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DYNAMIC ANALYSES OF SUSPENSION BRIDGE STRUCTURES

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I-7. Appendices

Appendix I-a

Cable Profiles of Suspension Bridges and their Associated Properties

A single flexible cable suspended between two fixed points is the simplest suspension bridge. The initial problem in such a case is to determine the form adopted by the cable when it is loaded solely by its own weight, and to find the tension in the cable at any point along its length. The solution of this problem provides a starting point for a consideration of the effects upon a suspended cable of extraneous applied forces, such as the dead weight of the stiffening structures of a practical suspension bridge. This appendix is devoted to the initial problem of determining the different cable profiles of suspension bridges and their associated properties, as well as discovering the most usable profile.

1. The Common Catenary

The curve in which a perfectly flexible uniform cable hangs when freely suspended between two fixed points is called a "catenary." "Perfectly flexible" means that the cable resists applied load by developing direct stresses only. It follows, therefore, that at any cross section the resultant cable force is tangential to the cable profile at that point and acts through the centroid of the cross section. "Uniform" indicates that the weight per unit length, w, of the cable is constant. This defines the classical problem of the common catenary which was first solved by James Bernouilli, in 1691; the earliest published solution was by David Gregory in 1697.

Consider a cable hanging symmetrically between two fixed points at the same level, as shown in Fig. I-a-i. Let 0 be the origin for the ordinates x and y. If the cable is treated as inextensible, the vertical equilibrium of the element of the cable shown in Fig. I-a-ii requires that

$$\frac{d}{ds}\left(T\,\frac{dy}{ds}\right) = -\overline{w}\,, \qquad \text{(I-a-1)}$$

where T is the tension in the cable, $\overline{w}$ is the weight of the cable per unit length of the cable curve and $\frac{dy}{ds}$ is the sine of the angle of inclination, i.e., $\sin\varphi$.

The horizontal component of cable tension, H_w, is constant since there are no acting longitudinal components of load.

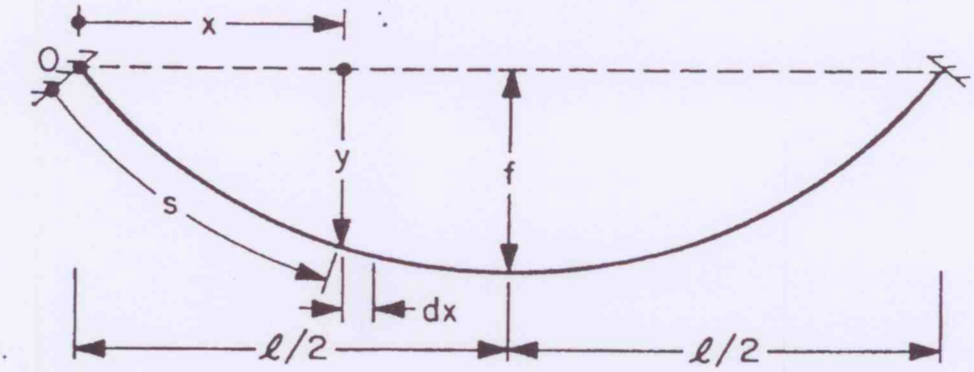
$$H_w = T\,\frac{dx}{ds} = \text{constant} \quad , \qquad \text{(I-a-2)}$$

where $\frac{dx}{ds} = \cos\varphi$. Consequently, Eq. I-a-1 is reduced to

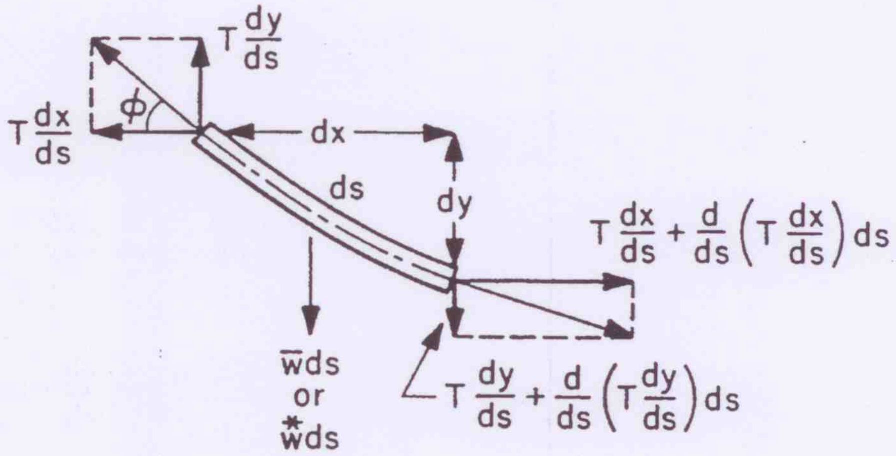
$$\left.\begin{aligned} H_w\,\frac{d^2y}{dx^2} &= -\overline{w}\,\frac{ds}{dx} \quad , \\ \text{or} \qquad & \\ H_w\,\frac{d^2y}{dx^2} &= -\overline{w}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} \quad . \end{aligned}\right\} \qquad \text{(I-a-3)}$$

Since $\overline{w}$ is constant, the solution of Eq. I-a-3 gives the Catenary. Integration of Eq. I-a-3 yields

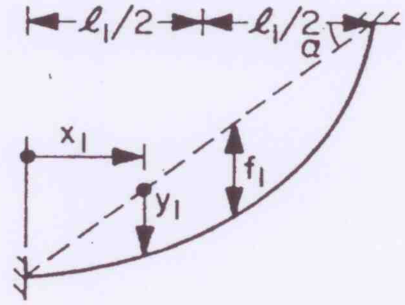
$$\sinh^{-1}\frac{dy}{dx} = -\frac{\overline{w}}{H_w}\,x + c_1 \quad ,$$



i-Definition Diagram



ii-Equilibrium of an Element



iii-Cable Profile in a Side Span

Fig. I-a

where c_1 is a constant of integration. But at $x = \frac{\ell}{2}$, $\frac{dy}{dx} = 0$, so that $c_1 = \frac{\overline{w}}{H_w}\frac{\ell}{2}$ and

$$\sinh^{-1}\frac{dy}{dx} = \frac{\overline{w}}{H_w}\left(\frac{\ell}{2} - x\right) . \qquad \text{(I-a-4)}$$

Integration again, the following can be obtained

$$y = -\frac{H_w}{\overline{w}}\cosh\left[\frac{\overline{w}}{H_w}\left(\frac{\ell}{2} - x\right)\right] + c_2 \quad ,$$

where c_2 is another constant of integration. The cable deflection at mid-span $\left(x = \frac{\ell}{2}\right)$ is the sag, f, and therefore $c_2 = f + \cancel{\chi}\ \left(\frac{H_w}{\overline{w}}\right)$ and

$$y = \frac{H_w}{\overline{w}}\left\{1 - \cosh\left[\frac{\overline{w}}{H_w}\left(\frac{\ell}{2} - x\right)\right]\right\} + f \ . \qquad \text{(I-a-5)}$$

This gives the shape of the curve adopted by the cable. When required, the length of the catenary is given by

$$s = \int_0^{\ell}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \ . \qquad \text{(I-a-6)}$$

Substituting $\frac{dy}{dx}$, obtained from Eq. I-a-5, in Eq. I-a-6 and integrating yields

$$s = 2\,\frac{H_w}{\overline{w}}\sinh\left(\frac{\overline{w}\ell}{2H_w}\right) \ . \qquad \text{(I-a-7)}$$

The tension at any point in the cable is given by Eq. I-a-2 or

$$T = H_w\frac{ds}{\cancel{dy}\ dx} = H_w\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} \ . \qquad \text{(I-a-8)}$$

Substituting the value $\frac{dy}{dx}$ derived from Eq. I-a-5, Eq. I-a-8 is reduced to

$$T = H_w \cosh\left[\frac{\overline{w}}{H_w}\left(\frac{\ell}{2} - x\right)\right]. \qquad \text{(I-a-9)}$$

This tension will be maximum at the ends of the span, where $x = 0$ or $x = \ell$, yielding

$$T_{max} = H_w \cosh\left(\frac{\overline{w}\ell}{2H_w}\right) \quad . \qquad \text{(I-a-10)}$$

All the above results depend upon a knowledge of the parameter $\frac{\overline{w}}{H_w}$ for their usefulness.

2. <u>The Parabolic Cable</u>

In many practical suspension bridges the total dead weight of the bridge, instead of being distributed as though uniform along the cables, is distributed more uniformly across the span. Of more practical importance than the common catenary, therefore, is the case of a cable suspended between two points and so loaded (or with a weight per unit length such) that the load per unit of span, ℓ, rather than the curve, is constant. Remarkably enough, although the catenary was understood at the end of the seventeenth century, this related yet simpler problem was not solved until one hundred years later. In 1794, a suspension bridge was proposed across the Neva, near Leningrad, and it was as a result of considering this proposed bridge that Nicholas Fuss published his solution that year.

Now, consider the cable, as before, to be perfectly flexible and inextensible. The vertical load on the element, ds, of the cable will be $\overset{*}{w}ds$ (instead of $\overline{w}ds$ which was for the common catenary). Again, the equilibrium of this element of the cable gives

$$T \frac{dx}{ds} = H_w = \text{constant} \quad , \tag{I-a-11}$$

and

$$\frac{d}{ds}\left(T \frac{dy}{ds}\right) = - \overset{*}{w} \quad . \tag{I-a-12}$$

Furthermore, Eqs. I-a-11 and I-a-12 give

$$H_w \frac{d^2y}{dx^2} = - \overset{*}{w} \frac{ds}{dx} \quad . \tag{I-a-13}$$

When $\overset{*}{w} \frac{ds}{dx}$ is constant, the profile of the cable is a parabola (which is the essence of the discovery made by Fuss).

However, for flat-sag cables of constant weight per unit length, the slope of the cable profile is everywhere small and, therefore

$$ds \simeq dx \quad .$$

The differential equation of the equilibrium curve is then accurately specified as

$$H_w \frac{d^2y}{dx^2} = - \overset{*}{w} \quad . \tag{I-a-14}$$

The solution of this differential equation, for the coordinate system shown in Fig. I-a-i, is the parabola

$$y = \frac{\overset{*}{w}\ell^2}{2H_w} \left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2 \right] \quad . \tag{I-a-15}$$

The cable deflection at mid-span $\left(x = \frac{\ell}{2}\right)$ is the sag , f , and the horizontal component of cable tension is

$$H_w = \frac{\overset{*}{w}\ell^2}{8f} \quad . \tag{I-a-16}$$

The tension at any point in the cable is given by Eq. I-a-8, and its value is

$$T = H_w \left[1 + \frac{\overset{*}{w}{}^2 \ell^2}{4H_w^2} \left(1 - 2\left(\frac{x}{\ell}\right)\right)^2 \right]^{\frac{1}{2}} \tag{I-a-17}$$

The maximum tension in the cable, occuring at either support, will be

$$T_{max} = \sqrt{H_w^2 + (\tfrac{1}{2}\overset{*}{w}\ell)^2} \quad . \tag{I-a-18}$$

With the aid of Eq. I-a-16, Eq. I-a-15 is more conveniently written as

$$y = \frac{4f}{\ell^2}\, x\,(\ell - x) \quad . \tag{I-a-19}$$

It is worthwhile to note that this equation is also valid for the parabolic cable shown in Fig. I-a-iii.

The length of the parabolic cable is given in general by Eq. I-a-6, and in this particular case the total length is therefore

$$s = \int_0^{\ell} \left[1 + \left\{ \frac{4f}{\ell}\left(1 - 2\left(\frac{x}{\ell}\right)\right)\right\}^2 \right]^{\frac{1}{2}} dx \quad . \tag{I-a-20}$$

It is convenient, and sufficiently accurate, to expand the integrand of Eq. 1-a-20 in a binomial series and then to carry out the integration term by term. If this is done, it is found that

$$s = \ell \left[1 + \frac{8}{3}\left(\frac{f}{\ell}\right)^2 - \frac{32}{5}\left(\frac{f}{\ell}\right)^4 + \ldots \right] , \tag{I-a-21}$$

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and for small $\frac{f}{\ell}$ ratios, it is sufficient to adopt

$$s \simeq \ell \left[1 + \frac{8}{3}\left(\frac{f}{\ell}\right)^2 \right] ,$$

for most practical purposes.

Similarly, in the more general case when the two ends are not on the same level, as shown in Fig. I-a-iii, this formula for s still holds provided that both y_1 and the sag f_1 are measured from the closing chord joined the two end supports.

3. Some Other Cases

In the case of the common catenary, $\overline{w}$ was constant measured along the cable; in the case of the parabolic cable, $\overset{*}{w}$ was constant measured along the span (horizontal) of the cable. In addition, there is the heterogeneous cable in which w is a variable, whether measured along the cable or the span. Shortly after solving the catenary problem, Bernouilli proceeded to solve this more general problem, inquiring into the law of the variation of w associated with various possible geometrical forms for the cable. The main result from this kind of approach concludes that w measured along the cable must vary so that $w\frac{ds}{dx}$, corresponding to $\overset{*}{w}$ measured along the span, is a constant. A further result of interest is that when $w\left(\frac{ds}{dx}\right)^3$ is constant, the curve is cycloid. Another example of a possible cable profile is the catenary of uniform strength developed by Gilbert in 1826, in which the cable's cross sectional area is proportional to the tension acting upon it. But this approach limits the spans of suspension bridge cables, which should be set by considerations other than mathematical limits.

4. Comparison of Cable Profiles

The cables of suspension bridges are commonly constructed with a uniform cross-sectional area, and thus, if allowed to hang freely, they would adopt the form of the common catenary given by Eq. I-a-5. But in practice they are often constructed at the site on a temporary platform, and the roadway is hung from them by vertical suspension rods so that when all is complete, and the structure is bearing its own weight, the form of the cables is more nearly parabolic. The aim of this erection procedure is to ensure that the dead weight of the whole bridge (roughly uniform measured along the span) be carried wholly by the cables and suspension rod without causing bending actions in any stiffening structures.

Thus practical interest naturally settles upon the parabolic rather than the catenary profile of cable, but there is another reason for this. The profiles of the two curves are very similar in terms of their ratios of span to sag which fall in the range common in suspension bridges (usually 8:1 or more). And since the cable profiles are alike, the loads in the cable and in any subsidiary structure of the real bridge will also be similar. In these circumstances it is natural to adopt the parabolic profile, with its greater simplicity and familiarity, as the standard one for suspension bridges, and this has become the general custom.

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