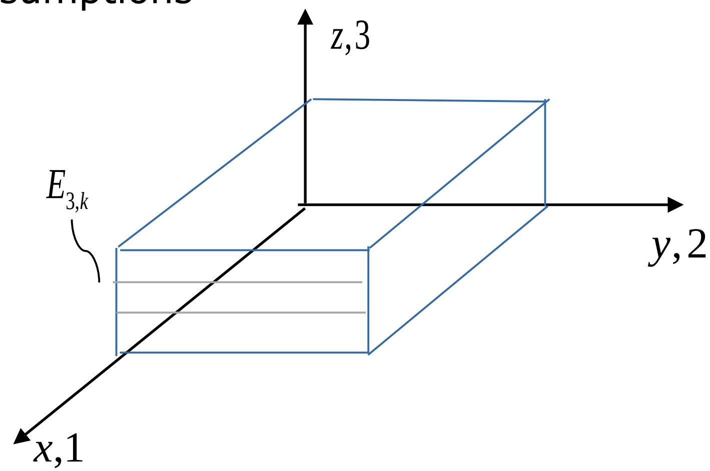
Plates

❖ Plates

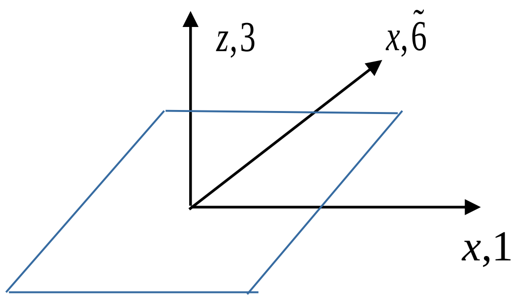
assumptions



$T_z \ll T_x, T_y$ (Plane stress)

- Kirchhoff plates
  - → Plane sections remain plane
  - $\perp$ to midline
  - → ignore T_4, T_5, S_4, S_5
- Piezoelectric polarized in z direction

➢ Constitutive relations for single lamina



Plane stress property

$$\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \\ \tilde{T}_6 \\ \tilde{D}_3 \end{bmatrix} = \begin{bmatrix} \tilde{c}_{11}^E & \tilde{c}_{12}^E & 0 & -\tilde{e}_{13} \\ \tilde{c}_{12}^E & \tilde{c}_{22}^E & 0 & -\tilde{e}_{23} \\ 0 & 0 & \tilde{c}_{66}^E & 0 \\ \tilde{e}_{31} & \tilde{e}_{32} & 0 & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_6 \\ \tilde{E}_3 \end{Bmatrix}$$

Plates

- rotations lead to fully coupled 4X4 material properties

$$\begin{bmatrix} \tilde{c}_{11}^{E} & \tilde{c}_{12}^{E} & \tilde{c}_{16} & -\tilde{e}_{13} \\ & \tilde{c}_{22}^{E} & \tilde{c}_{26} & -\tilde{e}_{23} \\ & & \tilde{c}_{66}^{E} & \tilde{e}_{63} \\ & & & \varepsilon_{33} \end{bmatrix}$$

2 strain-displacement for plate

$$u = u_o - z\frac{dw}{dx} \qquad S_1 = \frac{\partial u}{\partial x}$$

$$v = v_o - z\frac{dw}{dy} \quad \Rightarrow \quad S_2 = \frac{\partial v}{\partial y}$$

$$w = w_o \qquad S_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_1^o \\ S_2^o \\ S_6^o \end{bmatrix} + z\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_o}{\partial x} \\ \dfrac{\partial v_o}{\partial y} \\ \dfrac{\partial v_o}{\partial x} + \dfrac{\partial u_o}{\partial y} \end{bmatrix} + z\begin{bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$

Plates

Plugging into constitutive relation

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = [C]\begin{bmatrix} S_1^o \\ S_2^o \\ S_6^o \end{bmatrix} + z[C]\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \underbrace{\begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix}}\, E_3 \quad T_E$$

$$D_3 = [e]^T \begin{bmatrix} S_1^o \\ S_2^o \\ S_6^o \end{bmatrix} + z[e]^T \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} + \varepsilon_{33}^s E_3$$

$$\delta U_1^m = \int_V \vec{T}\delta s dV = \int_V [T_1 \quad T_2 \quad T_6]\delta \begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} dV$$

$$\delta U_1^m = \int_A \int_t [T_1 \quad T_2 \quad T_6]\begin{bmatrix} \delta S_1^o \\ \delta S_2^o \\ \delta S_6^o \end{bmatrix} + z[T_1 \quad T_2 \quad T_6]\begin{bmatrix} \delta\kappa_1 \\ \delta\kappa_2 \\ \delta\kappa_6 \end{bmatrix} dz dA$$

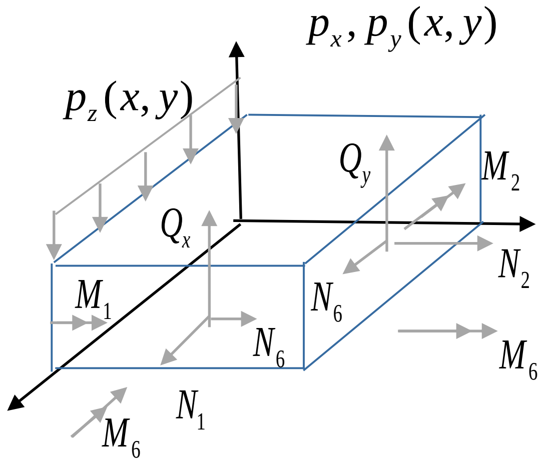
$$= \int_A [N_1 \quad N_2 \quad N_6]\begin{bmatrix} \delta S_1^o \\ \delta S_2^o \\ \delta S_6^o \end{bmatrix} + [M_1 \quad M_2 \quad M_6]\begin{bmatrix} \delta\kappa_1 \\ \delta\kappa_2 \\ \delta\kappa_6 \end{bmatrix} dA$$

$$N_1 = \int_t T_1 dz \qquad M_1 = \int_t z T_1 dz$$

$$N_2 = \int_t T_2 dz \qquad M_2 = \int_t z T_2 dz$$

$$N_6 = \int_t T_6 dz \qquad M_6 = \int_t z T_6 dz$$

Plates



$$q = \int_A D_3 dA$$

Plugging in material properties

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} S_1^o \\ S_2^o \\ S_6^o \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \sum_{k=1}^{n} \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix} E_{3k} h_k$$

$$A_{ij} = \sum_{k=1}^{n} \left(c_{ij}^E\right)_k \left(z_k - z_{k-1}\right)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(c_{ij}^E\right)_k \left(z_k^{\,2} - z_{k-1}^{\,2}\right)$$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = [B] \begin{bmatrix} S_1^o \\ S_2^o \\ S_6^o \end{bmatrix} + z[D]_{3\times 3} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \sum_{k=1}^{n} \frac{1}{2}\left(z_k + z_{k-1}\right) \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix} E_{3k} h_k$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(c_{ij}^E\right)_k \left(z_k^{\,3} - z_{k-1}^{\,3}\right)$$

$$h_k = z_k - z_{k-1}$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}_{6\times 6} \begin{Bmatrix} S^o \\ \kappa \end{Bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

Plates

- **Plates (continued)**
  - 6 strain-displacement

$$S_1^o = \frac{\partial u_o}{\partial x} \qquad \kappa_1 = -\frac{\partial^2 w}{\partial x^2}$$

$$S_2^o = \frac{\partial v_o}{\partial y} \qquad \kappa_2 = -\frac{\partial^2 w}{\partial y^2}$$

$$S_6^o = \frac{\partial u_o}{\partial x} + \frac{\partial u_o}{\partial y} \qquad \kappa_6 = -2\frac{\partial^2 w}{\partial x \partial y}$$

  - Actual strain $\quad S = S_o + z\kappa$

  - 6 stress-strain

$$N_1 = \int_{-t/2}^{t/2} T_1 dz = A\left(S_1^o + \nu S_2^o\right) \qquad M_1 = \int zT_1 dz = D\left(\kappa_1 + \nu\kappa_2\right)$$

constant thickness, isotropic where $A = \frac{Et}{1-\nu^2}$ $\qquad D = \frac{Et^3}{12\left(1-\nu^2\right)}$

$$N_2 = \int T_2 dz = A\left(S_2^o + \nu S_1^o\right) \qquad M_2 = \int zT_2 dz = D\left(\kappa_2 + \nu\kappa_1\right)$$

$$N_6 = \int T_6 dz = A\frac{(1-\nu)}{2}S_6^o \qquad M_6 = \int zT_6 dz = D\frac{(1-\nu)}{2}\kappa_6$$

Plates

- Add piezo

$$N_E = \int_{-t/2}^{t/2} T_E dz \qquad M_E = \int zT_E dz$$

$$[N]=[A]\{S_o\}+[B]\{\kappa\}-\{N_E\}$$

$$[M]=[B]\{S_o\}+[D]\{\kappa\}-\{M_E\}$$

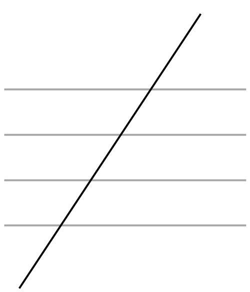
- Mechanical strain-displacement
  Mechanical stress-strain
  Piezo constitutive relationship

→ Mechanical Eqn. of Motion

→ Hamilton's Equation

→ Actuator, Sensor

- Piezo are sensor
  - sense change
  - use D_3 equations
  - assume $E_3 = 0$



$$q(t) = \int_A D_3 dA = \int_A \left(e_{31}S_1 + e_{32}S_2 + e_{36}S_6 + \varepsilon_3^s E_3\right) dA$$

$$q = \int_A \left[e_{31}\left(S_1^o + z\kappa_1\right) + e_{32}\left(S_2^o + z\kappa_2\right) + e_{36}\left(S_6^o + z\kappa_6\right)\right] dA$$

$z_k = z$ midplane of k-th active layer

Plates

- Mechanical Equations of Motion

the “equilibrium” equations ($F=ma$)

$$\frac{\partial N_1}{\partial x}+\frac{\partial N_2}{\partial y}=m\frac{\partial^2 u_o}{\partial t^2}-p_x(t) \qquad \frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}=m\frac{\partial^2 v_o}{\partial t^2}-p_y(t)$$

$$\frac{\partial^2 M_1}{\partial x^2}+2\frac{\partial^2 M_6}{\partial x\partial y}+\frac{\partial^2 M}{\partial y^2}=m\frac{\partial^2 w_o}{\partial t^2}-p_z(x,y,z)$$

$$m=\frac{mass}{area}=\rho t$$

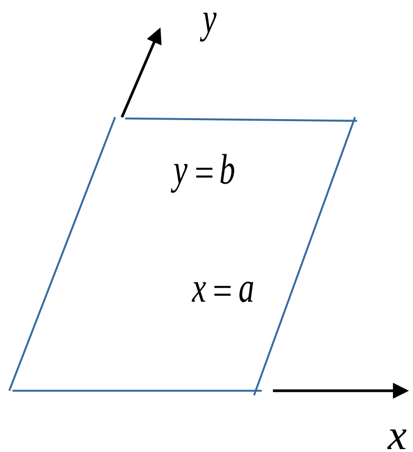
Plates

- No in-plane stress
  - Stress/bending coupling = 0 → B=0
  - Isotropic - Quasi-static

$$D\nabla^2\nabla^2 w = -\nabla^2 M^E - p_z$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

- B.C. options



On $x = a$

Clamped $w=0$ $u^o=0$ $v^o=0$ $\frac{\partial w}{\partial x}=0$

Simply supported $w=0$ $M_1=0$ $u^o=0$ $v^o=0$

Free $N_1=0$ $N_6=0$

$M_1=0$ $M_6=0$ $\frac{\partial M_6}{\partial y}+V_1=0$

$V_1=0$ (shear)

$$\rightarrow \frac{\partial M_1}{\partial x}+2\frac{\partial M_6}{\partial y}=0$$

Plates

- Principle of Virtual Work

$$\int_V D\delta E dV + \int_V T\delta S dV = \int_V F dU dV + \int_s t_n \delta U ds - q\delta\varphi$$

$$T = c^E S - eE \qquad\qquad D = eS + eE$$
$$\quad = c^E S - T^E$$

$$\int_V -(eS + eE)\delta E dV + \int_V \left[(c^E S - T^E)\delta S - F\delta U \right] dV - \int_s t_n \delta U ds - q d\varphi = 0$$

Coefficient of δE and δS → 0

$p_z(x,y)$

actuator $\int_V (\underbrace{S_t c^E}_{K} \delta S - F_t \cancel{\delta U} - \underbrace{T^E}_{\theta} \delta S) dV - \int_S t_n \delta U dS = 0$

sensor $\int_V (-\underbrace{T^E \delta E}_{\theta^T} - \underbrace{E_t e_t \delta E}_{C}) dV - \sum q\delta\varphi = 0$

Integrating by parts $\Pi_p = \int_V (\frac{1}{2} \overbrace{s_t c^E}^{K} s - \overbrace{T_t^E}^{\theta} s) dV - \int_s p_z(x,y) w ds = 0$

$$\int_V (\underbrace{-T^E E}_{\theta^T} - \frac{1}{2} \underbrace{E_t e_t E}_{C}) dV - \sum q\varphi = 0$$

Plates

- Substitute plate terms
  - looking at actuator equation

$$S = S^0 + z\kappa$$

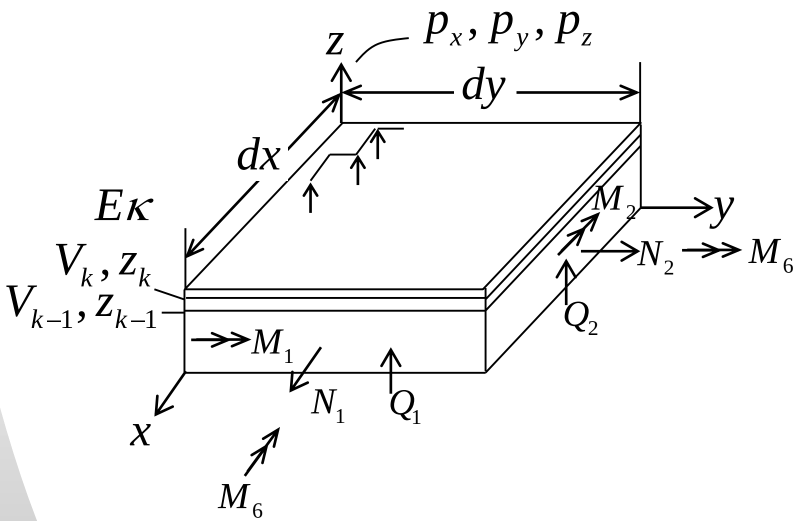
$$\Pi_P = \frac{1}{2}\int_A \left(S_t^0 A S^0 + S_t^0 B\kappa + \kappa_t B S^0 + \kappa_t D\kappa \right) dA$$

$$-\int_A \left(N_t^E S^0 + M_t^E \kappa \right) dA - \int_A p_z(x,y)\, w\, dA = 0$$

Internal energy

Piezo or thermal
Force & moment

Mechanical forcing



- Kirchhoff plate

$$u = u_0 - z\frac{dw}{dx} \quad (1a)$$

$$v = v_0 - z\frac{dw}{dx} \quad (1b)$$

$$w = w_0 - z\frac{dw}{dx} \quad (1c)$$

Plates

- Kinematics

  strain displacement

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_1^0 \\ S_2^0 \\ S_6^0 \end{bmatrix} + z \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{bmatrix} + z \begin{bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = L_0 \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}, \qquad (3)$$

$$L_0 = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & -z\dfrac{\partial^2}{\partial x^2} \\ 0 & \dfrac{\partial}{\partial y} & -z\dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & -2z\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix}$$

Plates

Also

$$E_{\kappa} = -\frac{1}{h_{\kappa}}\left(V_{\kappa} - V_{\kappa-1}\right)$$

$$\begin{bmatrix} E_n \\ \vdots \\ E_1 \end{bmatrix} = L_V \begin{bmatrix} V_n \\ \vdots \\ V_1 \end{bmatrix} \qquad (4)$$

- Energy Principle

$$\int_{t_1}^{t_2} \left[\delta T - \delta U_1^M + \delta U_1^E + \delta W_1^M - \delta W_1^E \right] dt = 0$$

- Kinetic Energy

$$\delta T = \int \rho \begin{bmatrix} \dot{u} & \dot{v} & \dot{w} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} dV \Rightarrow \int_A m(x,y) \begin{bmatrix} \dot{u}_0 & \dot{v}_0 & \dot{w}_0 \end{bmatrix} \begin{bmatrix} \delta\dot{u}_0 \\ \delta\dot{v}_0 \\ \delta\dot{w}_0 \end{bmatrix} dA$$

$$\parallel$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$\dot{u} = \dot{u}_0 \cancel{- z\frac{\partial \dot{w}_0}{\partial x}}$$

$$\dot{v} = \dot{v}_0 \cancel{- z\frac{\partial \dot{w}_0}{\partial y}}$$

$$\dot{w} = \dot{w}_0$$

Plates

- Mechanical energy

$$\delta U_1^M = \int_V T\delta S dV = \int_A \underbrace{\begin{bmatrix} N & M \end{bmatrix}}_{1\text{x}6} \begin{bmatrix} \delta S_0 \\ \delta\kappa \end{bmatrix} dV$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \int_t \begin{bmatrix} T_1 \\ T_2 \\ T_6 \\ zT_1 \\ zT_2 \\ zT_6 \end{bmatrix} dz$$

- Stress-strain

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} S_0 \\ \kappa \end{bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} S_0 \\ \kappa \end{bmatrix} - \begin{bmatrix} C \\ F \end{bmatrix} \begin{bmatrix} E_{2n} \\ \vdots \\ E_1 \end{bmatrix}$$

$$c_k = \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix}_k h_k \; , \quad F_k = \begin{bmatrix} e_{13} \\ e_{23} \\ e_{63} \end{bmatrix}_k h_k \frac{1}{2}\left(z_k + z_{k-1} \right)$$

$$C = \begin{bmatrix} C_n & C_{n-1} & \cdots & C_1 \end{bmatrix}$$

$$F = \begin{bmatrix} F_n & F_{n-1} & \cdots & F_1 \end{bmatrix}$$

Plates

plugging in

$$\delta U_1^M = \int_A \begin{bmatrix} N & M \end{bmatrix} \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} dA$$

$$= \int_A \left\{ \begin{bmatrix} S_0 & \kappa_0 \end{bmatrix} \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} - \begin{bmatrix} E_n & \cdots & E_1 \end{bmatrix} \begin{bmatrix} C^T & F^T \end{bmatrix} \begin{bmatrix} \delta S_0 \\ \delta \kappa \end{bmatrix} \right\} dA$$

- Work

$$\delta W_1^M = \int_A \begin{bmatrix} p_x & p_y & p_z \end{bmatrix} \begin{bmatrix} \delta u_o \\ \delta v_0 \\ \delta w_0 \end{bmatrix} dA$$

- Derivation of Equilibrium equation

$$\int_{t_1}^{t_2} \left[\delta T - \delta U_1^M + \delta W_1^m \right] dt +$$

$$\int_{t_1}^{t_2} \int_A \left\{ \begin{bmatrix} \dot{u}_0 & \dot{v}_0 & \dot{w}_0 \end{bmatrix} m(x,y) \begin{bmatrix} \delta u_o \\ \delta v_0 \\ \delta w_0 \end{bmatrix} - \begin{bmatrix} N & M \end{bmatrix} \begin{bmatrix} L_u \end{bmatrix} \begin{bmatrix} \delta u_o \\ \delta v_0 \\ \delta w_0 \end{bmatrix} + \begin{bmatrix} p_x & p_y & p_z \end{bmatrix} \begin{bmatrix} \delta u_o \\ \delta v_0 \\ \delta w_0 \end{bmatrix} \right\} dAdt$$

Plates

$$\begin{cases} \dfrac{\partial N_1}{\partial x}+\dfrac{\partial N_6}{\partial y}=m\dot{u}_0-p_x & \cdots \quad \delta u_0 \\ \dfrac{\partial N_6}{\partial x}+\dfrac{\partial N_2}{\partial y}=m\dot{v}_0-p_y & \cdots \quad \delta v_0 \\ \dfrac{\partial^2 M_1}{\partial x^2}+2\dfrac{\partial^2 M_6}{\partial x\partial y}+\dfrac{\partial^2 M_2}{\partial y^2}=m\ddot{w}_0-p_z & \cdots \quad \delta w_0 \end{cases}$$

- Electrical terms

$$\delta U_1^E=\int_V D\delta E dV$$

$$=\int_A\int_t\left\{[e]\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix}+\varepsilon_{33}E\right\}\delta E dz dA$$

$$\begin{bmatrix} e_{13} & e_{23} & e_{63} \end{bmatrix}$$

$$\delta U_1^E=\int_A\begin{bmatrix} S_0 & \kappa \end{bmatrix}\begin{bmatrix} C \\ F \end{bmatrix}\begin{bmatrix} \delta E_n \\ \vdots \\ \delta E_1 \end{bmatrix}+\begin{bmatrix} E_n & \cdots & E_1 \end{bmatrix}[\varepsilon]\begin{bmatrix} \delta E_n \\ \vdots \\ \delta E_1 \end{bmatrix}dA\ , \qquad [\varepsilon]=\begin{bmatrix} \varepsilon_{33_n}^S h_n & & \\ & \ddots & \\ & & \varepsilon_{33_1}^S h_1 \end{bmatrix}$$

Plates

- Electric work

$$\delta W_1^E = \int_V q\delta\varphi dV = \int_A \begin{bmatrix} q_n & \cdots & q_1 \end{bmatrix} \begin{bmatrix} \delta V_n \\ \vdots \\ \delta V_1 \end{bmatrix} dA$$

$$\int_{t_1}^{t_2} \left(\delta U_1^E - \delta W_1^E \right) dt = 0$$

$$\begin{bmatrix} E \end{bmatrix} = L_v \begin{bmatrix} V_n \\ \vdots \\ V_1 \end{bmatrix}$$

$$\int_A \left\{ \begin{bmatrix} S_0 & \kappa \end{bmatrix} \begin{bmatrix} C \\ F \end{bmatrix} L_v + \begin{bmatrix} E_n & \cdots & E_1 \end{bmatrix} [\varepsilon] L_v - \begin{bmatrix} q_n & \cdots & q_0 \end{bmatrix} \right\} \begin{bmatrix} \delta V_n \\ \vdots \\ \delta V_1 \end{bmatrix} dA = 0$$

- Electrical equation of motion

$$L_v^T \begin{bmatrix} C^T & F^T \end{bmatrix} \begin{bmatrix} S_0 \\ \kappa \end{bmatrix} + L_v^T [\varepsilon] L_v [V] = [q]$$

- Everything so far

$\int_A \;\Rightarrow\;$ Section equations