Inverse Problems

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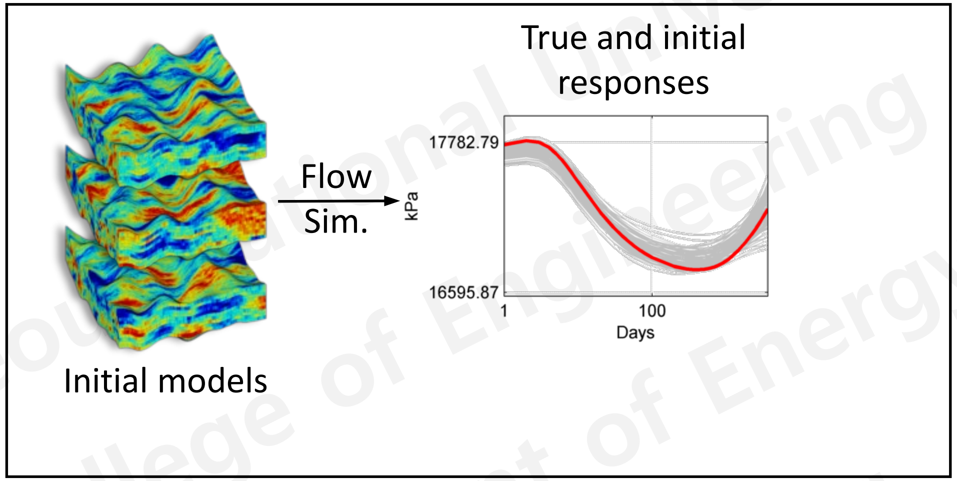
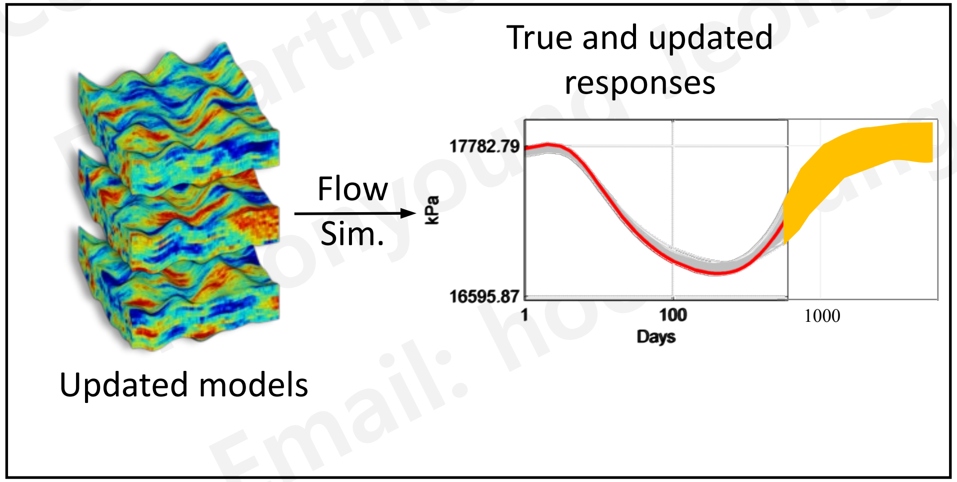
What is an Inverse Problem?

- Forward problem
  - ✓ Given model parameters ($\mathbf{m}$) of a system, a forward model (f) calculates (or simulates) the responses ($\mathbf{d}$) of the system
  - ✓ A forward problem is represented by equations, initial conditions, and boundary conditions of a system
  - ✓ $\mathbf{d} = f(\mathbf{m})$
- Inverse problem
  - ✓ Given model parameters of a system, an inverse model finds the most probable model parameters honoring observed data by calibrating the model parameters
  - ✓ The objective of an inverse problem is to find $\mathbf{m}$ minimizing the discrepancy (or gap, difference) between $\mathbf{d}_{obs}$ and $f(\mathbf{m})$
  - ✓ The discrepancy can be expressed in many ways, which is called an objective function
- Terminology
  - ✓ $\mathbf{m}_{prior}$: an initial solution, before inversion
  - ✓ $\mathbf{m}_{posterior}$: optimal solution, after inversion

History Matching is a Typical Inverse Problem





History Matching

- To calibrate model parameters
  - ✓ Rock properties
  - ✓ Fluid properties
  - ✓ Properties between rock and fluid
  - ✓ Fault
  - ✓ Initial Conditions
  - ✓ Aquifer
- To find the most probable models conditioning to observed data

Things to be Careful about HM

- Undetermined problem
  - ✓# of unknowns > # of equations
  - ✓Non-unique
  - ✓No single exact solution ➔ Multiple solutions
- Error
  - ✓Forward modeling error
  - ✓Measurement error

Manual vs. Automatic HM

- Manual HM
  - ✓To calibrate model parameters manually based on the intuition and experience of engineers
  - ✓Critical step to understand reservoirs
- Automatic (or Assisted) HM
  - ✓To calibrate model parameters using optimization algorithms

A Bayesian Inversion Framework (1)

- We will derive the objective function for inversion using the Bayesian theory
- Inversion is to find $\mathbf{m}$ maximizing $\mathrm{P}(\mathbf{m}|\mathbf{d}_{obs})$
  - ✓ $\mathrm{P}(\mathbf{m}|\mathbf{d}_{obs})$ is a probility of $\mathbf{m}$ given $\mathbf{d}_{obs}$
  - ✓ $\mathbf{d}_{obs}$ is an event that already happened
- Typical Four assumptions
  - ✓ The measurement errors in $\mathbf{d}_{obs}$ follows a normal distribution with zero means
    - $\mathbf{d}_{obs} = \mathbf{d}_{true} + \epsilon_{measurement}$
  - ✓ The forward modeling errors in $f(\mathbf{m}_{true})$ follows a normal distribution with zero means
    - $\mathbf{d}_{true} = f(\mathbf{m}_{true}) + \epsilon_{modeling}$
  - ✓ The sum of $\epsilon_{measurement}$ and $\epsilon_{modeling}$ follows a normal distribution with zero means
    - $\mathbf{d}_{obs} = f(\mathbf{m}_{true}) + \epsilon$
    - $\epsilon \sim N(0, \mathbf{C}_d)$
  - ✓ $\mathbf{m}_{true}$ follows a normal distribution with mean $\mathbf{m}_{prior}$ and covariance matrix $\mathbf{C}_\mathbf{m}$
    - $\mathbf{m}_{true} \sim N(\mathbf{m}_{prior}, \mathbf{C}_\mathbf{m})$

Univariate Normal Distribution

- $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma}}$

Multivariate Normal Distribution

- $f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\mathbf{C_x})}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{x})^T \mathbf{C_x} (\mathbf{x}-\boldsymbol{\mu}_\mathbf{x})}$ where $\mathbf{x} = [x_1 \quad \cdots \quad x_N]^T$ and $\boldsymbol{\mu}_\mathbf{x} = [\mu_{x_1} \quad \cdots \quad \mu_{x_N}]^T$

A Bayesian Inversion Framework (2)

- Which event ($\mathbf{d}_{obs}$ and $\mathbf{m}_{true}$) happens earlier in $\mathbf{d}_{obs} = f(\mathbf{m}_{true}) + \epsilon$
  - ✓ $\mathbf{d}_{obs}$ is bottomhole pressure, flow rates, GOR
  - ✓ $\mathbf{m}_{true}$ already happened, and $\mathbf{d}_{obs}$ happens later
- What is the probability of $\mathbf{d}_{obs}$ when $\mathbf{m}_{true}$ exists?
  - ✓ $\epsilon = \mathbf{d}_{obs} - f(\mathbf{m}_{true}) \sim N(0, C_d)$
  - ✓ $P(\mathbf{d}_{obs}|\mathbf{m}_{true}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\mathbf{C_d})}} e^{-\frac{1}{2}(\mathbf{d}_{obs}-f(\mathbf{m}_{true}))^T \mathbf{C_d}(\mathbf{d}_{obs}-f(\mathbf{m}_{true}))}$
- What is the probability of $\mathbf{m}_{true}$?
  - ✓ $\mathbf{m}_{true} \sim N(\mathbf{m}_{prior}, \mathbf{C_m})$
  - ✓ $P(\mathbf{m}_{true}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\mathbf{C_m})}} e^{-\frac{1}{2}(\mathbf{m}_{true}-\mathbf{m}_{prior})^T \mathbf{C_m}(\mathbf{m}_{true}-\mathbf{m}_{prior})}$

A Bayesian Inversion Framework (3)

- We want to find $\mathbf{m}_{true}$, so $\mathbf{m}_{true}$ is replaced with $\mathbf{m}$
- We know $P(\mathbf{d}_{obs}|\mathbf{m})$ and $P(\mathbf{m})$, but we want to know $\mathrm{P}(\mathbf{m}|\mathbf{d}_{obs})$
- Bayesian theorem
  - ✓ $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$
  - ✓ $\mathrm{P}(\mathbf{m}|\mathbf{d}_{obs}) = \frac{\mathrm{P}(\mathbf{d}_{obs}|\mathbf{m})\mathrm{P}(\mathbf{m})}{\mathrm{P}(\mathbf{d}_{obs})} \approx e^{-\frac{1}{2}(\mathbf{d}_{obs}-f(\mathbf{m}))^T\mathbf{C_d}(\mathbf{d}_{obs}-f(\mathbf{m}))}\ e^{-\frac{1}{2}(\mathbf{m}-\mathbf{m}_{prior})^T\mathbf{C_m}(\mathbf{m}-\mathbf{m}_{prior})}$

$$= e^{-\frac{1}{2}\left[\left(\mathbf{d}_{obs}-f(\mathbf{m})\right)^T\mathbf{C_d}\left(\mathbf{d}_{obs}-f(\mathbf{m})\right)+(\mathbf{m}-\mathbf{m}_{prior})^T\mathbf{C_m}(\mathbf{m}-\mathbf{m}_{prior})\right]}$$

Maximum Likelihood Estimation and Maximum A Posterior

- Maximum A Posterior (MAP)
  - ✓ $\mathrm{P}(\mathbf{m}|\mathbf{d}_{obs}) \approx e^{-\frac{1}{2}\left[(\mathbf{d}_{obs}-f(\mathbf{m}))^T \mathbf{C_d}(\mathbf{d}_{obs}-f(\mathbf{m}))+(\mathbf{m}-\mathbf{m}_{prior})^T \mathbf{C_m}(\mathbf{m}-\mathbf{m}_{prior})\right]}$
  - ✓ We want to find **m** maximizing $\mathrm{P}(\mathbf{m}|\mathbf{d}_{obs})$
  - ✓ Find **m** minimizing $(\mathbf{d}_{obs} - f(\mathbf{m}))^T \mathbf{C_d}(\mathbf{d}_{obs} - f(\mathbf{m})) + (\mathbf{m} - \mathbf{m}_{prior})^T \mathbf{C_m}(\mathbf{m} - \mathbf{m}_{prior})$
  - ✓ $O(\mathbf{m}) = (\mathbf{d}_{obs} - f(\mathbf{m}))^T \mathbf{C_d}(\mathbf{d}_{obs} - f(\mathbf{m})) + (\mathbf{m} - \mathbf{m}_{prior})^T \mathbf{C_m}(\mathbf{m} - \mathbf{m}_{prior})$
- Maximum Likelihood Estimation (MLE)
  - ✓ Likelihood probability $\approx e^{-\frac{1}{2}(\mathbf{d}_{obs}-f(\mathbf{m}))^T \mathbf{C_d}(\mathbf{d}_{obs}-f(\mathbf{m}))}$
  - ✓ Find **m** minimizing $(\mathbf{d}_{obs} - f(\mathbf{m}))^T \mathbf{C_d}(\mathbf{d}_{obs} - f(\mathbf{m}))$
  - ✓ $O(\mathbf{m}) = (\mathbf{d}_{obs} - f(\mathbf{m}))^T \mathbf{C_d}(\mathbf{d}_{obs} - f(\mathbf{m}))$
  - ✓ Find **m** without considering the prior probability
  - ✓ Possible if **m** does not change much compared to $\mathbf{m}_{prior}$

Procedure of HM

① $m_k = [k_1 \ k_2 \ \cdots \ k_N]^T$ ($m_0 = m_{prior}$)

② Calculate the objective function $O(m_k)$

i) Open the simulation data file

ii) Replace your keywords with $k_1, \cdots, k_N$

ex) In the data file

PERMX

100 ← k_1

200 ← k_2

iii) Run the simulation

iv) Read the result

v) Calculate and return $J(m_k)$

$$O(m_k) = \frac{1}{2}(d_{obs} - g(m))^T C_D^{-1} (d_{obs} - g(m)) + \frac{1}{2}(m - m_{prior}) C_M^{-1} (m - m_{prior})$$

③ Calculate $\nabla O(m_k)$ using FDM or a stochastic gradient

④ Update x_{k+1} using your gradient-based optimization method

⑤ Repeat ①~④ until convergence criteria are satisfied.

$$N_d - 5\sqrt{2N_d} \le 2\,O(m_{MAP}) \le N_d + 5\sqrt{2N_d}$$

from Eq 8.6 in Inverse Theory for Petroleum Reservoir characterization and History Matching

C_d

$d = [p_1 \ p_2 \ p_3]^T$ where p_1, p_2, p_3 are bottomhole pressures at Wells 1, 2, 3

$$C_d = \begin{bmatrix} \sigma_{p_1}^2 & \sigma_{p_1,p_2} & \sigma_{p_1,p_3} \\ \sigma_{p_2,p_1} & \sigma_{p_2}^2 & \sigma_{p_2,p_3} \\ \sigma_{p_3,p_1} & \sigma_{p_3,p_2} & \sigma_{p_3}^2 \end{bmatrix}$$

Normally, covariance between different data is assumed to be zero

$$C_d = \begin{bmatrix} \sigma_{p_1}^2 & 0 & 0 \\ 0 & \sigma_{p_2}^2 & 0 \\ 0 & 0 & \sigma_{p_3}^2 \end{bmatrix} \rightarrow (d_{obs} - g(m))^T C_d^{-1} (d_{obs} - g(m))$$

$$= \frac{(p_{1,obs} - p_1)^2}{\sigma_{p_1}^2} + \frac{(p_{2,obs} - p_2)^2}{\sigma_{p_2}^2} + \frac{(p_{3,obs} - p_3)^2}{\sigma_{p_3}^2}$$

← 일종의 weight : 오차분산이 크면 less weighted

C_m

$m = [k_1 \ k_2 \ k_3]^T$ where k_1, k_2, k_3 are permeability values at cells 1, 2, 3

$$C_m = \begin{bmatrix} \sigma_{k_1}^2 & \sigma_{k_1,k_2} & \sigma_{k_1,k_3} \\ \sigma_{k_2,k_1} & \sigma_{k_2}^2 & \sigma_{k_2,k_3} \\ \sigma_{k_3,k_1} & \sigma_{k_3,k_2} & \sigma_{k_3}^2 \end{bmatrix}$$

$$\sigma_{k_1,k_2} = Cov(k_1, k_2) = \sigma^2 - \gamma(h)$$