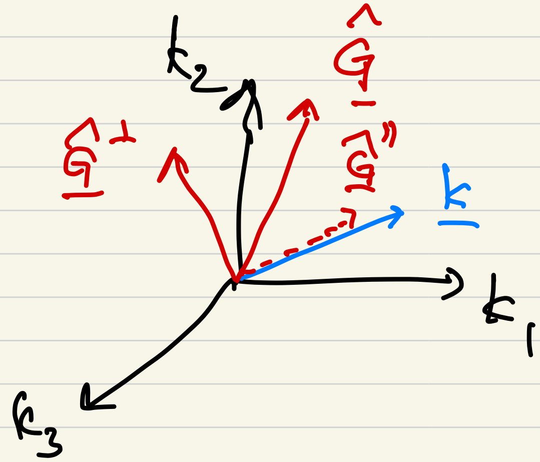
- continuity eq : $\boxed{k_j \hat{u}_j = 0}$
- Navier-Stokes eqs.

$$\frac{\partial u_j}{\partial t} + \underbrace{\frac{\partial}{\partial x_k}(u_j u_k)}_{G_j \to \hat{G}_j} = \nu \frac{\partial^2 u_j}{\partial x_k \partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_j}$$



$$\hat{G}_j = \hat{G}_j^{\parallel} + \hat{G}_j^{\perp}$$

$\hat{G}_j^{\perp}$ ← normal to $\underline{k}$

$\hat{G}_j^{\parallel}$ ↑ parallel to $\underline{k}$

$$\hat{G}_j^{\parallel} = e_j(\underline{e}\cdot\hat{\underline{G}}) \qquad e_j = k_j/k \qquad (k = |\underline{k}|)$$

$$= e_j(e_k \hat{G}_k) = \frac{k_j k_k}{k^2}\hat{G}_k$$

$$\hat{G}_j^{\perp} = \hat{G}_j - \hat{G}_j^{\parallel} = \delta_{jk}\hat{G}_k - \frac{k_j k_k}{k^2}\hat{G}_k = \underline{\left(\delta_{jk} - \frac{k_j k_k}{k^2}\right)}\hat{G}_k$$

$\equiv P_{jk}$: projection tensor

$$\text{FT} \longrightarrow \boxed{\frac{d\hat{u}_j}{dt} + \hat{G}_j = -\nu k^2 \hat{u}_j - \frac{1}{\rho} i k_j \hat{p}}$$

$$k_j \hat{u}_j = 0$$

$$k_j \cdot (\quad \text{"} \quad)$$

$$\longrightarrow \quad 0 = -\frac{1}{\rho} i k^2 \hat{p} - k_j \hat{G}_j$$

$$-\frac{1}{\rho} i k^2 \hat{p} = k_j \hat{G}_j$$

$$\rightarrow \boxed{-\frac{1}{\rho} i k_j \hat{p} = \frac{k_j k_k}{k^2} \hat{G}_k = \hat{G}_j^{\parallel}}$$

$$\boxed{\frac{d\hat{u}_j}{dt} + \nu k^2 \hat{u}_j = \hat{G}_j^{\parallel} - \hat{G}_j = -\hat{G}_j^{\perp}} = -P_{jk} \hat{G}_k$$

Final period of decay of isotropic turbulence
in which Re is very low $\rightarrow$ convection is negligible

$$\Rightarrow \frac{d\hat{u}_j}{dt} = -\nu k^2 \hat{u}_j \quad \Rightarrow \hat{u}_j(\underline{k}, t) = \hat{u}_j(\underline{k}, 0)\, e^{-\nu k^2 t}$$

Thus, in the final period of decay, each Fourier coeff. evolves independently of all other modes, decaying exponentially in time.

high-wavenumber modes decays more rapidly than low- " " do.

$$\hat{G}_j(\underline{k},t) = FT\left(\frac{\partial}{\partial x_k}(u_j u_k)\right) = i k_k FT(u_j u_k)$$

$$= i k_k FT\left[\left(\sum_{\underline{k}'} \hat{u}_j(\underline{k}') e^{i\underline{k}'\cdot\underline{x}}\right)\left(\sum_{\underline{k}''} \hat{u}_k(\underline{k}'') e^{i\underline{k}''\cdot\underline{x}}\right)\right]$$

(using orthogonality)

$$= i k_k \underbrace{\sum_{\underline{k}'} \hat{u}_j(\underline{k}') \hat{u}_k(\underline{k}-\underline{k}')}_{}$$ convolution sum

Back to N-S eq. in wave space.

$$\boxed{\left(\frac{d}{dt} + \nu k^2\right) \hat{u}_j(\underline{k},t) = -i k_l P_{jk}(\underline{k}) \sum_{\underline{k}'} \hat{u}_k(\underline{k}',t)\, \hat{u}_l(\underline{k}-\underline{k}',t)}$$

convection term is nonlinear and non-local in wavenumber space, involving the interaction of wavenumber triads, $\underline{k}$, $\underline{k}'$ and $\underline{k}''$ such that $\underline{k}'' = \underline{k} - \underline{k}'$.

6.5 Velocity spectra.

- Two pt. corr. $R_{ij}(\underline{r}, t) = \langle u_i(\underline{x}, t)\, u_j(\underline{x}+\underline{r}, t) \rangle$

$$R_{ij}(\underline{r}) = \iiint_{-\infty}^{\infty} \underbrace{\phi_{ij}(\underline{k})}_{} e^{i\underline{k}\cdot\underline{r}}\, d\underline{k}$$

↳ velocity spectrum tensor

$$= \hat{u}_i(-\underline{k}, t)\, \hat{u}_j(\underline{k}, t) = \hat{u}_i^*(\underline{k}, t)\, \hat{u}_j(\underline{k}, t)$$

$$\phi_{ij}(\underline{k}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} R_{ij}(\underline{r})\, e^{-i\underline{k}\cdot\underline{r}}\, d\underline{r}$$

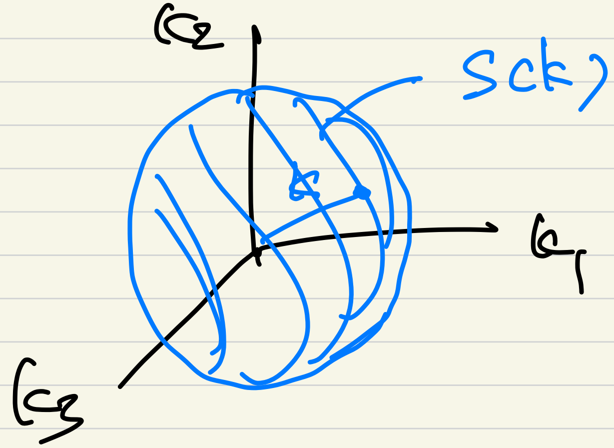
$$\left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle = \iiint_{-\infty}^{\infty} k_k k_l\, \phi_{ij}(\underline{k})\, d\underline{k}$$

$$\rightarrow \boxed{\varepsilon} = \iiint_{-\infty}^{\infty} 2\nu k^2 \cdot \frac{1}{2} \phi_{ii}(\underline{k})\, d\underline{k} = \iiint_{-\infty}^{\infty} \nu \boxed{k^2} \underline{\phi_{ii}(\underline{k})}\, d\underline{k}.$$

- Energy spectrum ft. $E(k) \equiv \oint \frac{1}{2} \phi_{ii}(\underline{k})\, dS(k)$

integration over the surface of the sphere

$S(k)$: sphere in wavenumber space



$k = |\underline{k}|$

$$R_{ij}(0) = \langle u_i u_j \rangle = \iiint_{-\infty}^{\infty} \phi_{ij}(\underline{k})\, d\underline{k}$$

$$k = \frac{1}{2}\langle u_i u_i \rangle = \frac{1}{2}\iiint_{-\infty}^{\infty} \phi_{ii}(\underline{k})\, d\underline{k} = \int_0^{\infty} E(k)\, dk$$

turb. kinetic energy

$$\varepsilon = \iiint_{-\infty}^{\infty} \nu k^2 \phi_{ii}(\underline{k})\, d\underline{k} = \int_0^{\infty} 2\nu k^2 E(k)\, dk$$

dissipation

- In isotropic turbulence, $\phi_{ij}(\underline{k})$ can depend only on $\underline{k}$.

$$\rightarrow \phi_{ij}(\underline{k}) = A(k)\,\delta_{ij} + B(k)\,k_i k_j$$

$$\phi_{ij}(\underline{k}) = \hat{u}_i^*(\underline{k})\,\hat{u}_j(\underline{k})$$

continuity : $k_j \hat{u}_j = 0$

k_j : (")

$$\rightarrow k_j \phi_{ij} = 0 = A k_i + B k_i k^2 \rightarrow B = -A/k^2$$

$$E(k) = \oint \tfrac{1}{2}\phi_{ii}(\underline{k})\, dS(\underline{k})$$

$$= \tfrac{1}{2}(3A + k^2 B)\,4\pi k^2 = (6A\pi k^2 + 2k^4 B\pi) = 4\pi k^2 A$$

$$\rightarrow A = E(k)/4\pi k^2$$

isotropic turbulence

$$\Rightarrow \boxed{\phi_{ij}(\underline{k}) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)} = \frac{E(k)}{4\pi k^2} P_{ij}(\underline{k})$$

- Taylor's hypothesis

R_{ij} ? $R_{ij} = \langle u_i(\underline{x},t)\, u_j(\underline{x}+\underline{r},t)\rangle$

$\vec{V}\underline{e}_1$ HW probe

hot wire

flying hot wire moving at V

probe position $\underline{X}(t) = \underline{x}_0 + \underline{e}_1 V t$

velocity measured $\underline{U}^{(m)}(t) = \underline{U}(\underline{X}(t),t) - \underline{e}_1 V$

Then, the temporal auto correlation

$$R_{ij}^{(m)}(s) = \left\langle \left(U_i^{(m)}(t) - \langle U_i^{(m)}(t)\rangle\right)\left(U_j^{(m)}(t+s) - \langle U_j^{(m)}(t+s)\rangle\right)\right\rangle$$

separation time

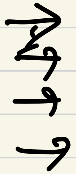
$$= \langle u_i(\underline{X}(t),t)\, u_j(\underline{X}(t+s),t+s)\rangle$$

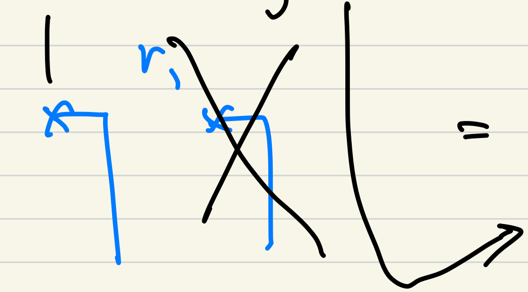
$$= \langle u_i(\underline{X}(t),t)\, u_j(\underline{X}(t)+\underline{e}_1 r_1,\ t+\tfrac{r_1}{V})\rangle \qquad r_1 = Vs$$

In homogeneous turbulence,

$$R_{ij}^{(m)}(s) = \langle u_i(\underline{x}_0+\underline{e}_1 V t, 0)\, u_j(\underline{x}_0+\underline{e}_1 V t+\underline{e}_1 r_1, 0)\rangle$$

$u(t) \to R_{ij}^{(M)}(s)$ $= \langle u_i(\underline{x}_0, 0)\, u_j(\underline{x}_0 + \underline{e}_1 r_1, 0) \rangle$





$= R_{ij}(\underline{e}_1 r_1)$: approximation of spatial corr. by temporal corr.

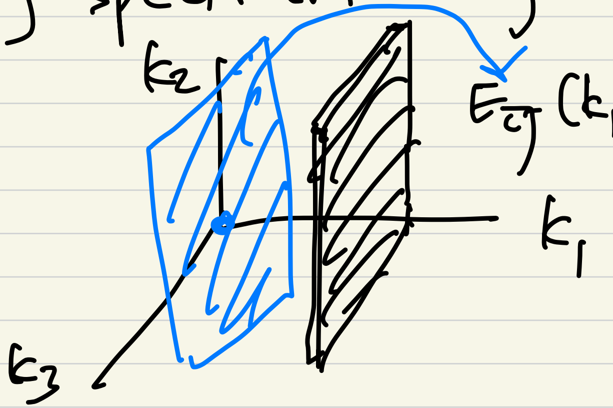
Taylor hypothesis : $\frac{\partial}{\partial t} = -V \frac{\partial}{\partial x_1}$

or, frozen turbulence approximation,

In grid turb. w/ $u'/\langle U_1 \rangle \ll 1$, Taylor hypothesis is valid.

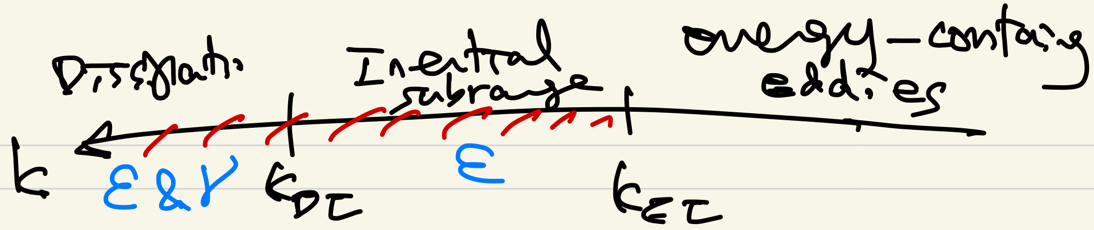
In free shear flows, " " fails.

• 1-D energy spectrum $E_{ij}(k_1)$ cf. $\Phi_{ij}(\underline{k})$, $E(k)$



$E_{ij}(k_1 = 0) \neq 0$

$E(k=0) = 0$



- Kolmogorov spectra

For $k > k_{EI}$, $E(k)$ is a universal ft. of k, ε and ν

$$\rightarrow E(k) = (\varepsilon \nu^5)^{\frac{1}{4}} \underbrace{\varphi(k\eta)}$$ from dimensional analysis

a universal non-dimen. ft. : Kolmogorov spectral ft.

In the inertial subrange,

$E(k)$ is a ft of ε and k.

$$\rightarrow E(k) = C \varepsilon^{2/3} k^{-5/3}$$: Kolmogorov $-\frac{5}{3}$ spectrum,

$C = 1.5$

$$E_{11}(k_1) = C_1 \varepsilon^{2/3} k_1^{-5/3}$$ $C_1 = 0.49$

$$E_{22}(k_1) = C_1' \varepsilon^{2/3} k_1^{-5/3}$$ $C_1' = \frac{4}{3} C_1 \doteqdot 0.65$

- Model spectrum

$$E(k) = C\varepsilon^{2/3} k^{-5/3} f_L(kL) f_\eta(k\eta)$$

shape of dissipation range tends to unity for small kL.

non-dimensional fct.

shape of energy-containing range tends to unity for large kL

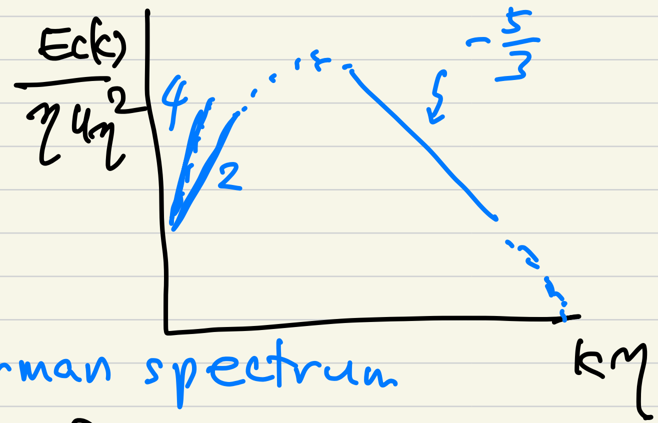
In the inertial subrange, f_L & f_η are unity.

$$f_L(kL) = \left[kL / \{ (kL)^2 + c_L \}^{\frac{1}{2}} \right]^{\frac{5}{3} + P_0}$$

$P_0 = 2$: $E(k) \sim k^2$ for small k

better

$P_0 = 4$: $E(k) \sim k^4$ for small k : von Karman spectrum



$$f_\eta(k\eta) = \exp\left\{ -\beta\left(\left[(k\eta)^4 + c_\eta^4 \right]^{\frac{1}{4}} - c_\eta \right) \right\}$$

$\beta = 5.2$

$c_L = 6.78$

$c_\eta = 0.40$