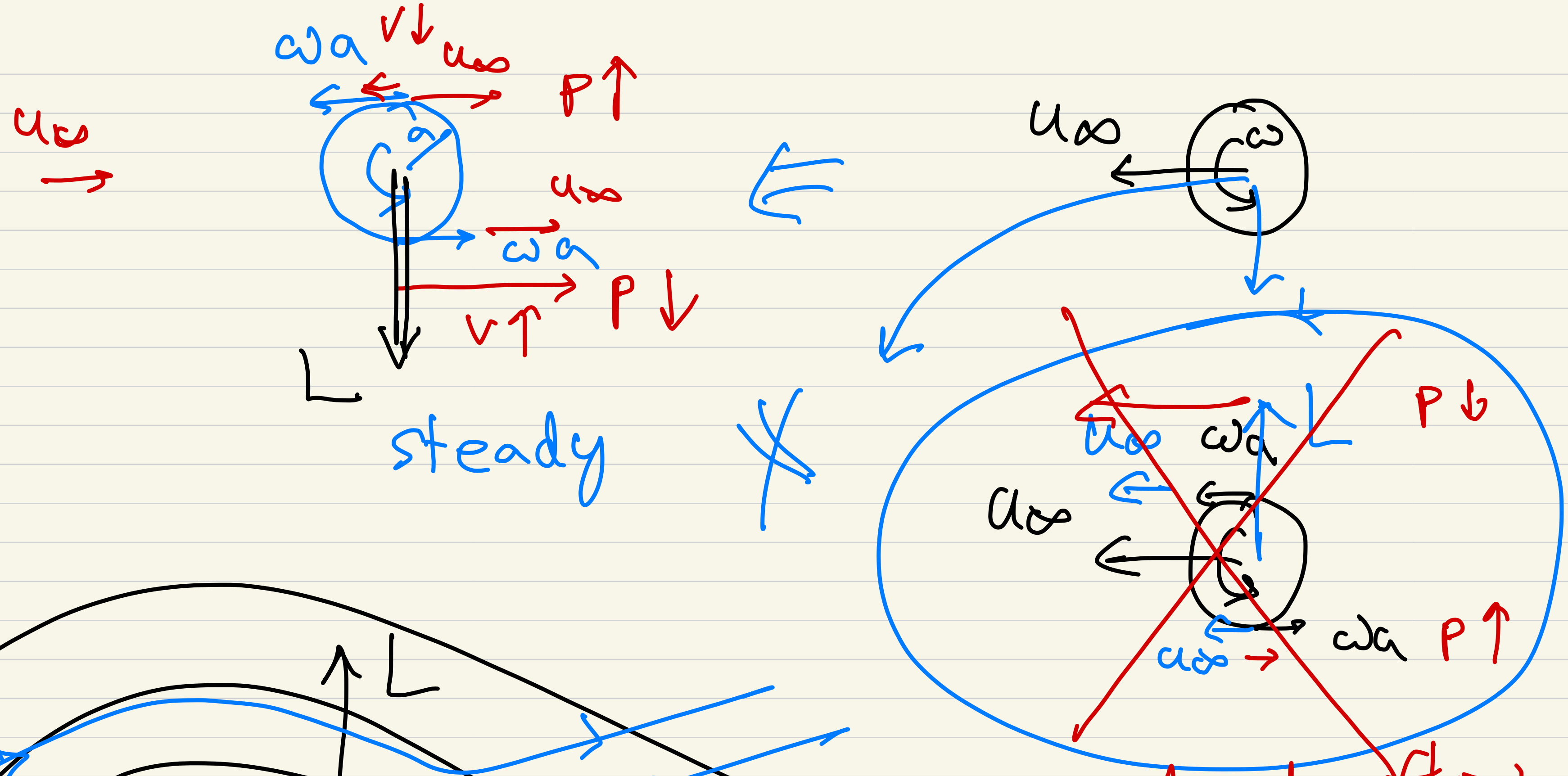
Kutta (1902)
Joukowski (1906) →

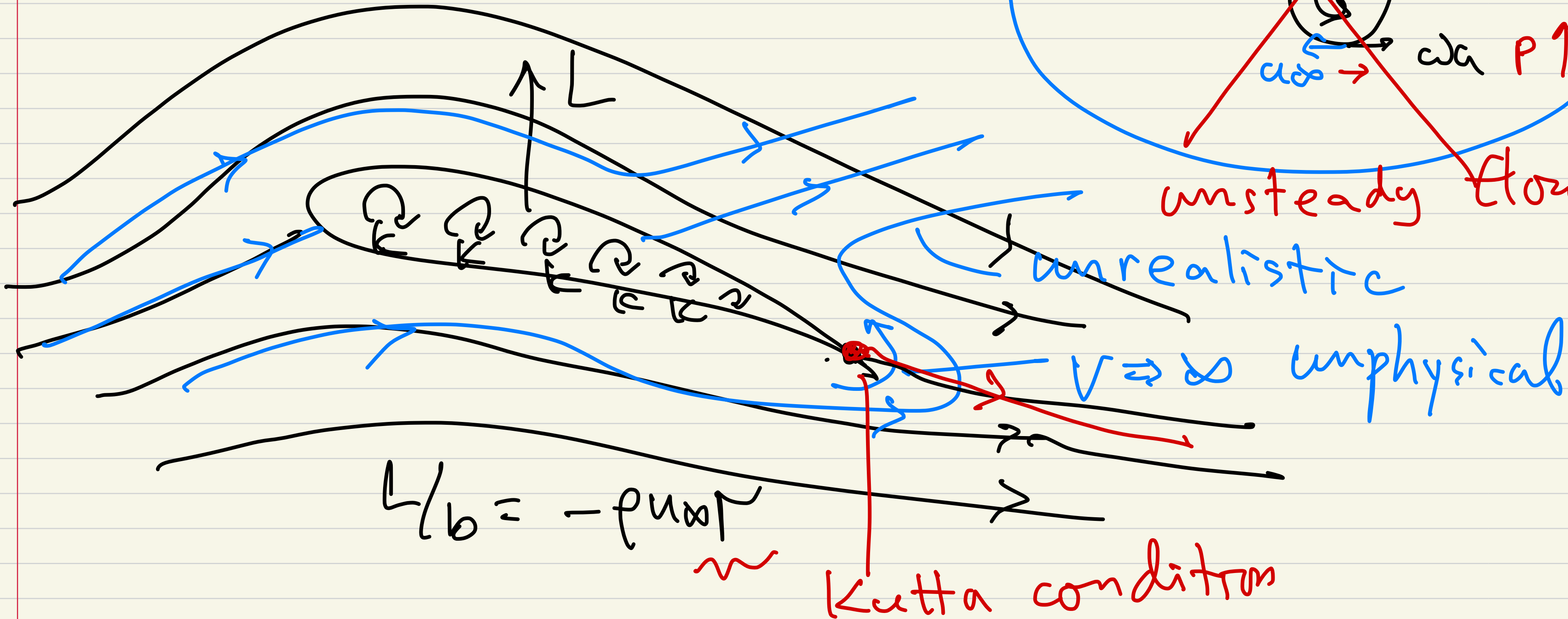


↳ indep. of a

"Kutta – Joukowski lift theorem"

According to inviscid theory, the lift per unit depth of any cylinder of any shape immersed in a uniform stream equals $\rho u_\infty \Gamma$, where Γ is the total net circulation contained within the body shape.
The direction of the lift is 90° from the stream direction rotating opposite to the circulation.





$$L/b = -\rho u_\infty \Gamma$$