Concrete in Plasticity

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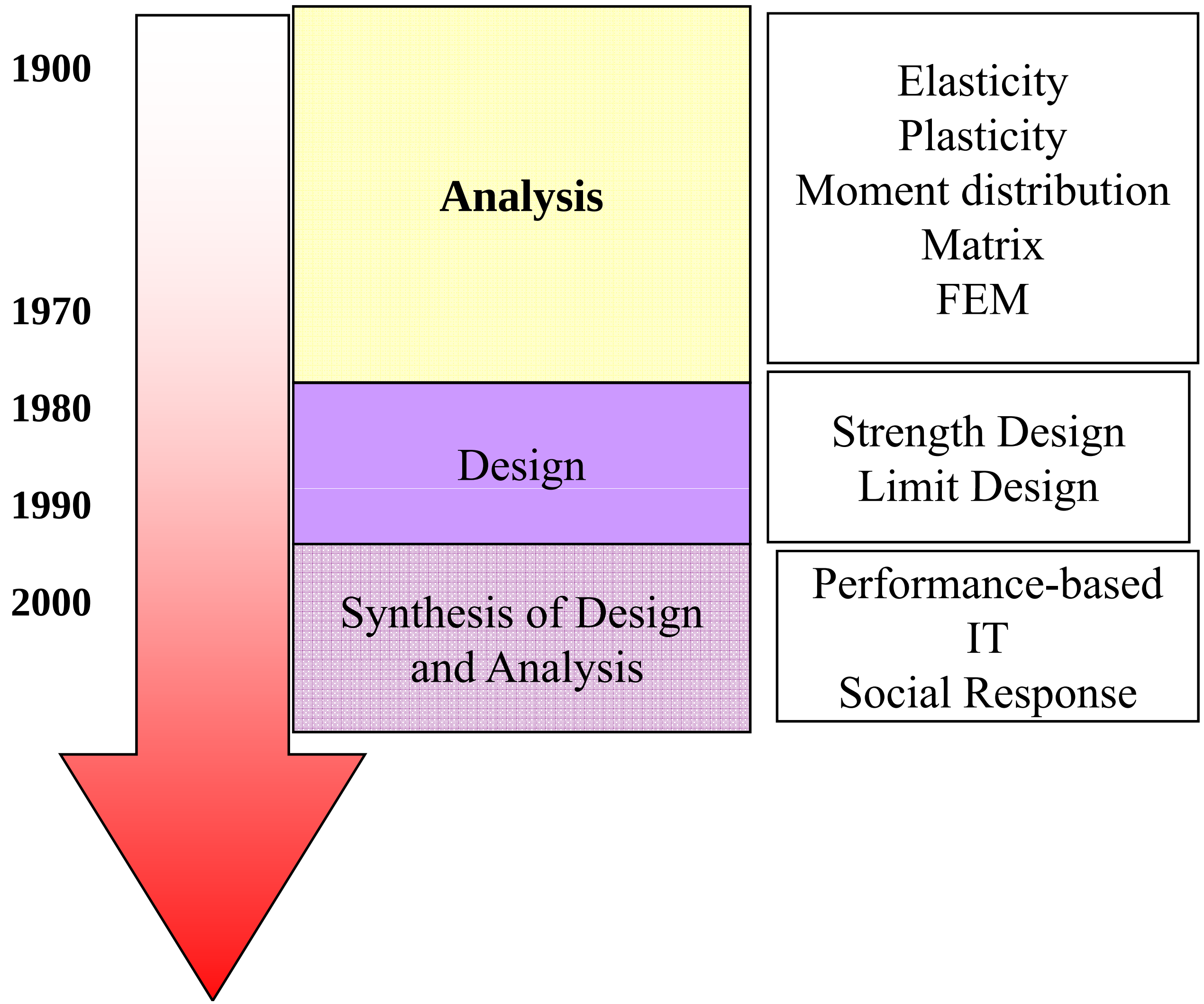
From analysis to Design and Synthesis

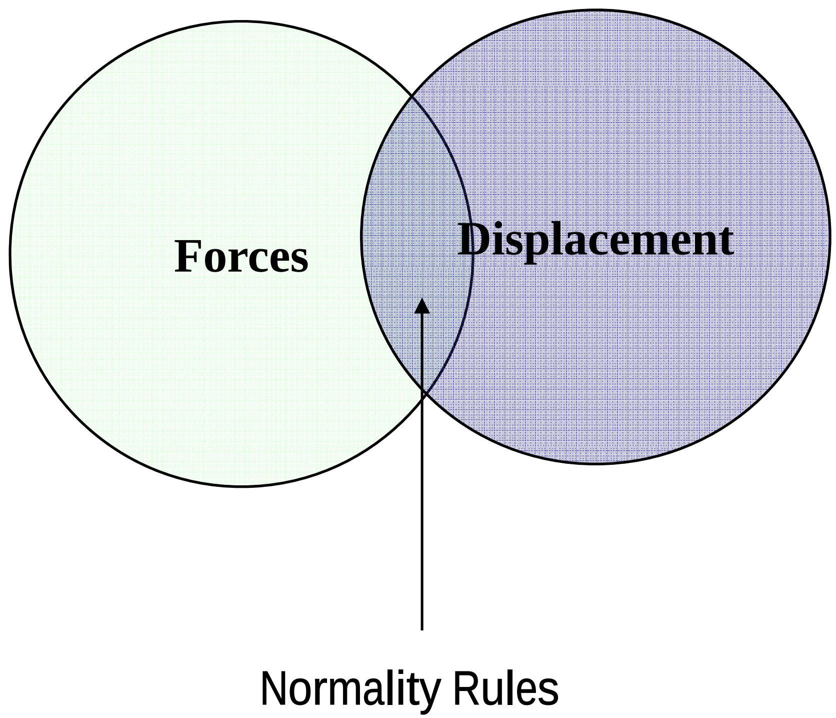


History of Theory of Plasticity

- **Gvosdev**
- **Drucker and Prager**
- **W.F Chen**
- **Johansen: Yield Line Theory**
- **Hillerborg: Strip Method**
- **The Technical University of Denmark:**
- **Nielsen and Braestrup**
- **Swiss Federal Institute of Technology in Zurich:**
- **Mueller and Marti**
- **Vecchio and Collinse in Canada**
- **T.T.C.Hsu in U.S.A**

Limit Theory for Design of Reinforced Concrete

- Lower Bound
- Equilibrium
- Yield Condition
- Upper Bound
- Geometry
- Work equation

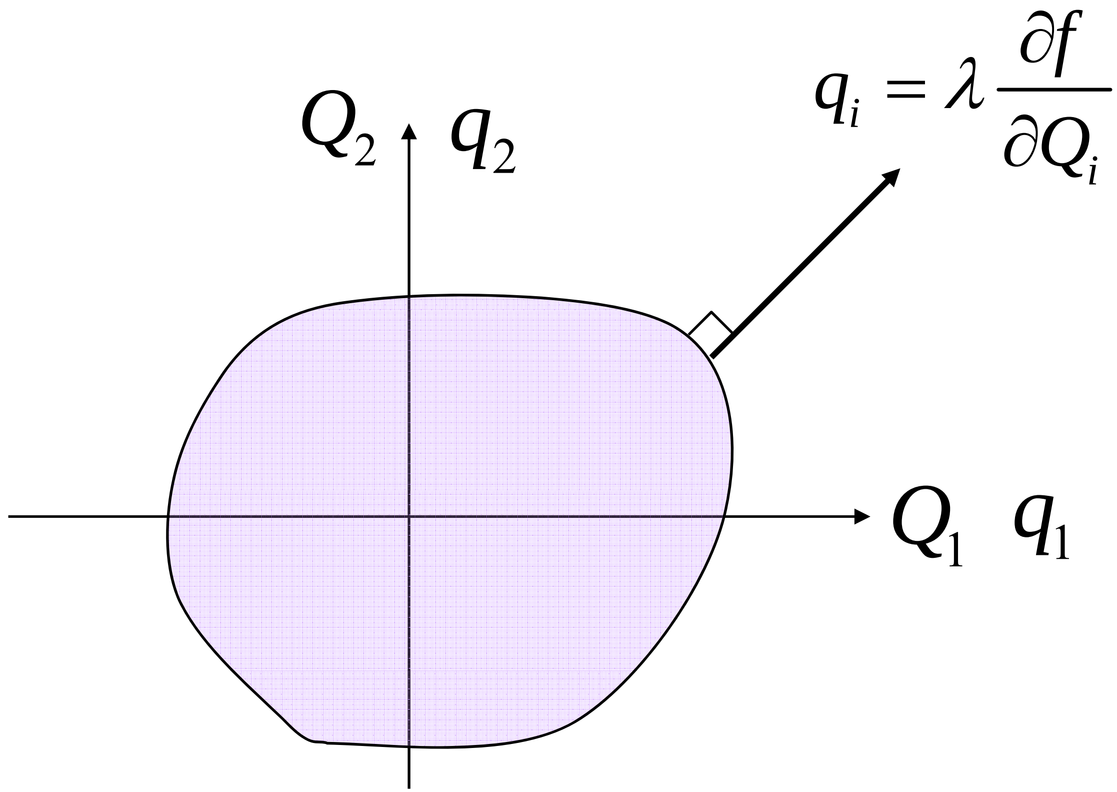


Contents

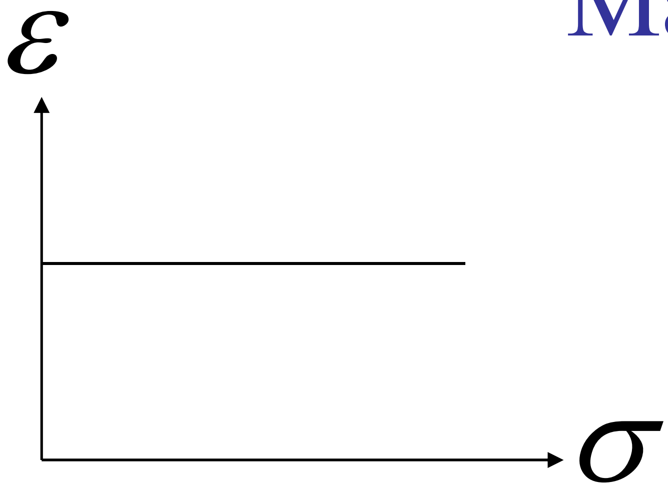
Theory of Plasticity

- **1.1 Constitutive Equations**

Von Mises's Flow Rule



Extreme Principles for Rigid-Plastic Materials



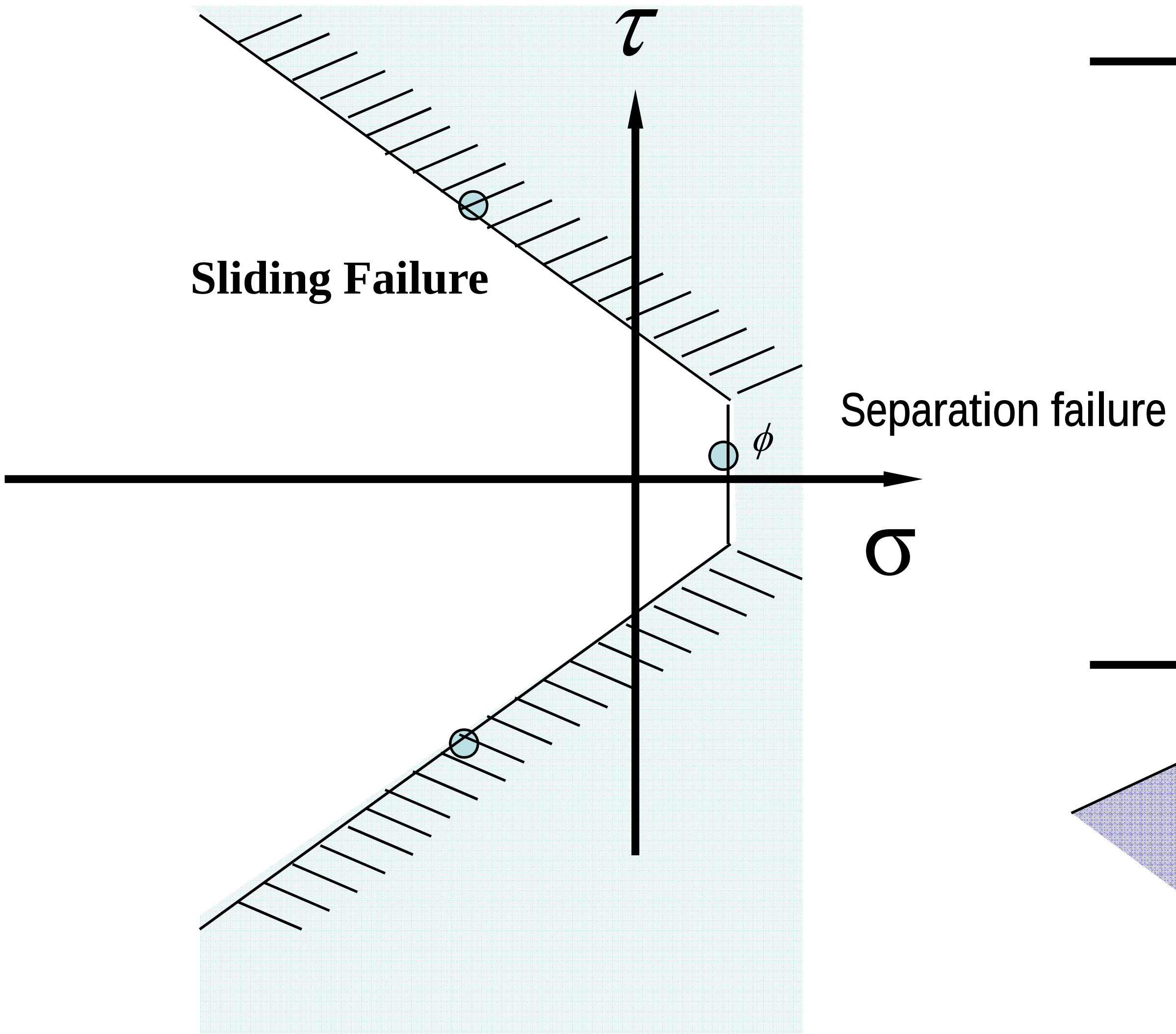
–Lower Bound Theorem
-Upper Bound Theorem
-Uniqueness Theorem

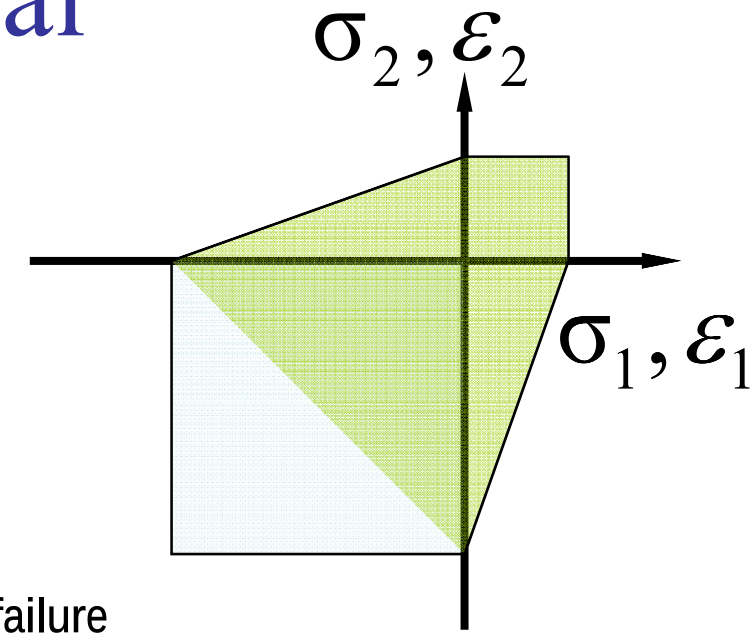
Yield Conditions

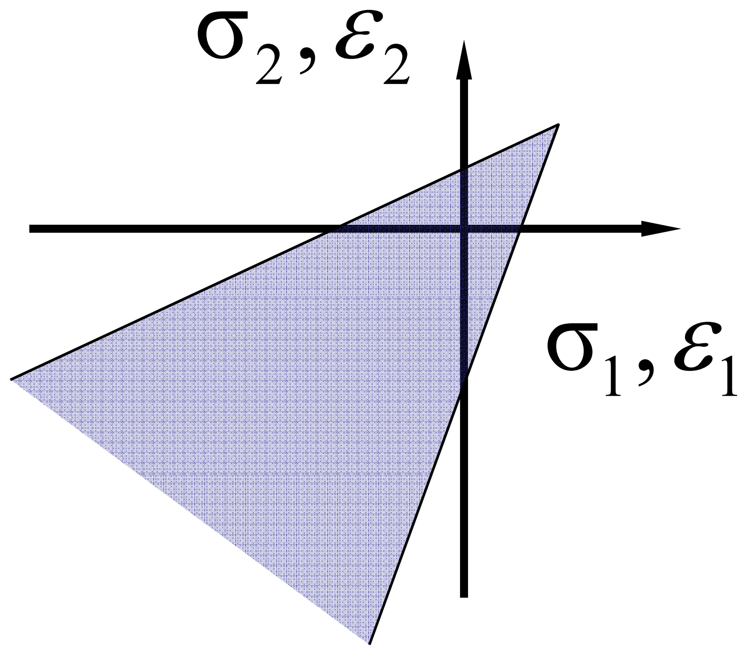
- Concrete
- Frictional hypothesis

- Reinforcing bar
- Disks
- Slabs

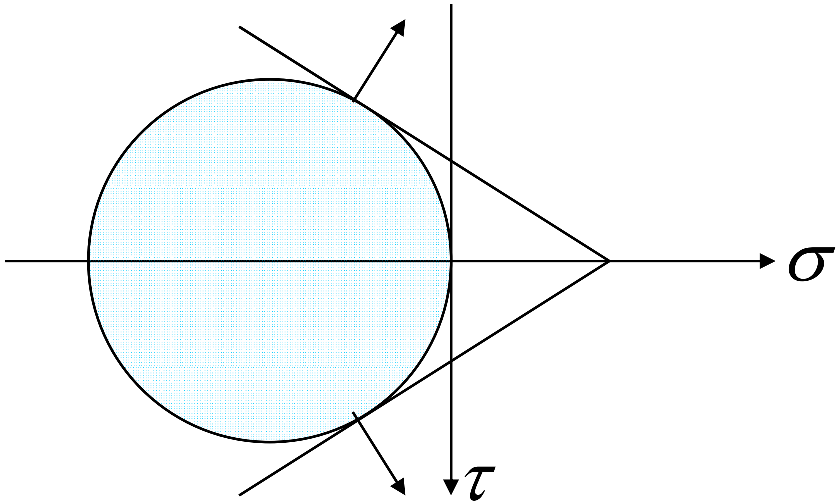
Mohr Coulomb Material





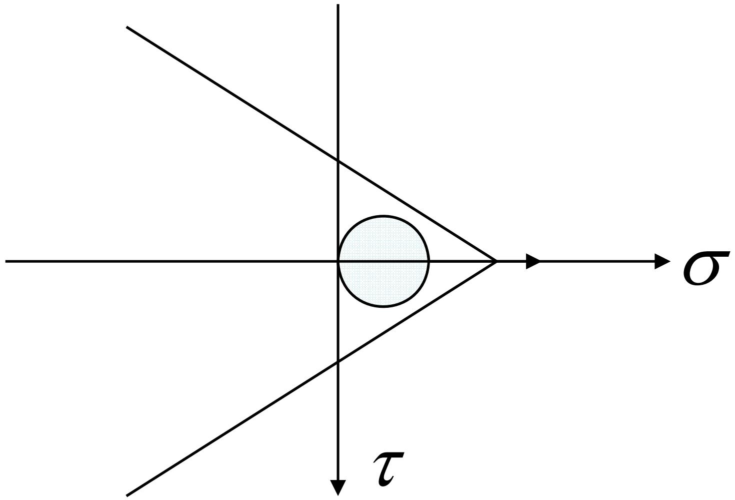


Determination of Parameters for More-Coulomb material

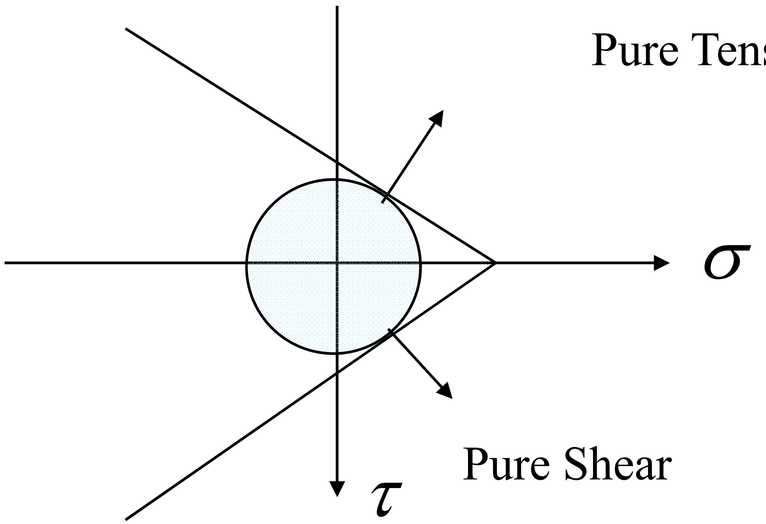


Pure Compression

$$k\sigma_1 - \sigma_3 = f_c$$

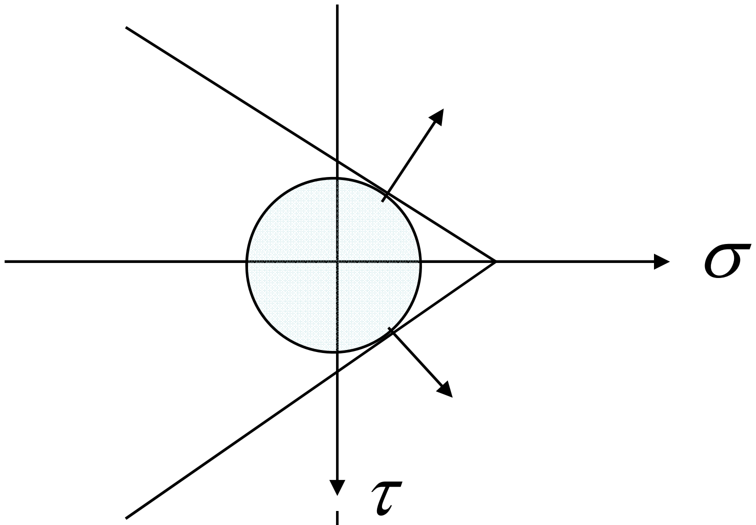


Pure Tension

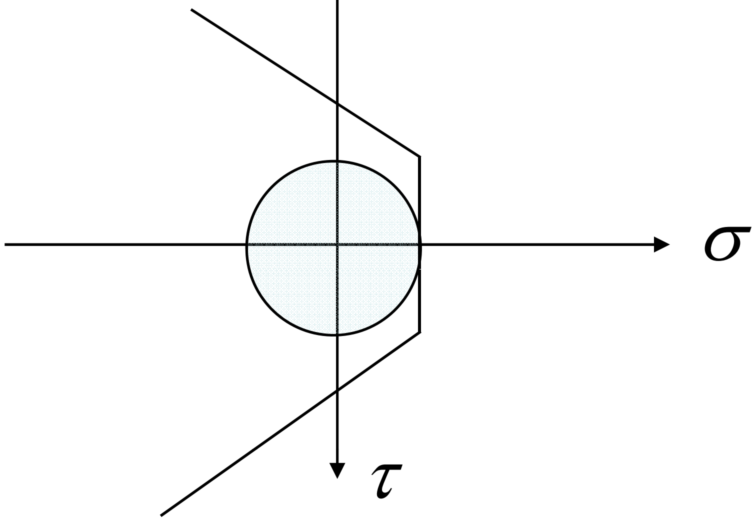


Pure Shear

Failure Modes in Pure Shear

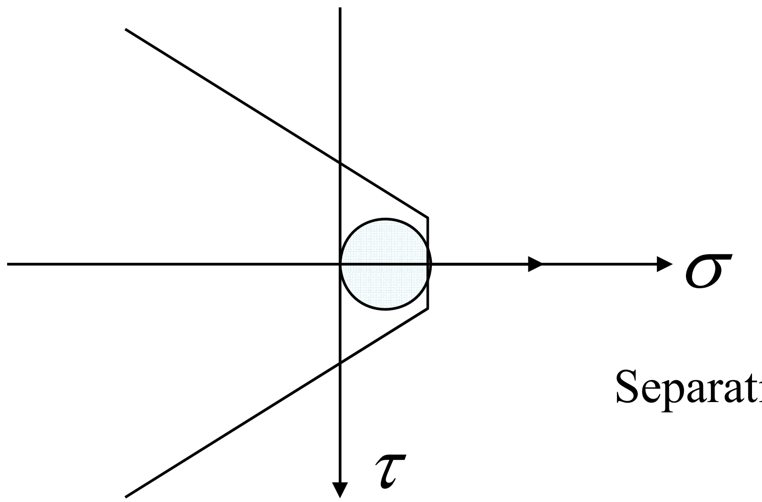


Sliding failure

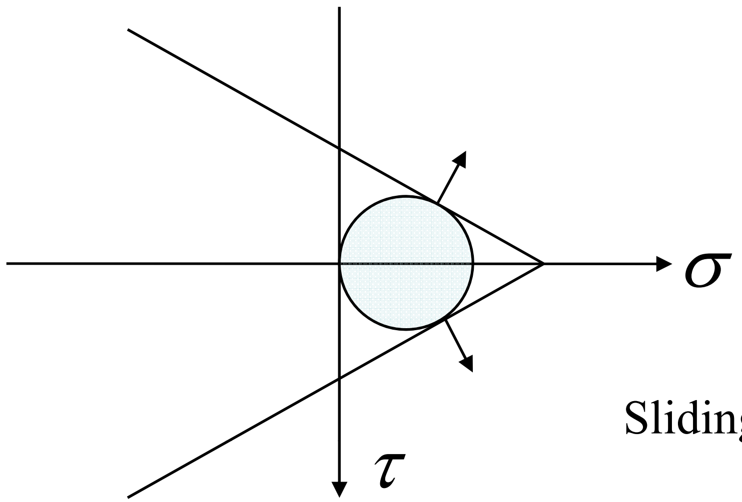


Separation failure

Failure Modes in Pure Tension

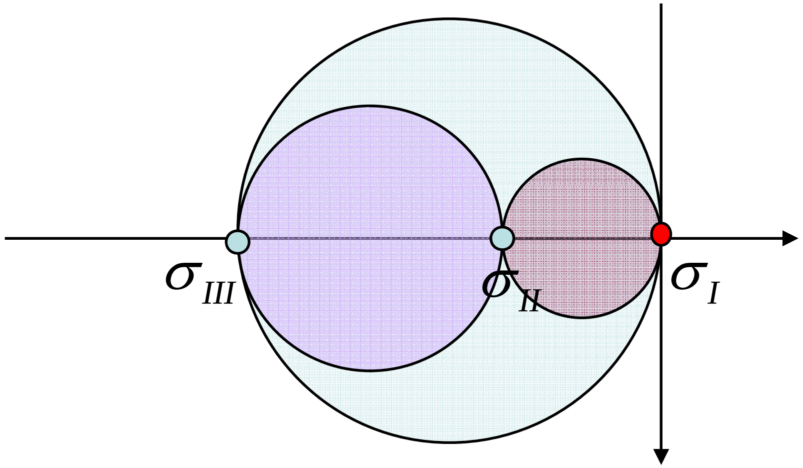


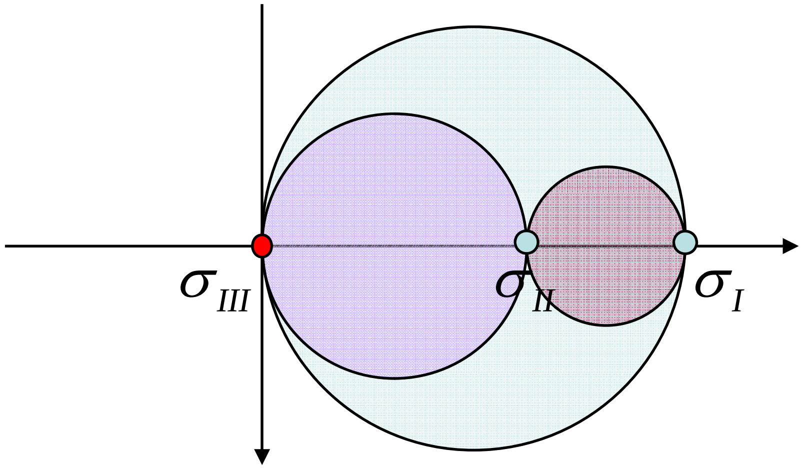
Separation failure

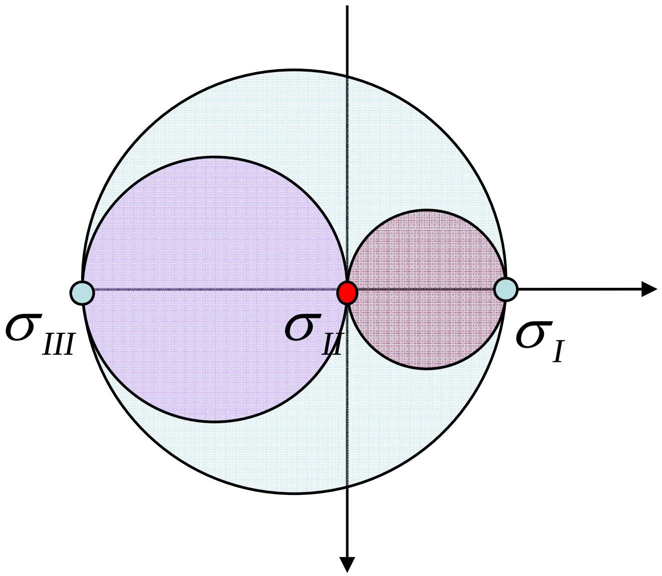


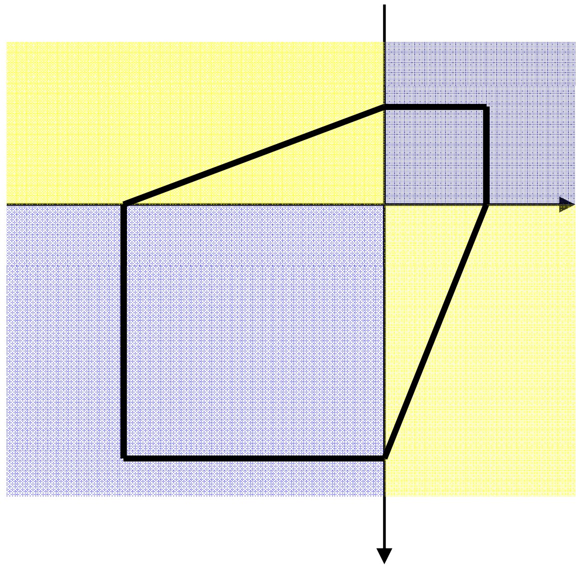
Sliding failure

Yield Condition in Plane Stress

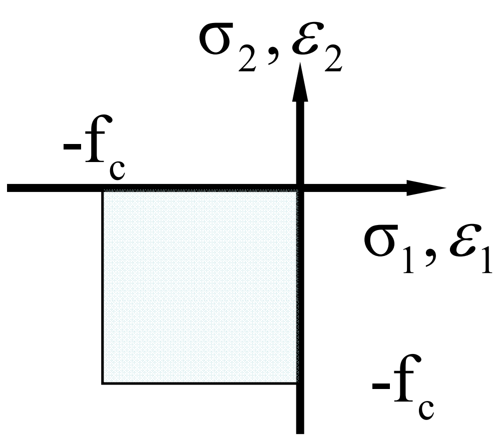


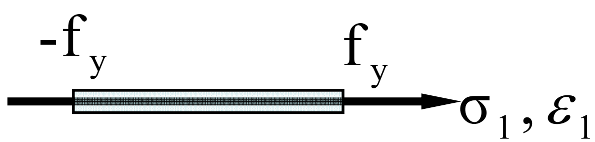


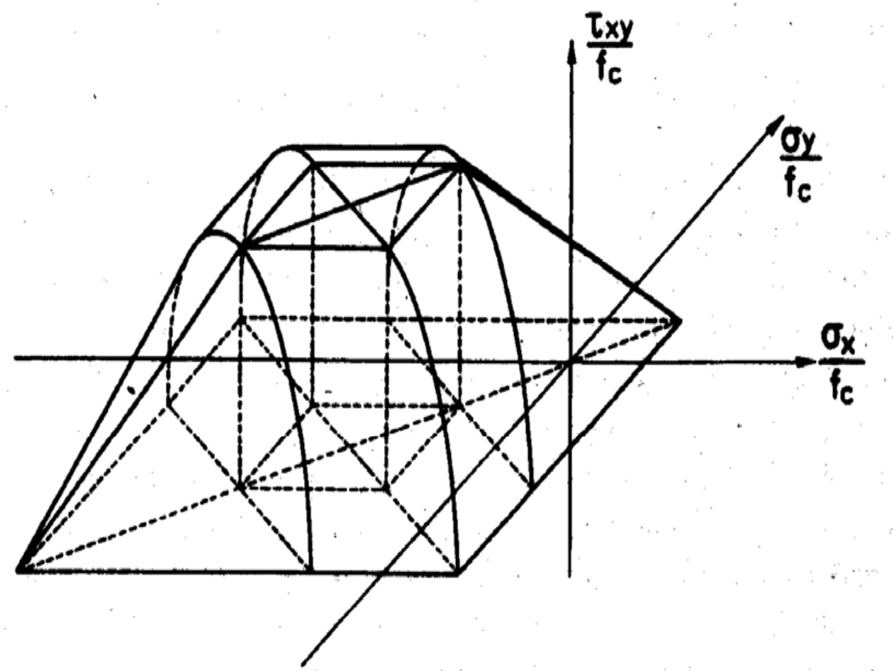




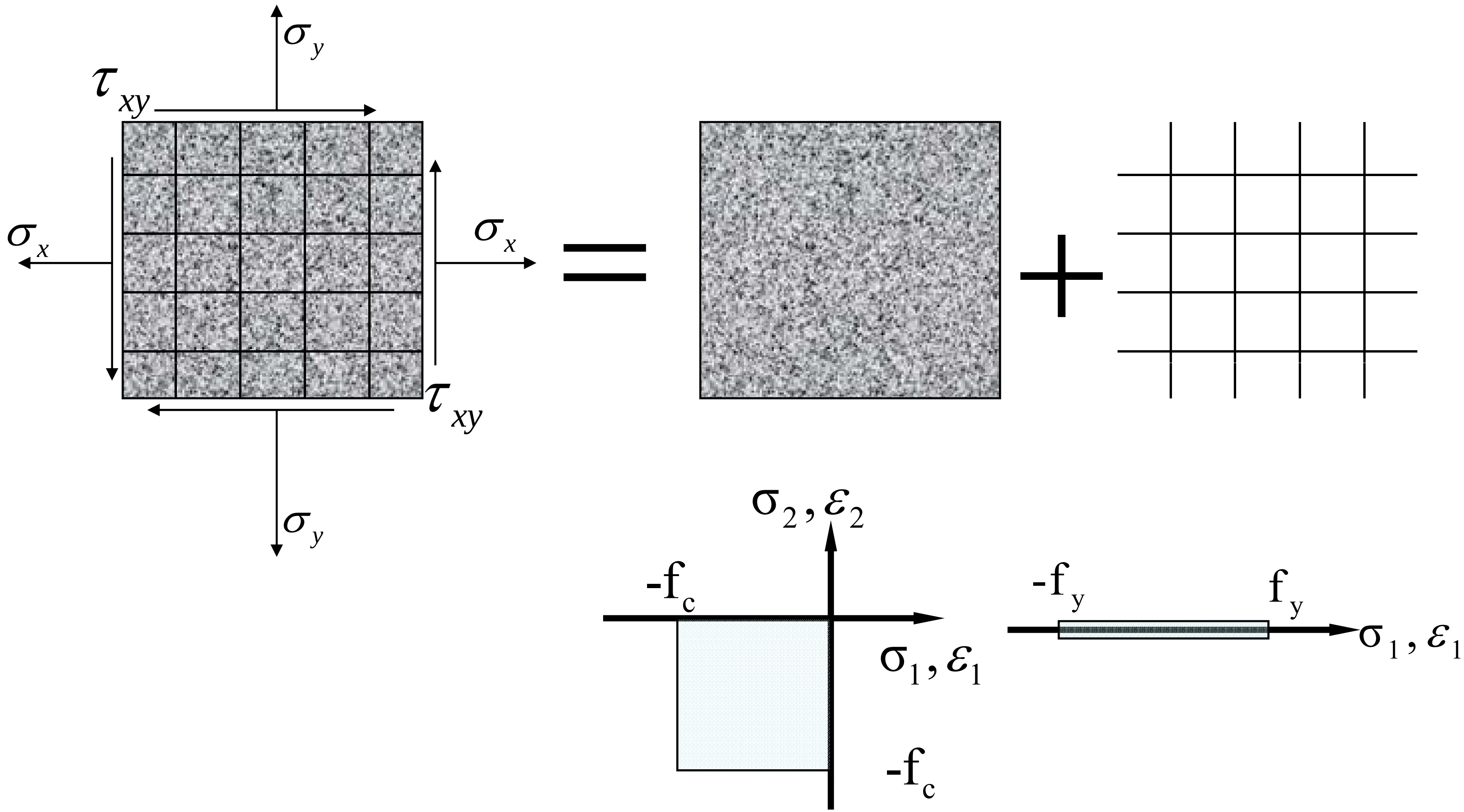
Yield condition for Disk



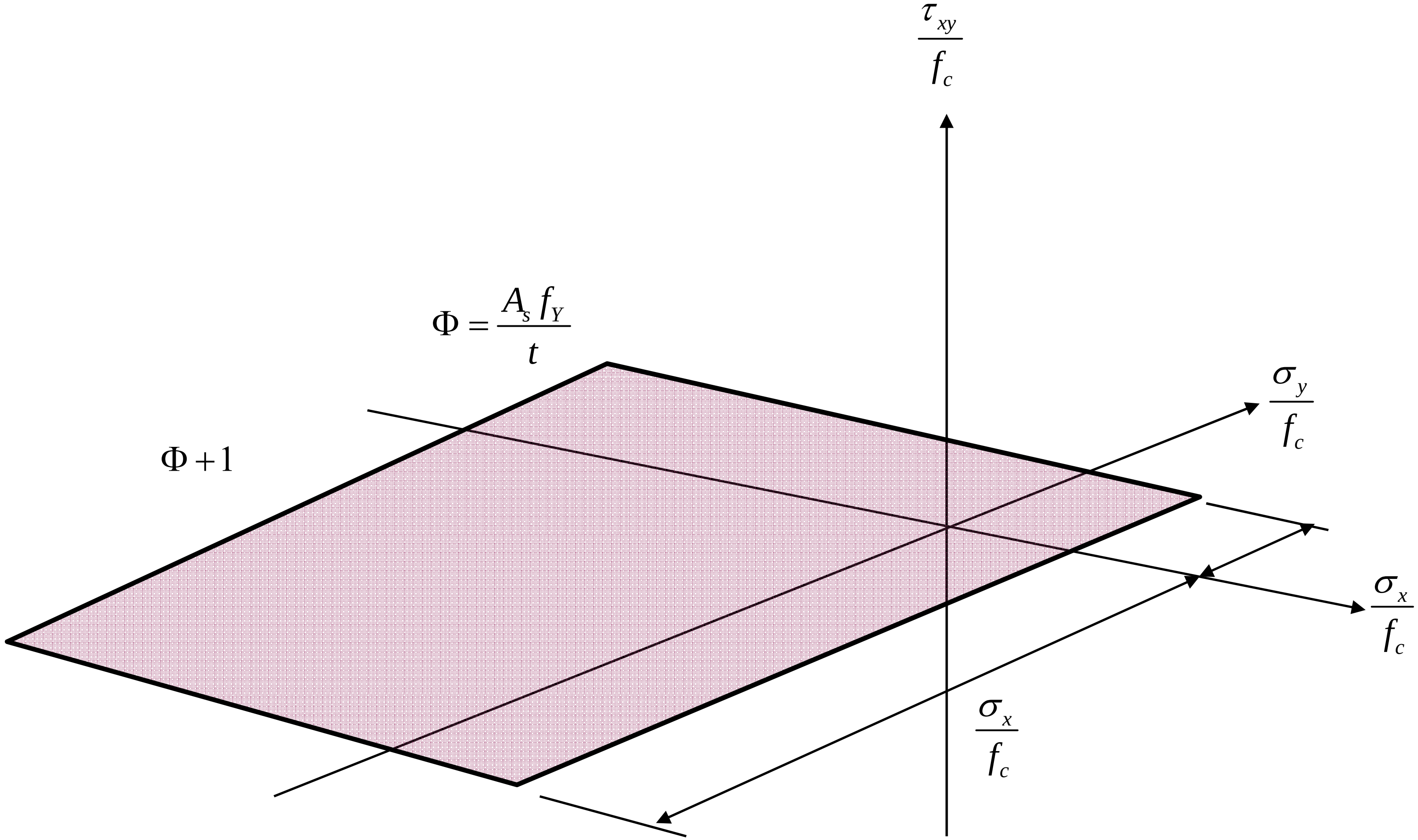




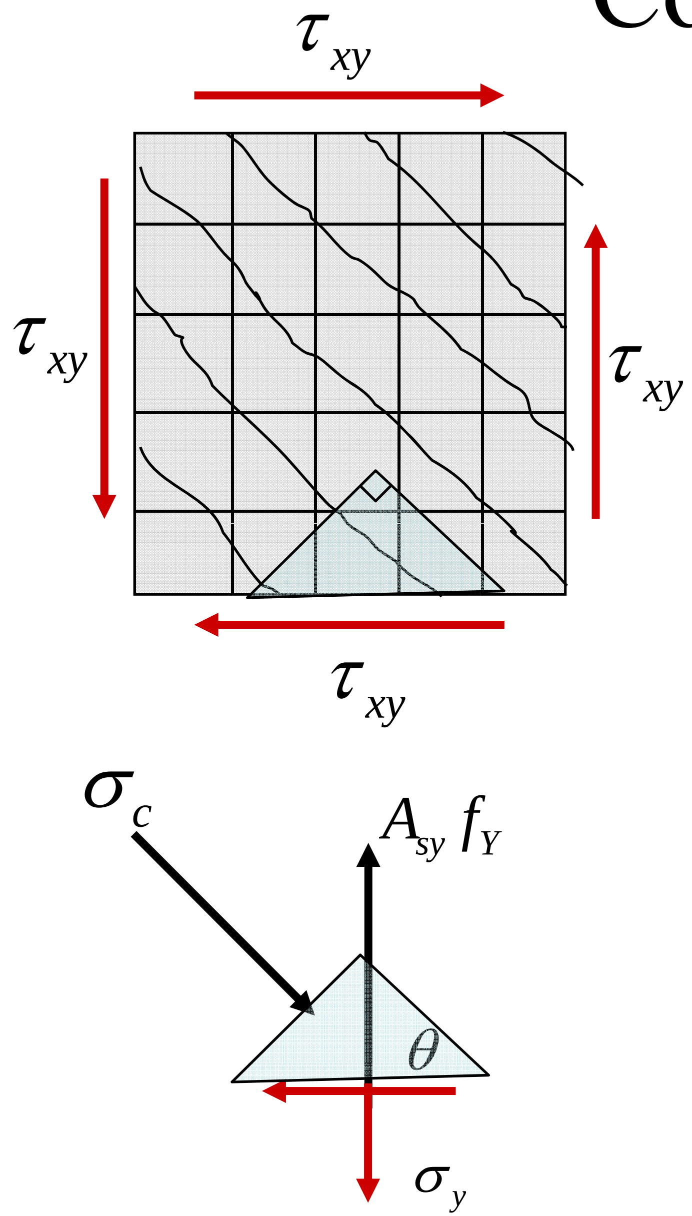
Disks in Plane Stresses

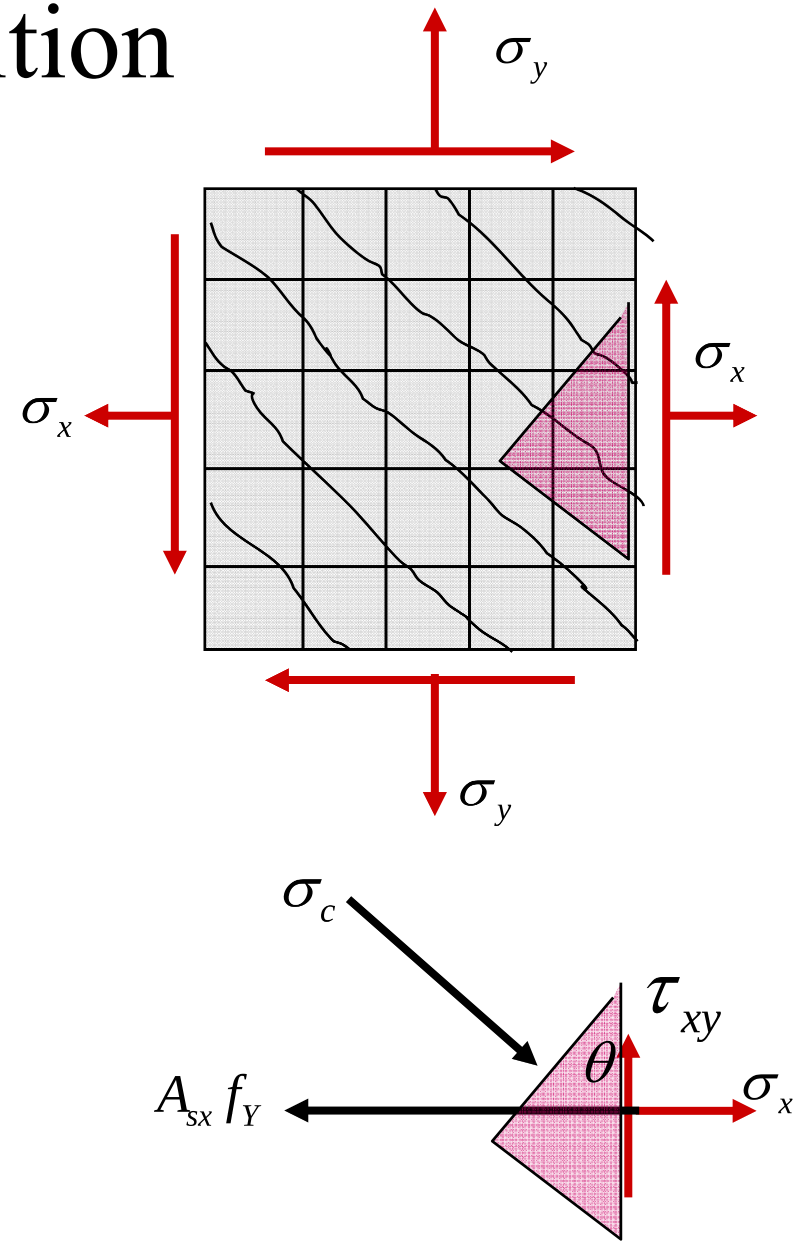


Yield Condition for Disk

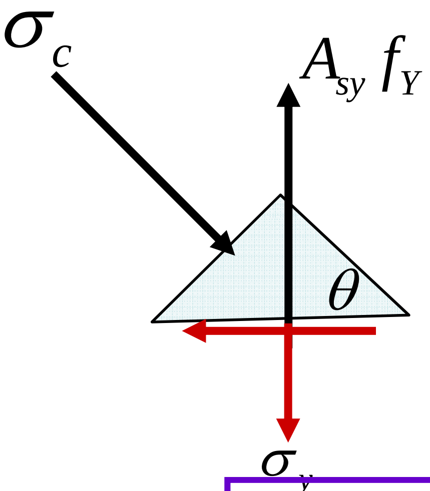


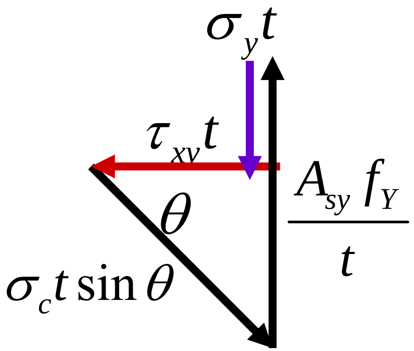
Free Body Diagram for Yield Condition





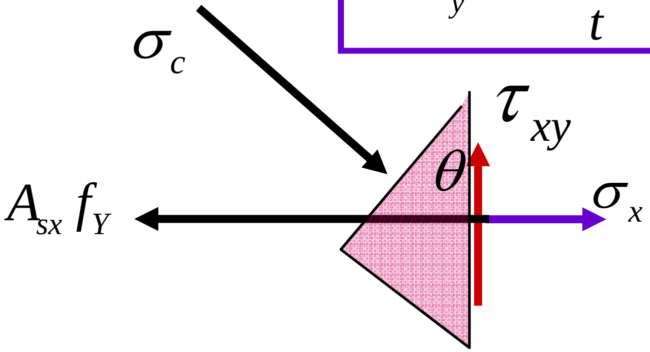
Shear strength

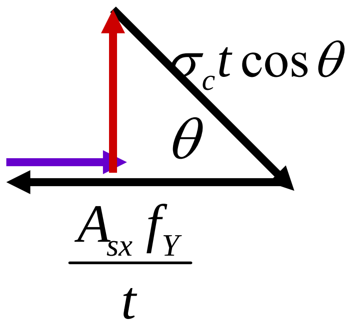




$$-\left(\Phi f_c - \sigma_x\right)\left(\Phi f_c - \sigma_y\right) + \tau_{xy}^2 = 0$$

$$\sigma_y t = \frac{A_{sy}\, f_Y}{t} - \sigma_c t \sin^2\theta$$

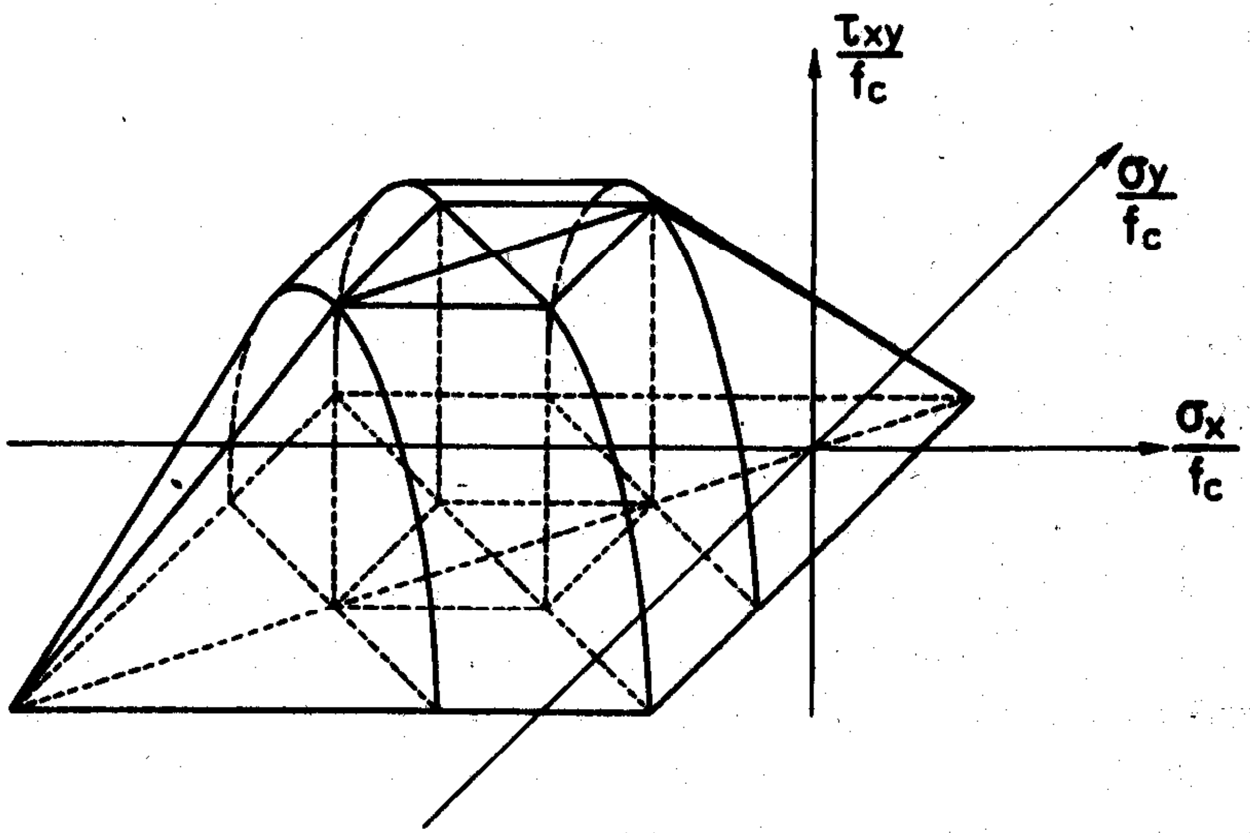




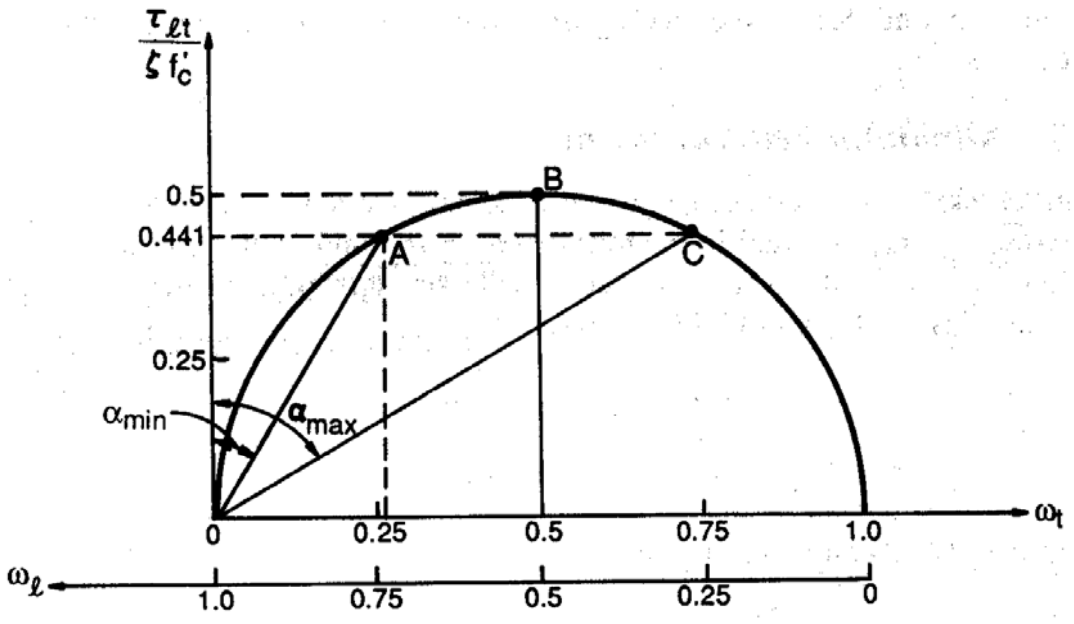
$$\tau_{xy} t = \sigma_c t \sin\theta \cos\theta$$

$$\sigma_x t = \frac{A_{sx}\, f_Y}{t} - \sigma_c t \cos^2\theta$$

Shear Strength of Disk



Relationship between Shear Strength and Reinforcement Ratio in Disk



NOTE: CIRCULAR CURVE SATISFIES $\omega_\ell + \omega_t = 1$;
α_{min} AND α_{max} ACCORDING TO CEB CODE

Reinforcement in Disk based on Plasticity Solution

$$\frac{A_{sx} f_Y}{t} = \sigma_x + |\tau_{xy}|$$

$$\frac{A_{sy} f_Y}{t} = \sigma_y + |\tau_{xy}|$$

$$\sigma_x \le \sigma_y$$

$$\sigma_x \ge -|\tau_{xy}|$$

$$\sigma_x \sigma_y \le \tau_{xy}^2$$

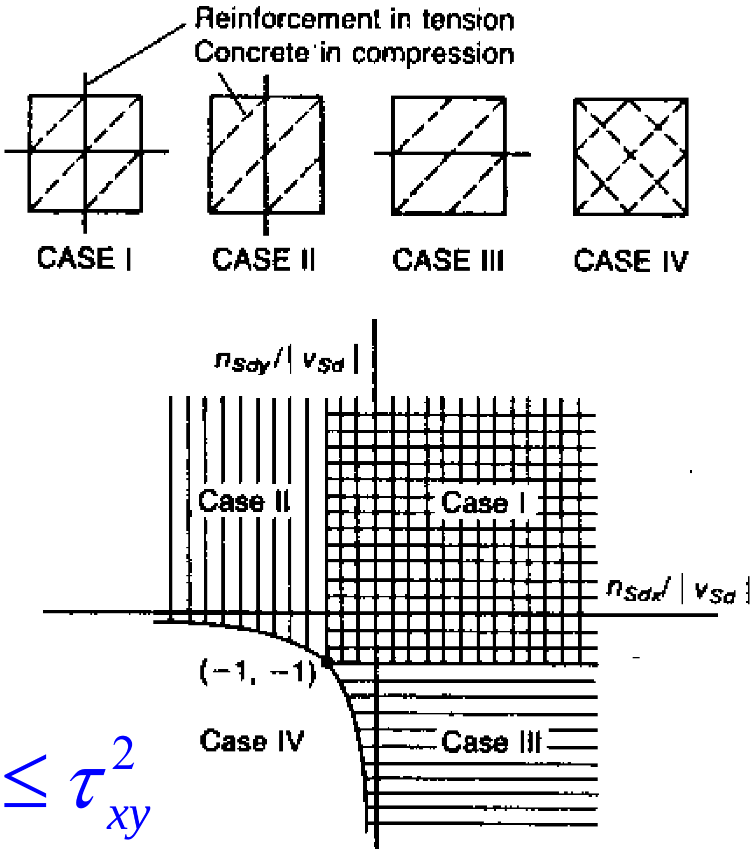
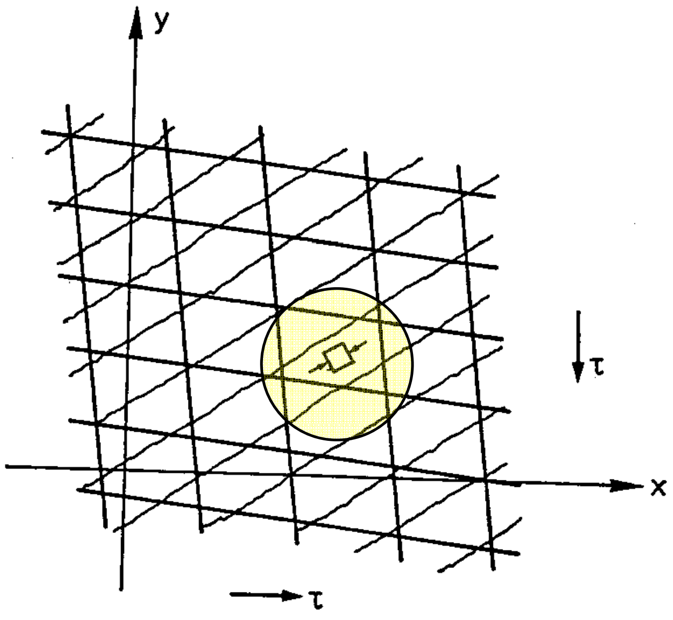
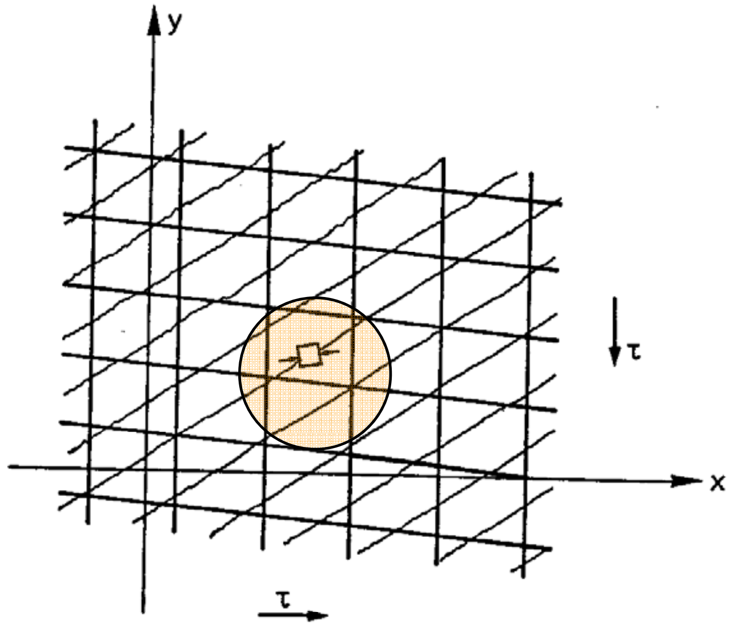


Fig. 6.5.4. Systems of resistance

Failure Mechanisms in Disks



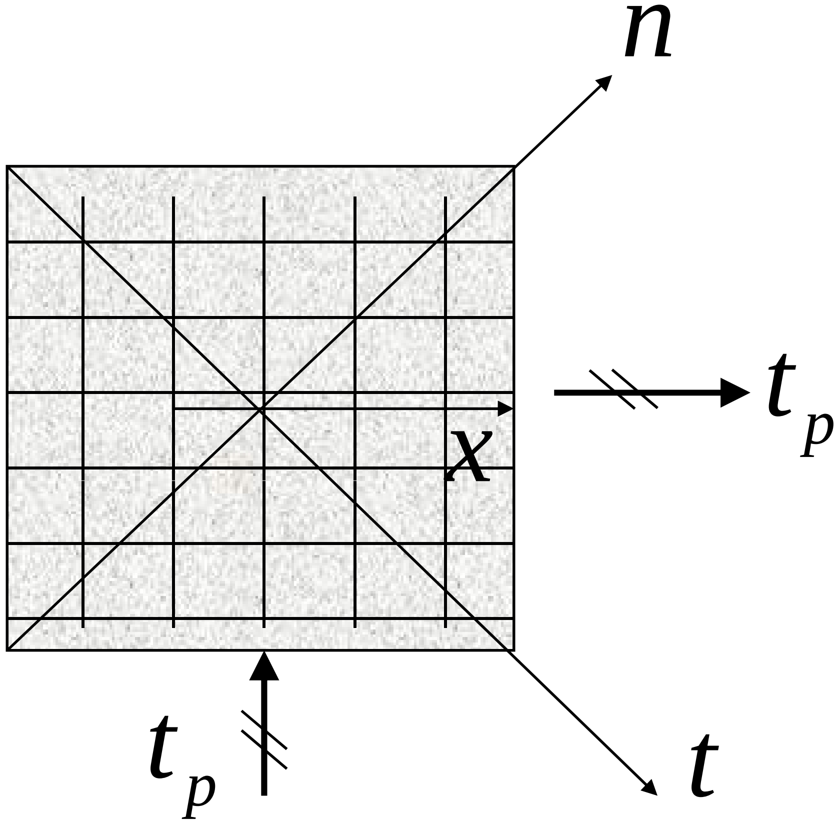
Failure mechanism for underreinforced element.



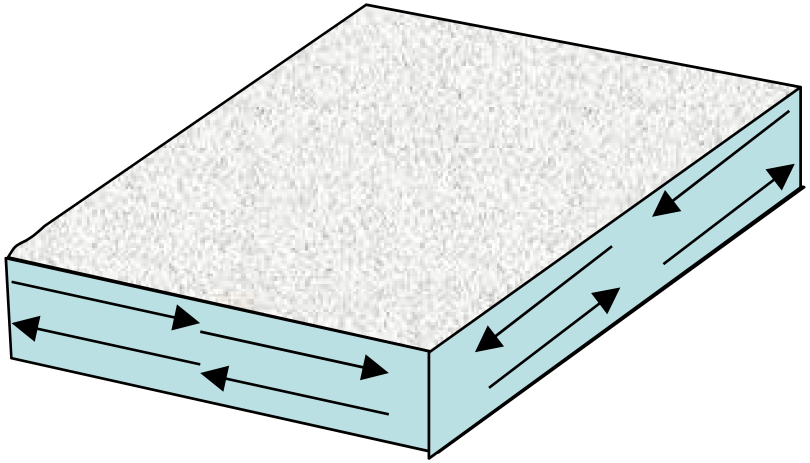
Failure mechanism for element overreinforced in the x-direction.

Yield conditions for Slabs

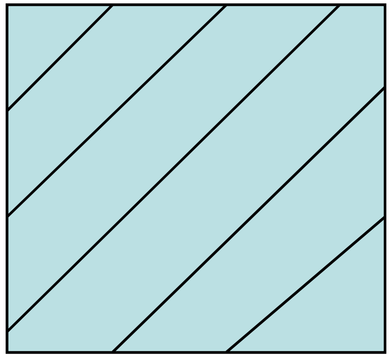
- Pure bending -> yield line theory and strip method
- Pure torsion -> Sandwich model



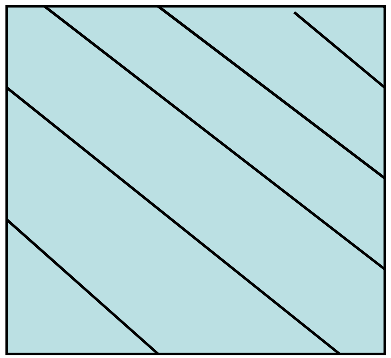
Torsion in slabs



$$t_p = \left(1 - 2\Phi_o\right) A_s f_Y h$$



Top panel



bottom panel

The Theory of Plain Concrete

- Constitutive Equations
- Dissipation Works
- Lines of Discontinuity
- Stress Fields
- Strain Fields
- Applications
- Concentrated Loadings

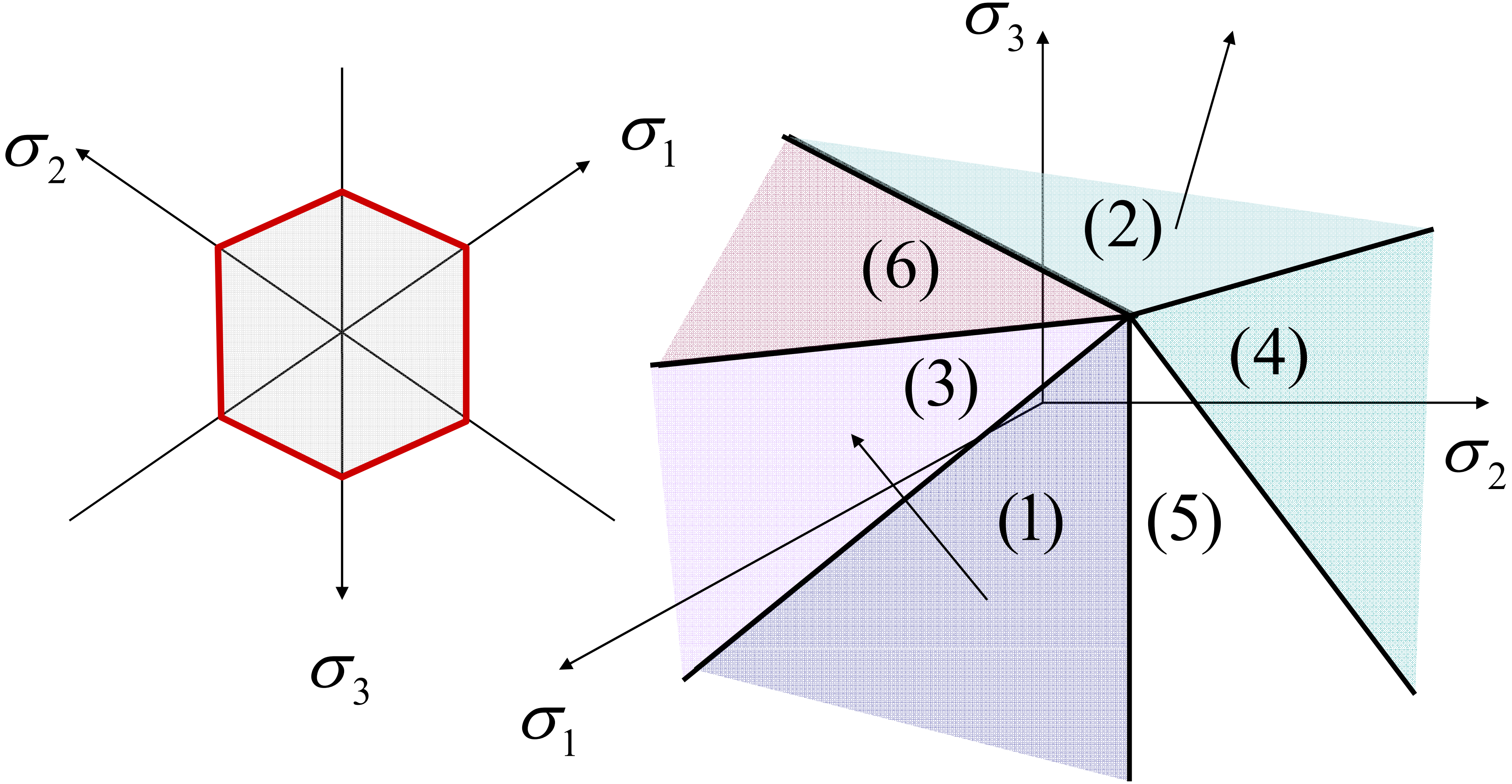
Constitutive Equations

- Coulomb Material
- Yield Condition ***6 Surfaces***
- Strain Vectors

$$k\sigma_1 - \sigma_3 = f_c, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$
$$k\sigma_3 - \sigma_1 = f_c, \quad \sigma_3 \geq \sigma_2 \geq \sigma_1$$
$$k\sigma_1 - \sigma_2 = f_c, \quad \sigma_1 \geq \sigma_3 \geq \sigma_2$$
$$k\sigma_2 - \sigma_1 = f_c, \quad \sigma_2 \geq \sigma_3 \geq \sigma_1$$
$$k\sigma_2 - \sigma_3 = f_c, \quad \sigma_2 \geq \sigma_1 \geq \sigma_3$$
$$k\sigma_3 - \sigma_2 = f_c, \quad \sigma_3 \geq \sigma_1 \geq \sigma_2$$

$$q_i = \lambda \frac{\partial f}{\partial Q_i}$$

Yield Surface in 3-D



Plastic Strain Vector

- Along planes

$$k\sigma_1 - \sigma_3 = f_c, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$q_i = \lambda \frac{\partial f}{\partial Q_i}$$

	ε_1	ε_2	ε_3
1	λk	0	$-\lambda$
2	$-\lambda$	0	λk
3	λk	$-\lambda$	0
4	$-\lambda$	λk	0
5	0	λk	$-\lambda$
6	0	$-\lambda$	λk

	ε_1	ε_2	ε_3
1	λk	0	$-\lambda$
2	$-\lambda$	0	λk
3	λk	$-\lambda$	0
4	$-\lambda$	λk	0
5	0	λk	$-\lambda$
6	0	$-\lambda$	λk

$$\frac{\sum \varepsilon^+}{\sum \left|\varepsilon^-\right|} = k$$

- Along edges

	ε_1	ε_2	ε_3
1/5	$\lambda_1 k$	$\lambda_2 k$	$-(\lambda_1+\lambda_2)$
4/5	$-\lambda_1$	$(\lambda_1+\lambda_2)k$	$-\lambda_2$
4/2	$-(\lambda_1+\lambda_2)$	$\lambda_2 k$	$\lambda_1 k$
2/6	$-\lambda_1$	$-\lambda_2$	$(\lambda_1+\lambda_2)k$
3/6	$\lambda_1 k$	$-(\lambda_1+\lambda_2)$	$\lambda_2 k$
1/3	$(\lambda_1+\lambda_2)k$	$-\lambda_2$	$-\lambda_1$

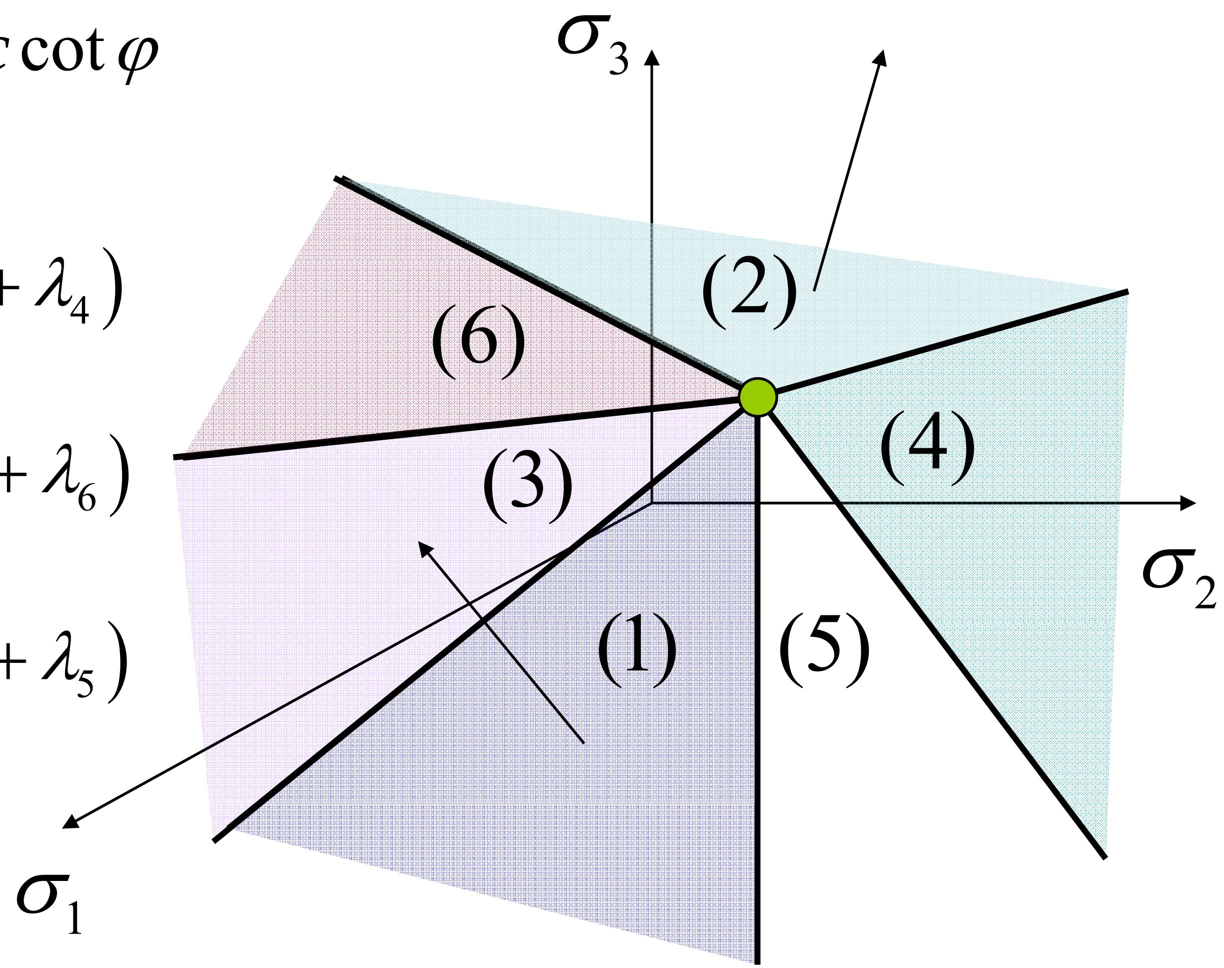
Plastic Strain Vector at Apex

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{f_c}{k-1} = c\cot\varphi$$

$$\varepsilon_1 = (\lambda_1 + \lambda_3)k - (\lambda_2 + \lambda_4)$$

$$\varepsilon_2 = (\lambda_4 + \lambda_5)k - (\lambda_3 + \lambda_6)$$

$$\varepsilon_3 = (\lambda_2 + \lambda_6)k - (\lambda_1 + \lambda_5)$$



Dissipation Work

Along plane 1

$$W = \sigma_1\varepsilon_1 + \sigma_1\varepsilon_1 + \sigma_1\varepsilon_1 = \sigma_1\lambda k - \sigma_3\lambda = \lambda f_c$$

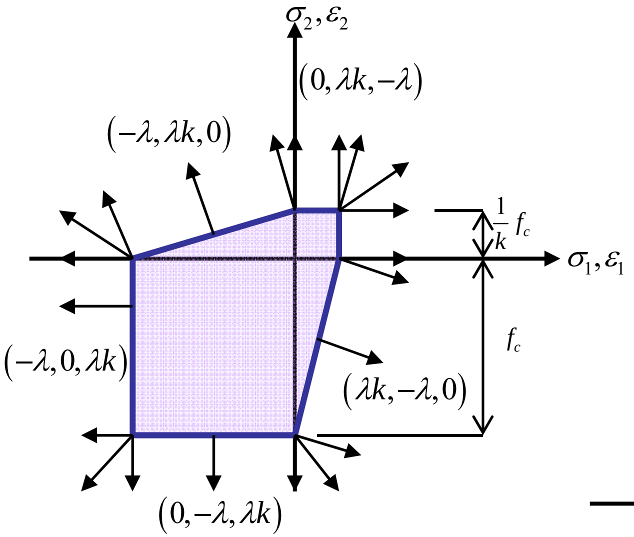
Along the edge 1/5

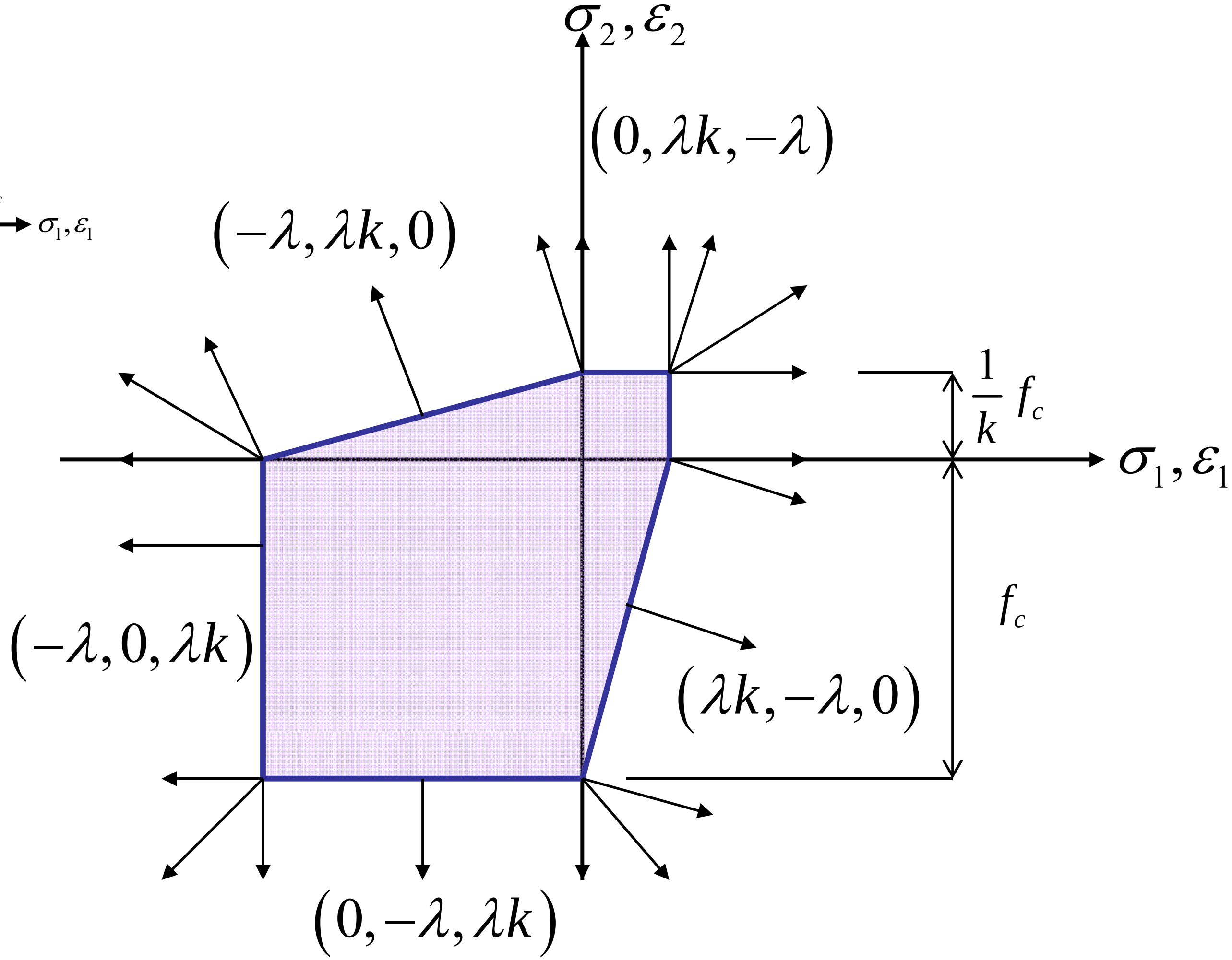
$$W = \left(\lambda_1 + \lambda_2\right) f_c$$

At the apex

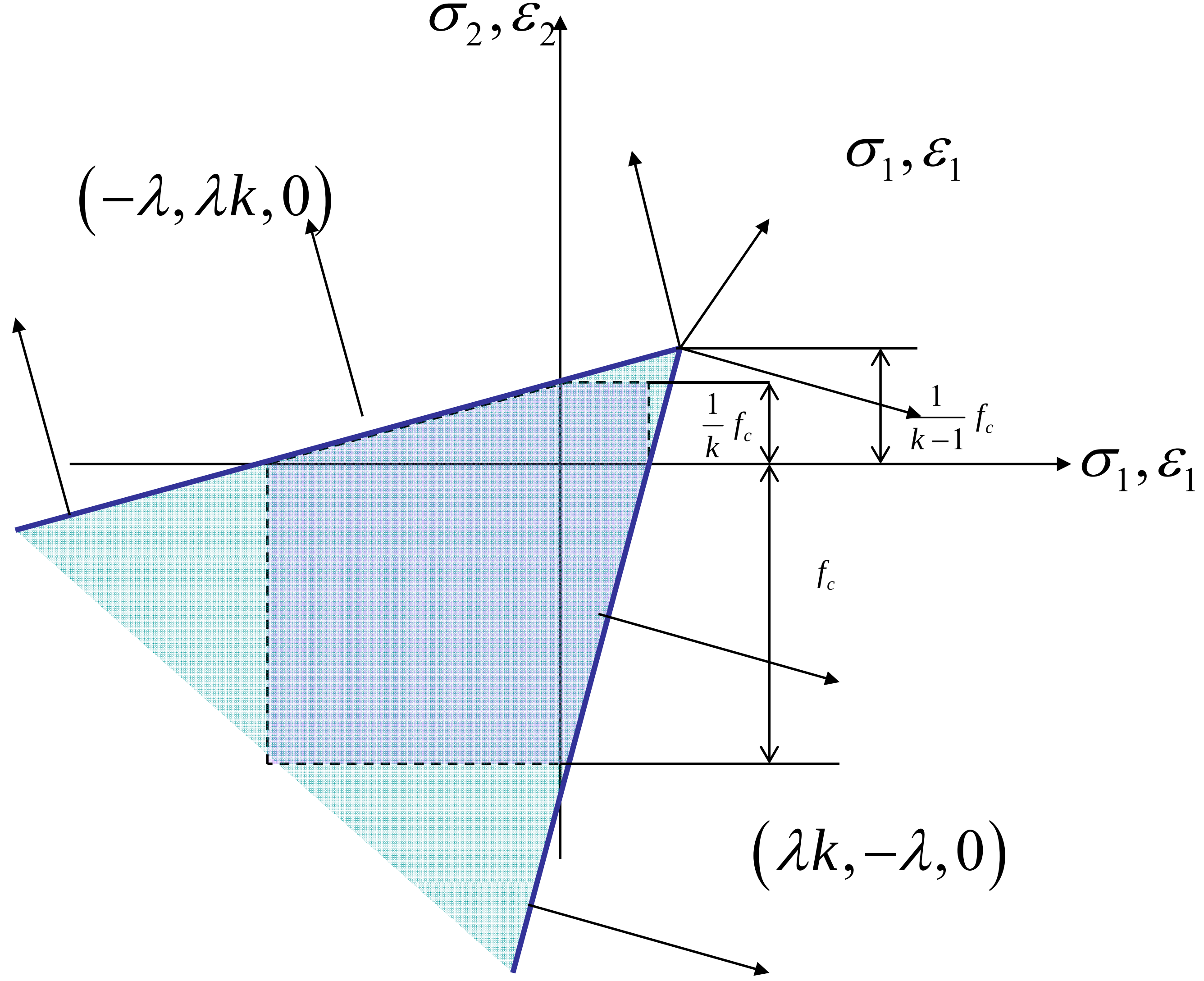
$$W = \left(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6\right) f_c$$

Yield Condition and Flow Rule for Coulomb Material in Plane Stress

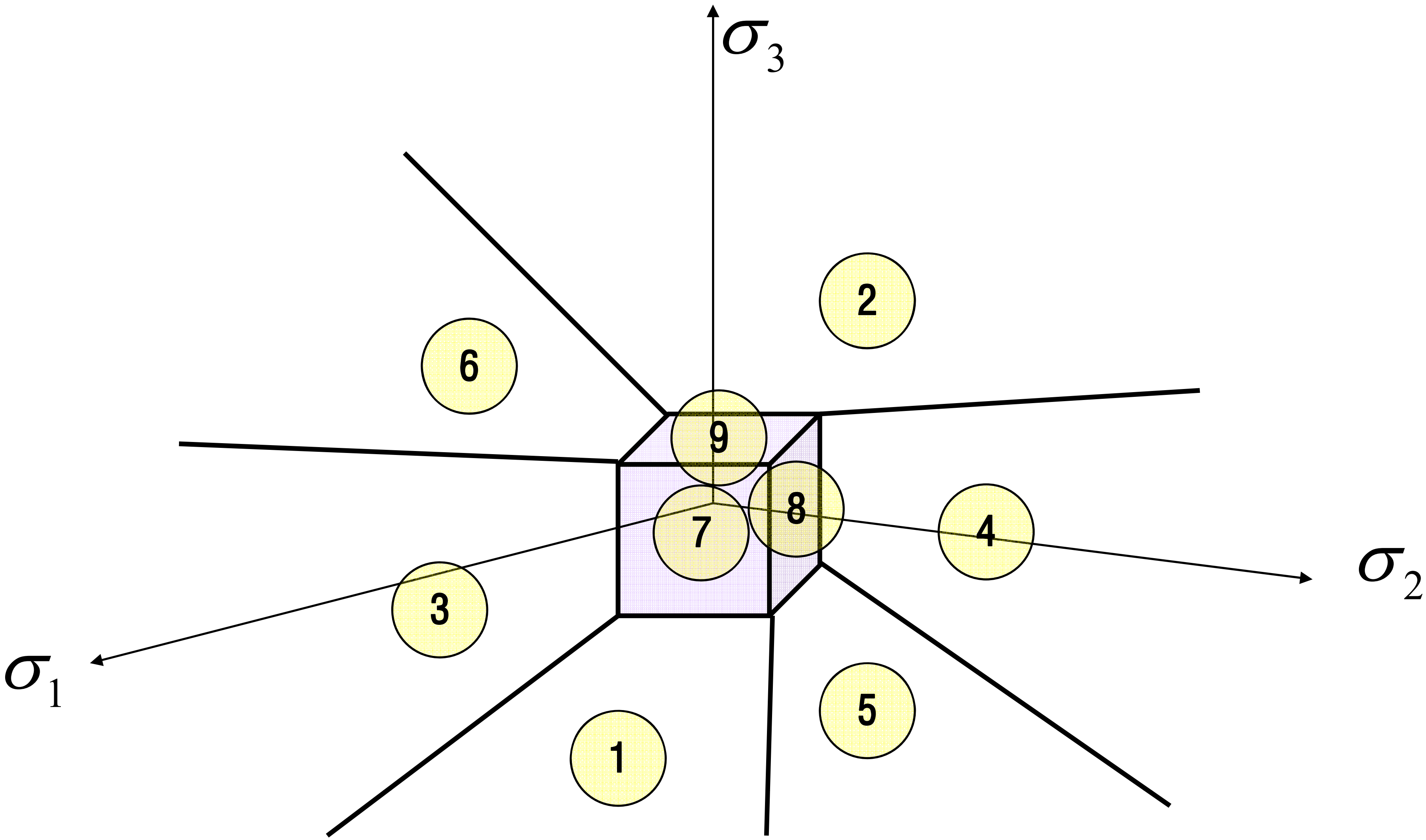




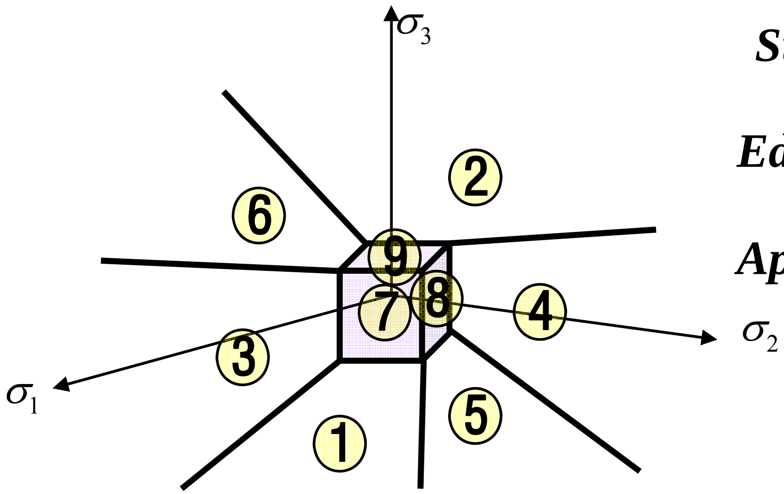
Plane Strain



Modified Coulomb Material



Modified Coulomb Material



Surfaces: **7, 8, 9**

Edges: **1/7, 3/7, 6/9, 2/9, 4/8, 5/8, 8/7, 7/9, 8/9**

Apex: **4/5/8, 7/8/9/1, 1/3/7, 3/7/6/9, 2/6/9,**
8/9/2/4, 7/8/9

$$\frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} > k$$

Dissipation Works

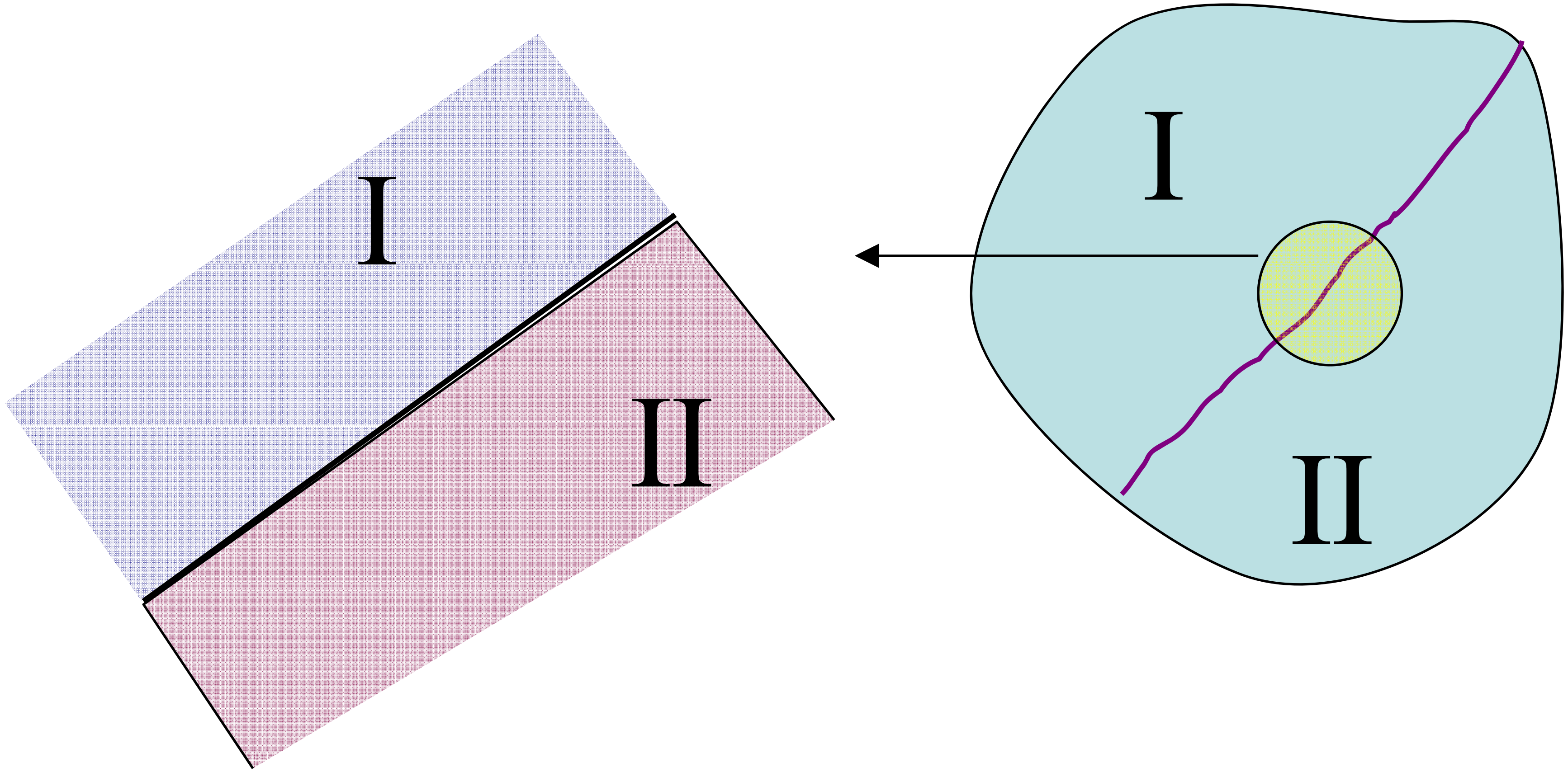
$$When \quad \frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} = k$$

$$W = f_c \sum \left|\varepsilon^{-}\right|$$

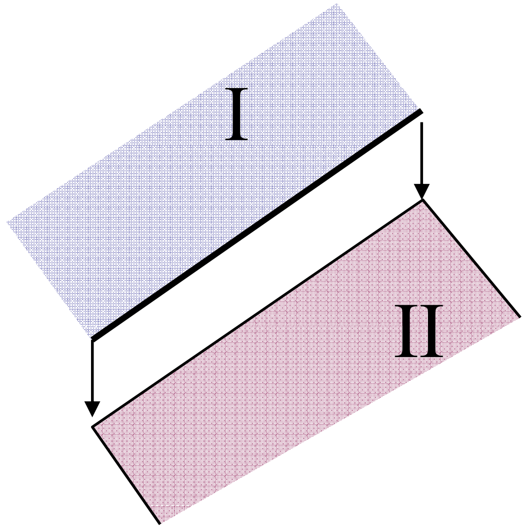
$$When \quad \frac{\sum \varepsilon^{+}}{\sum \left|\varepsilon^{-}\right|} > k$$

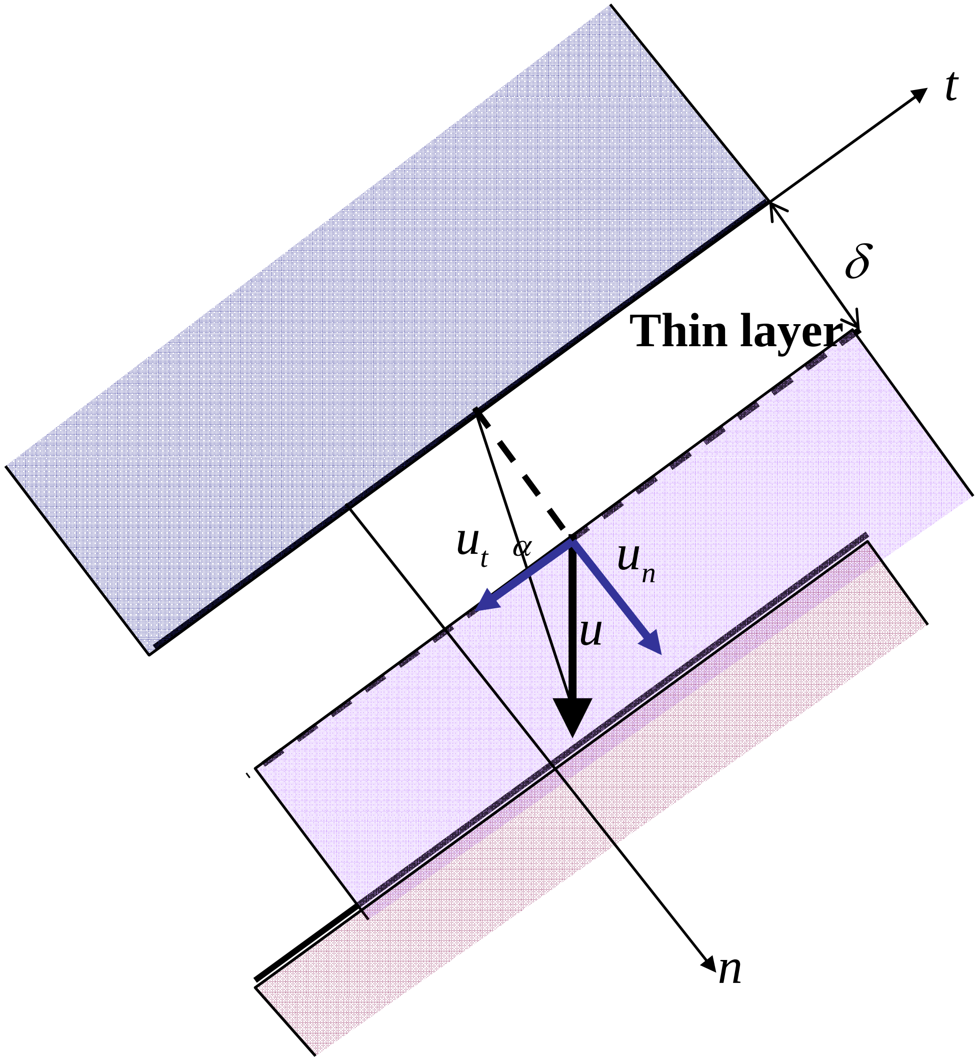
$$W = f_c \sum \left|\varepsilon^{-}\right| + f_t \left(\sum \varepsilon^{+} - k \sum \left|\varepsilon^{-}\right|\right)$$

Planes and Lines of Discontinuity



Cracking

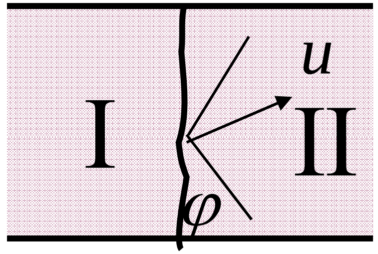


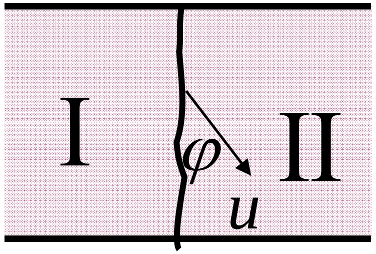


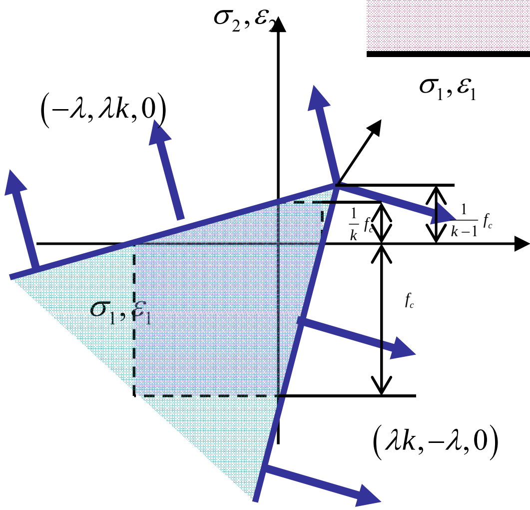
$$\varepsilon_{\max} = \varepsilon^{+} = \frac{1}{2}\frac{u}{\delta}\left(\sin\alpha + 1\right)$$

$$\varepsilon_{\min} = \varepsilon^{-} = \frac{1}{2}\frac{u}{\delta}\left(\sin\alpha - 1\right)$$

Dissipation work : Plane Strain





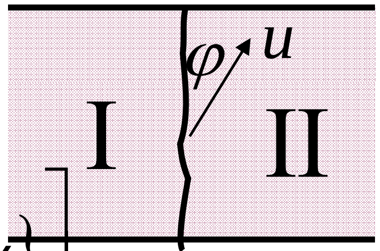


Along surfaces

$$W = f_c \frac{1}{2}\frac{u}{\delta}(1-\sin\varphi)$$

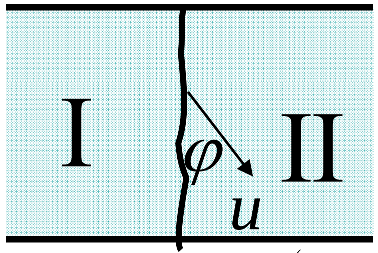
$$W_l = ub\frac{1}{2} f_c (1-\sin\varphi)$$

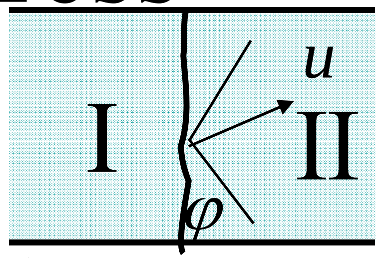
At apex $\varphi \le \alpha \le \pi - \alpha$



$$W_l = ub\frac{1}{2} f_c \left[1-\sin\alpha + \frac{f_t}{f_c}\{-(k-1)+(k+1)\sin\alpha\}\right]$$

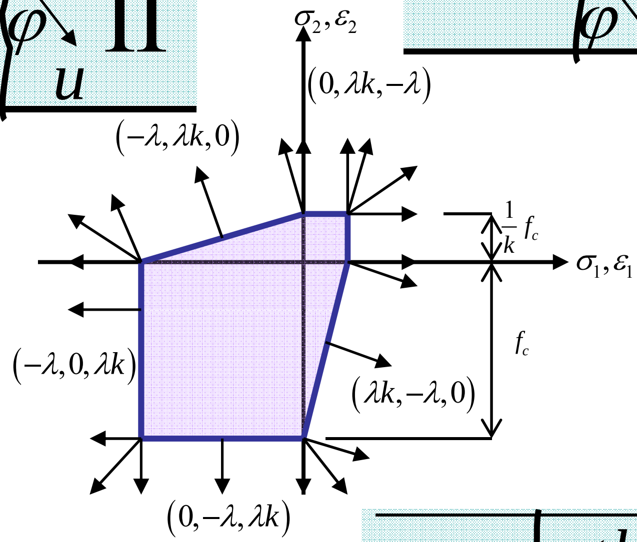
Dissipation work : Plane Stress





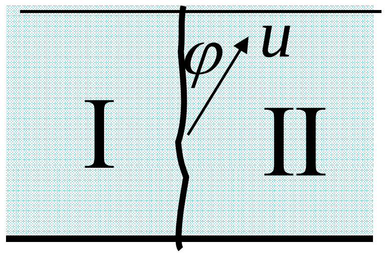
At apex at A $\quad \varphi \le \alpha \le \pi - \alpha$

$$W_l = \frac{1}{2}\frac{f_c}{k}ub\left(1+\sin\alpha\right)$$



Along other apexes and surfaces

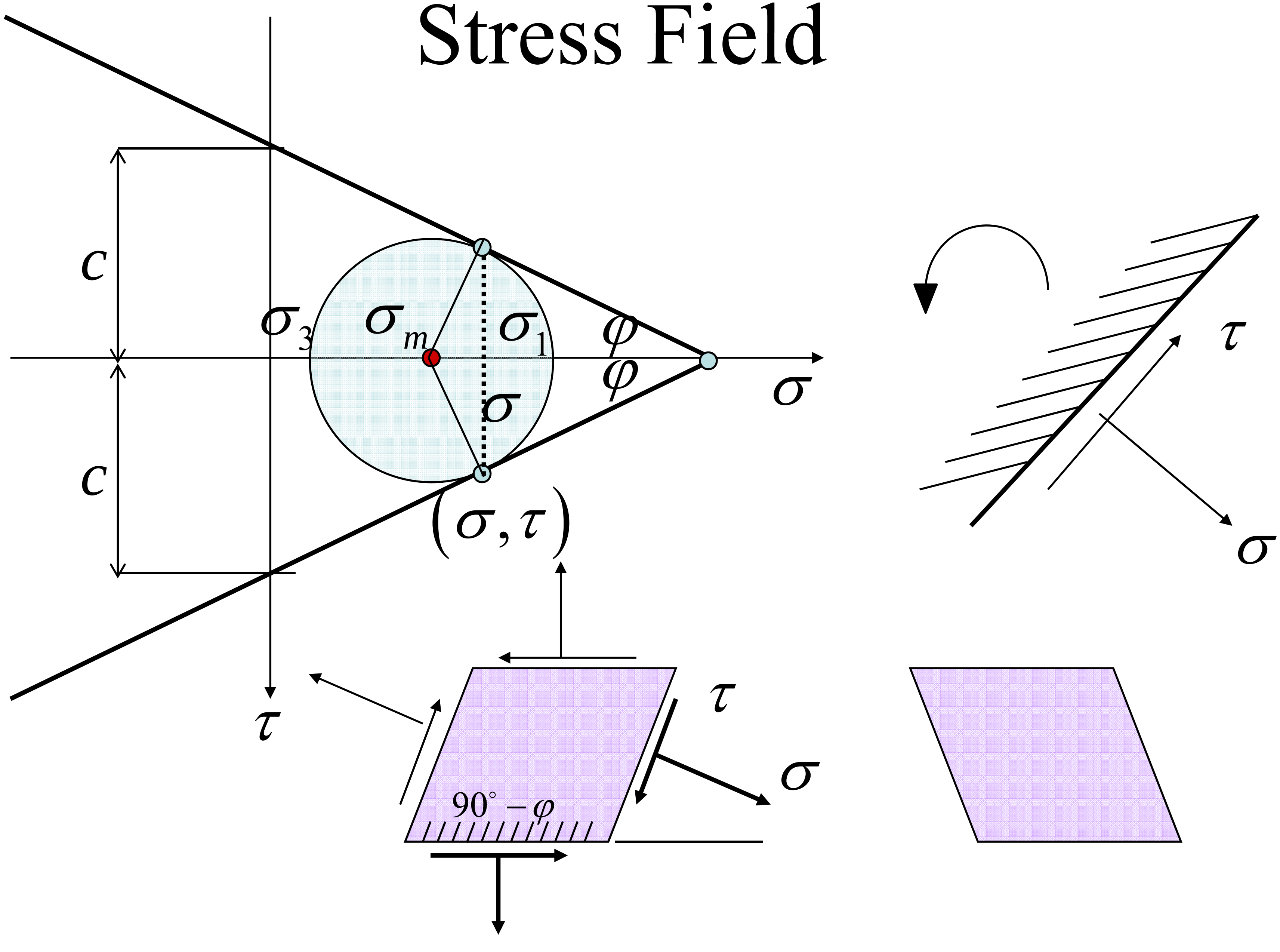
$\alpha \le \varphi \quad$ and $\quad \alpha \ge \pi - \varphi$



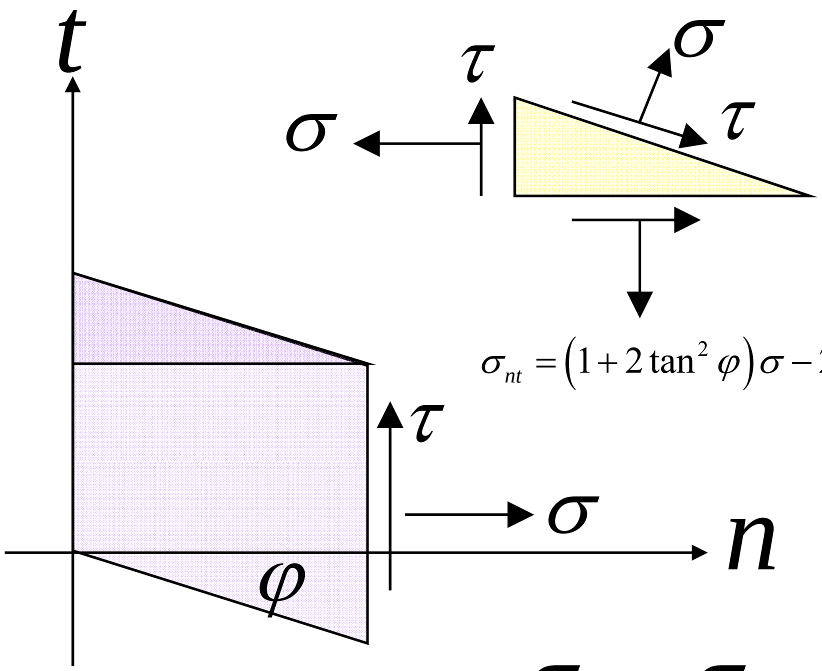
$$W_l = \frac{1}{2} f_c ub\left(1-\sin\alpha\right)$$

Coulomb Material

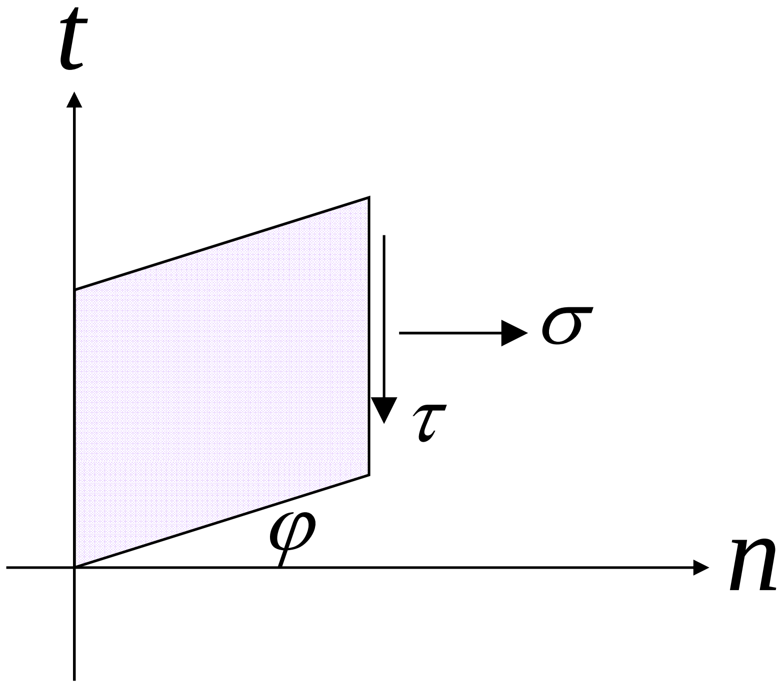
Stress Field



Stress on Failure section



$$\sigma_{nt} = \left(1 + 2\tan^2\varphi\right)\sigma - 2c\tan\varphi$$



$$\sigma_n = \sigma$$

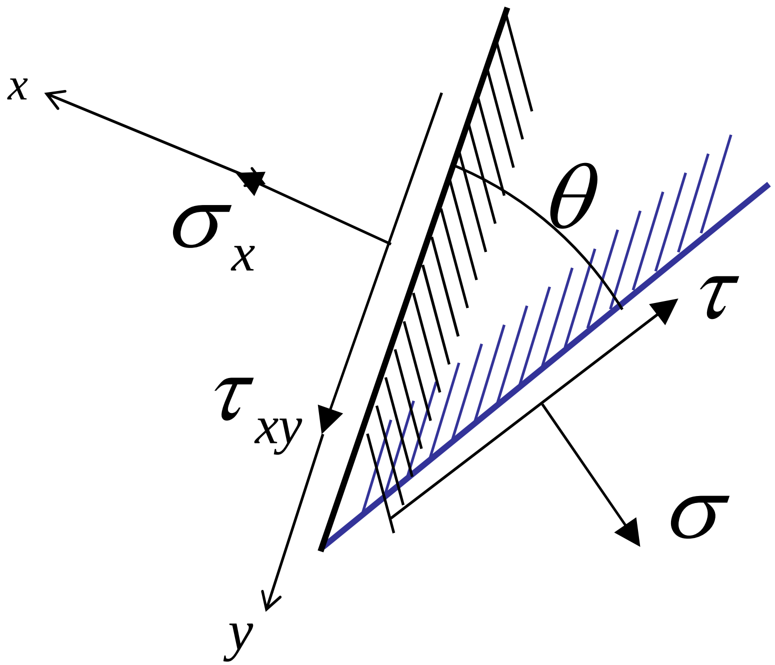
$$\tau_{nt} = \tau = c - \sigma\tan\varphi$$

Stresses from Failure Section

$$\sigma_x = c\cot\varphi + \left(\sigma_m - c\cot\varphi\right)\left[1 - \sin\varphi\sin\left(2\theta + \varphi\right)\right]$$

$$\sigma_y = c\cot\varphi + \left(\sigma_m - c\cot\varphi\right)\left[1 + \sin\varphi\sin\left(2\theta + \varphi\right)\right]$$

$$\tau_{xy} = -\left(\sigma_m - c\cot\varphi\right)\sin\varphi\cos\left(2\theta + \varphi\right)$$



Failure section from a given stresses

For given σ_x and τ_{xy}

$$\tan\beta = -\frac{\tau_{xy}}{\sigma_x - c\cot\varphi}$$

Angle θ is determined.

$$\cos\left(2\theta+\varphi-\beta\right)=\frac{\sin\beta}{\sin\varphi}$$

$$\sigma_m = c\cot\varphi + \frac{\sigma_x - c\cot\varphi}{1-\sin\varphi\sin\left(2\theta+\varphi\right)}$$

Stress σ and τ are calculated.

$$\sigma = \left(\sigma_m + c\cot\varphi\right)\cos^2\varphi$$

Equilibrium Equations

Rankine Zone

$$\frac{\partial \sigma}{\partial x} \pm \frac{\partial\left(c-\sigma \tan \varphi\right)}{\partial y}=0$$

$$\frac{\partial\left[\left(1+2 \tan^2 \varphi\right)\sigma-2c \tan \varphi\right]}{\partial y} \pm \frac{\partial\left(c-\sigma \tan \varphi\right)}{\partial x}=0$$

$$\Rightarrow \quad \frac{\partial \sigma}{\partial x}=\frac{\partial \sigma}{\partial y}=0$$

Prandtl Zone

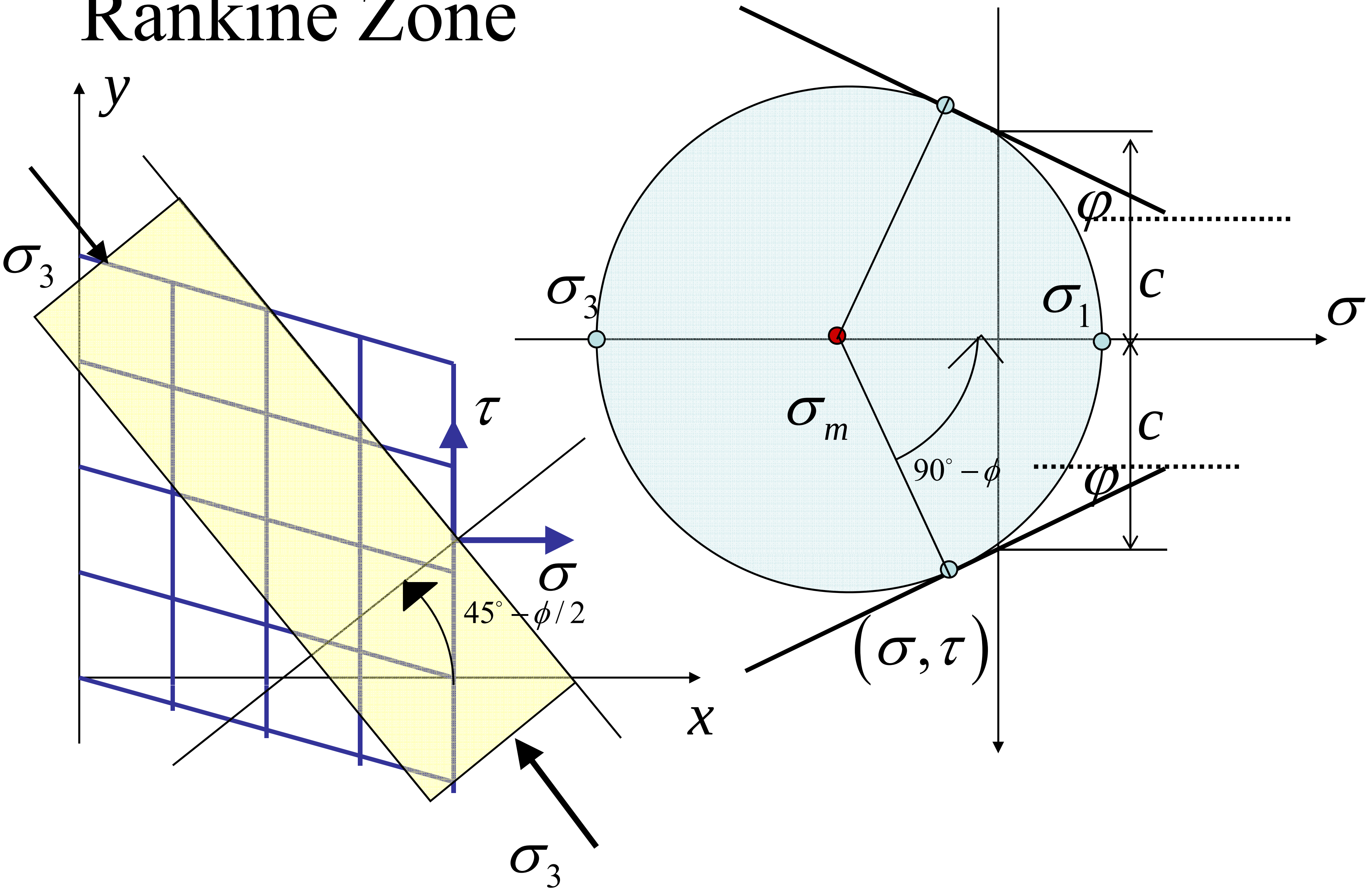
$$\frac{\partial \sigma}{\partial r}=0$$

$$\frac{\partial \sigma}{\partial \theta} \pm 2\left(c-\sigma \tan \varphi\right)=0$$

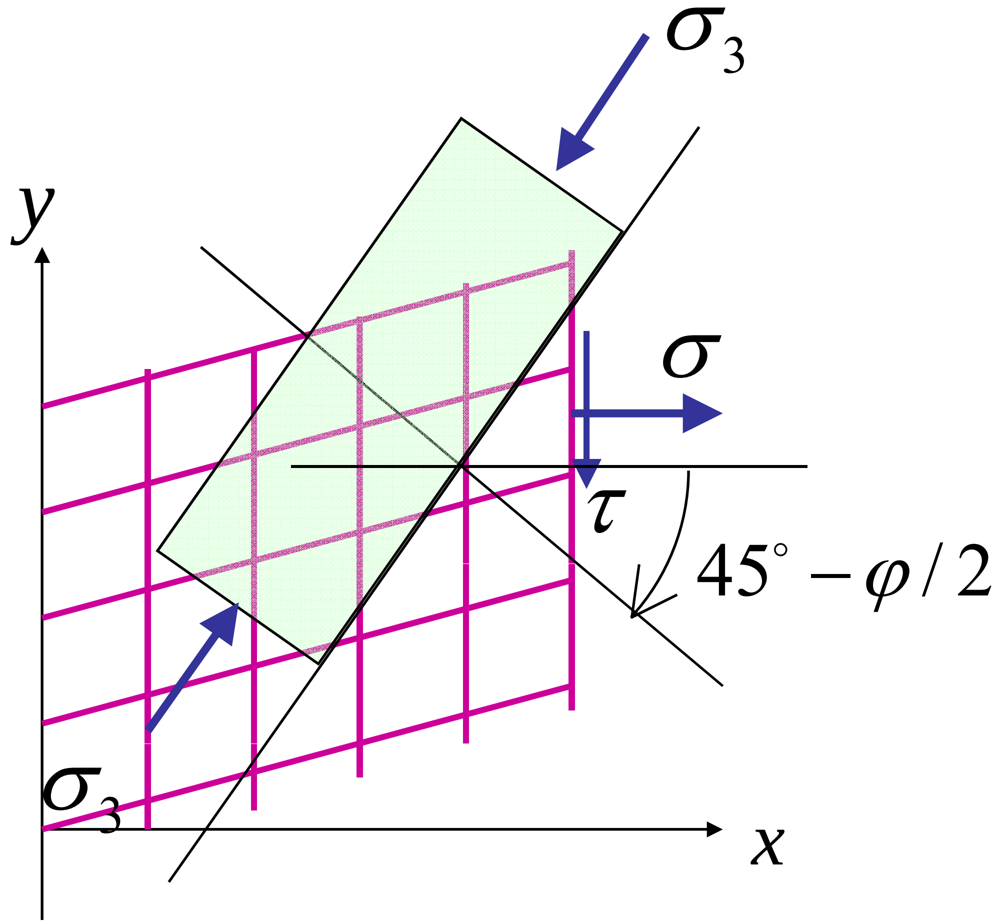
$$\Rightarrow \quad \sigma=-Ce^{\pm 2\theta \tan \varphi}+c \cot \varphi$$

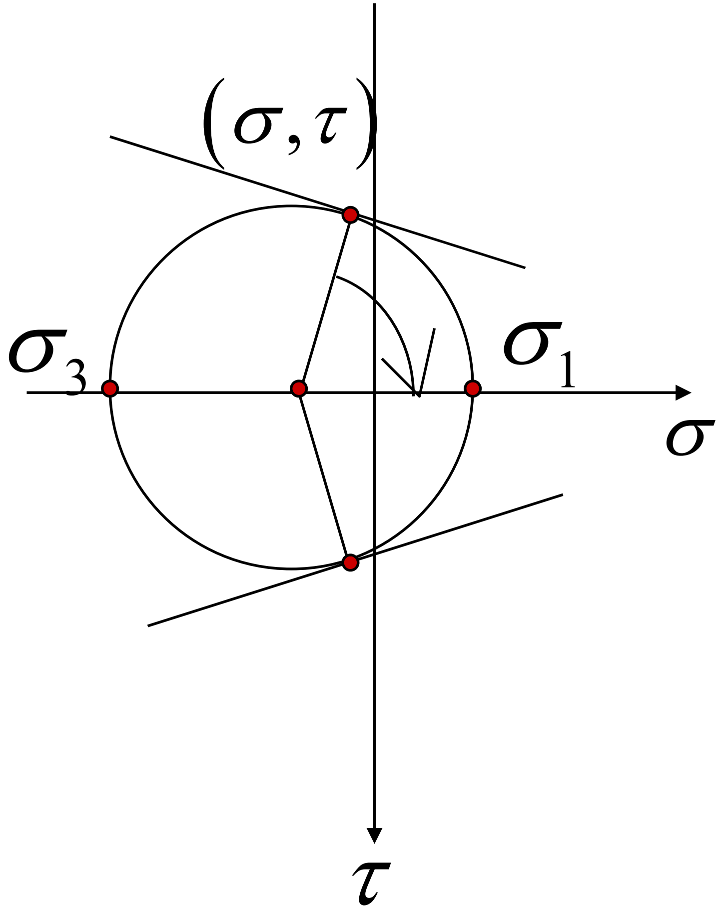
$$\tau_{r\theta}=\pm C \tan \varphi \, e^{\pm 2\theta \tan \varphi}$$

Rankine Zone

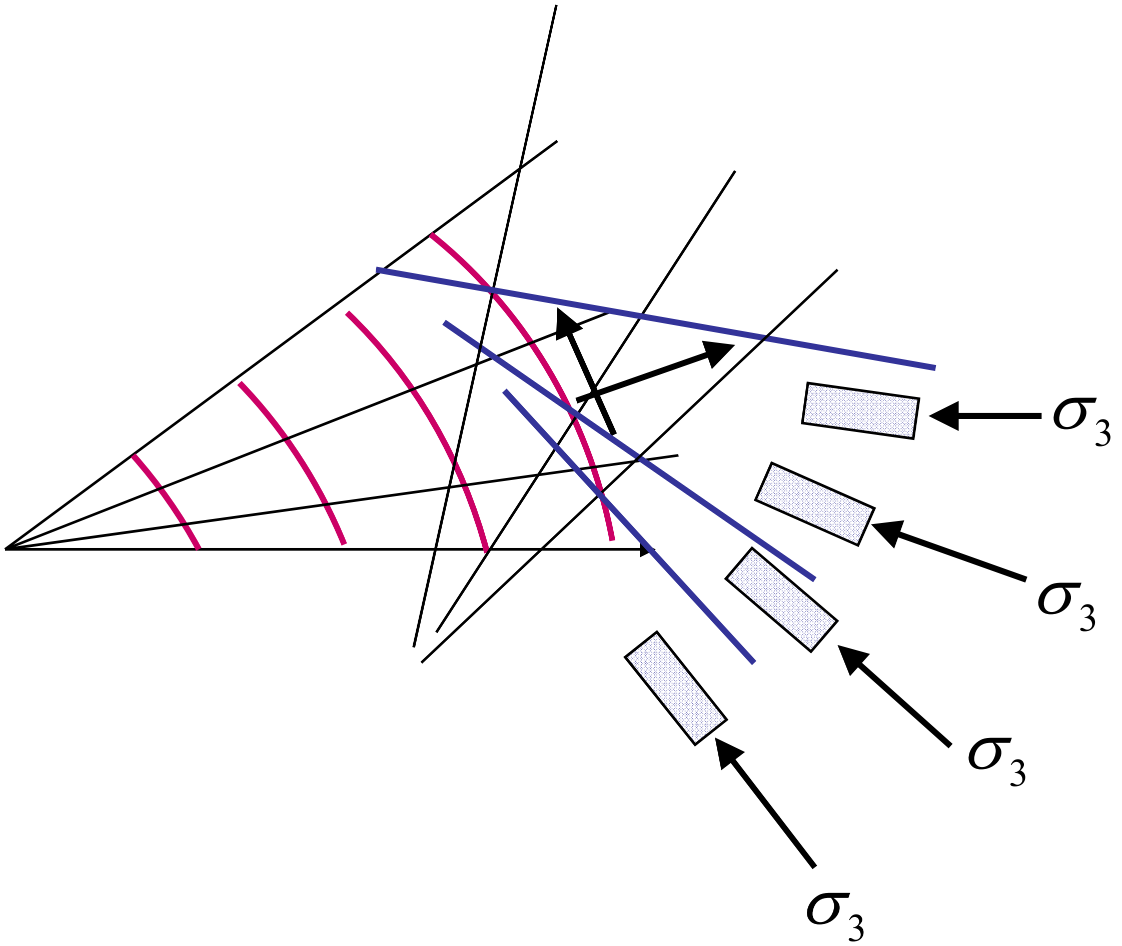


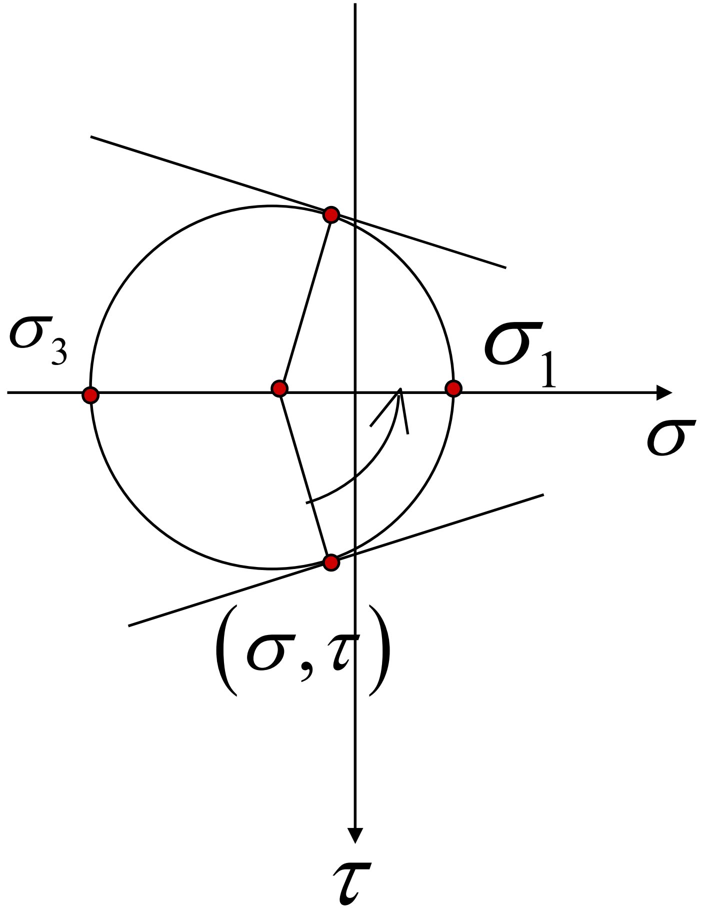
Strut and Rankine Zone



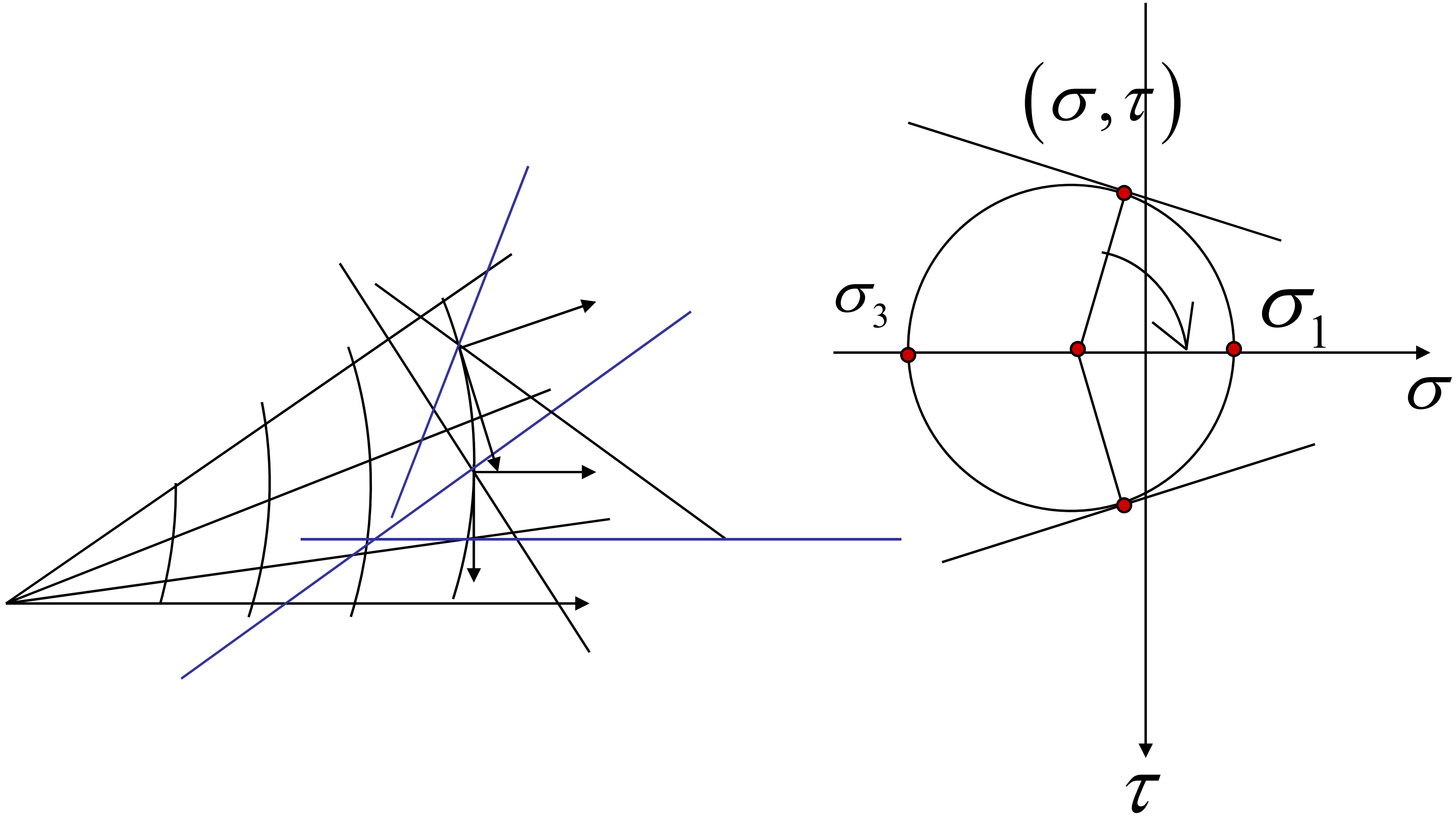


Prandtl Zone



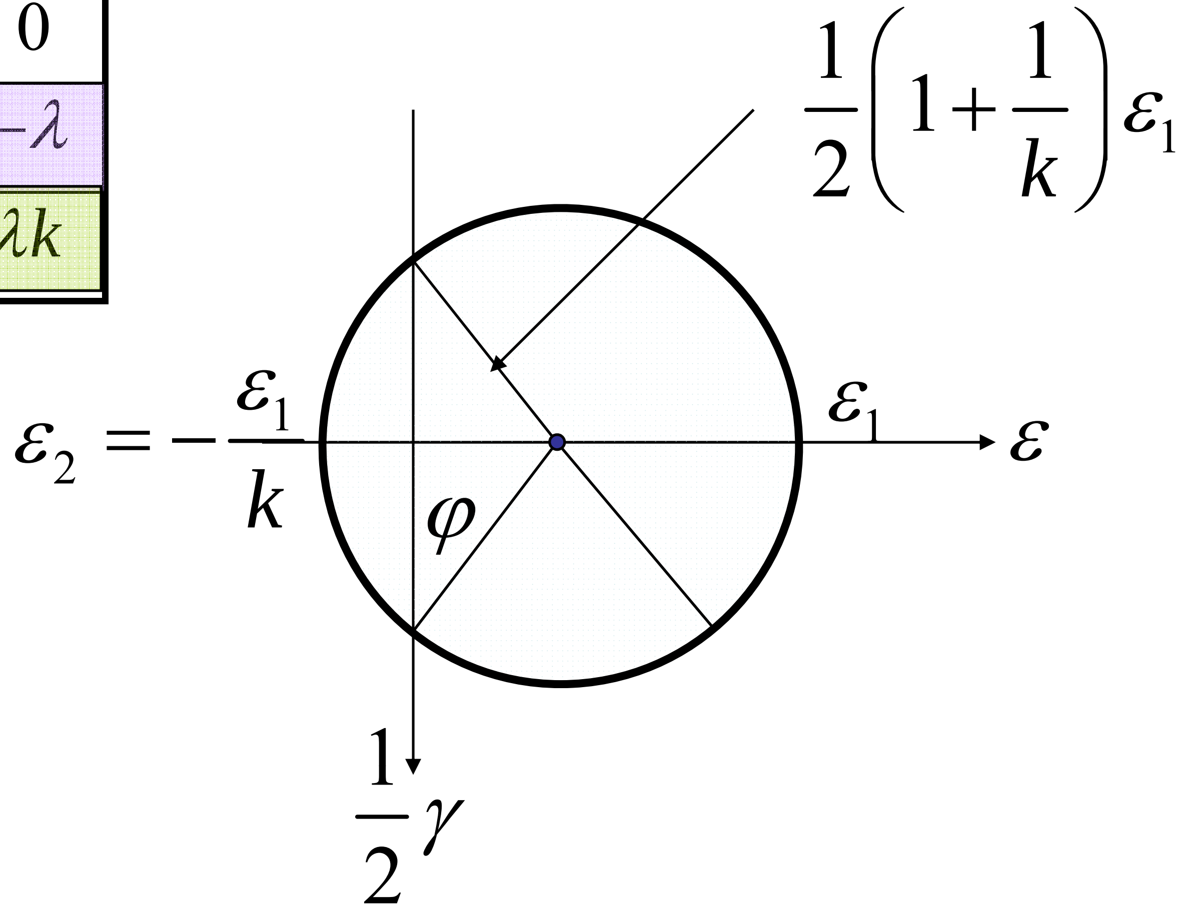


Prandtle Zone



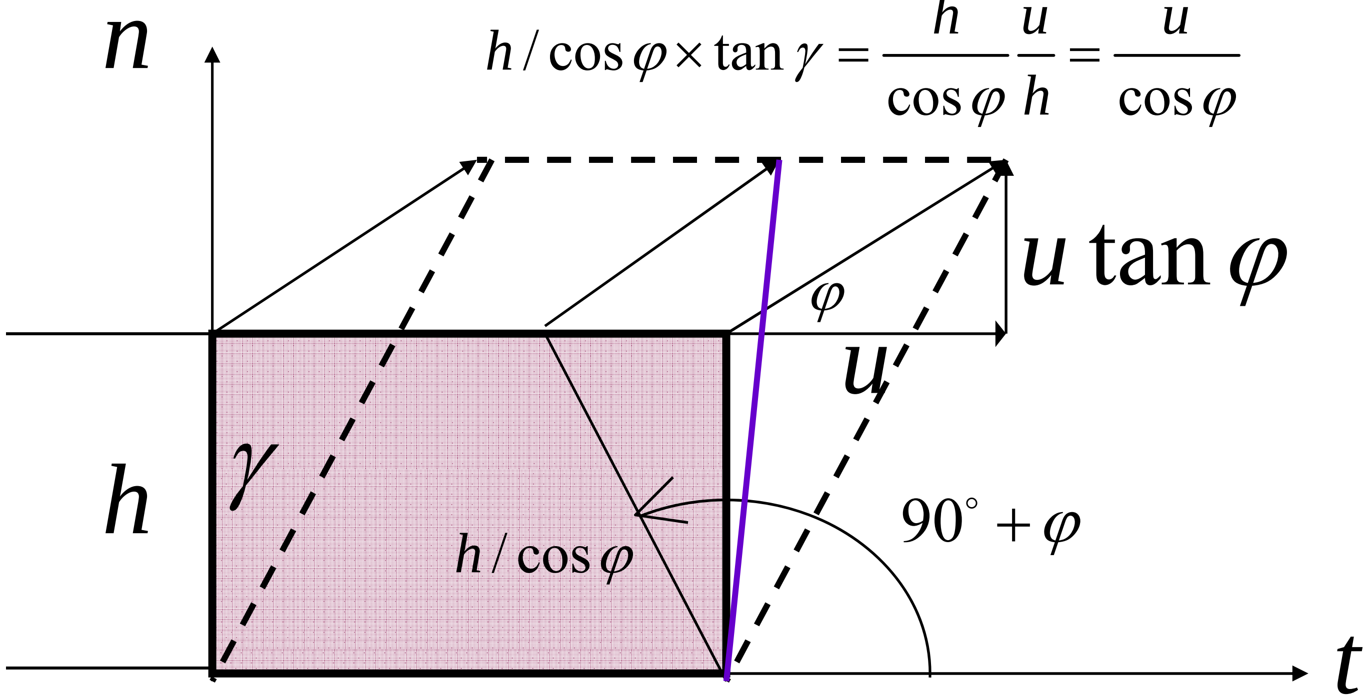
Strain Fields

	ε_1	ε_2	ε_3
1	λk	0	$-\lambda$
2	$-\lambda$	0	λk
3	λk	$-\lambda$	0
4	$-\lambda$	λk	0
5	0	λk	$-\lambda$
6	0	$-\lambda$	λk



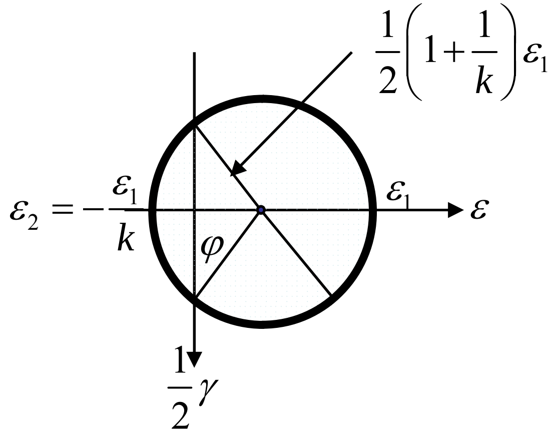
Homogeneous Strain Field

$$h/\cos\varphi \times \tan\gamma = \frac{h}{\cos\varphi}\frac{u}{h} = \frac{u}{\cos\varphi}$$

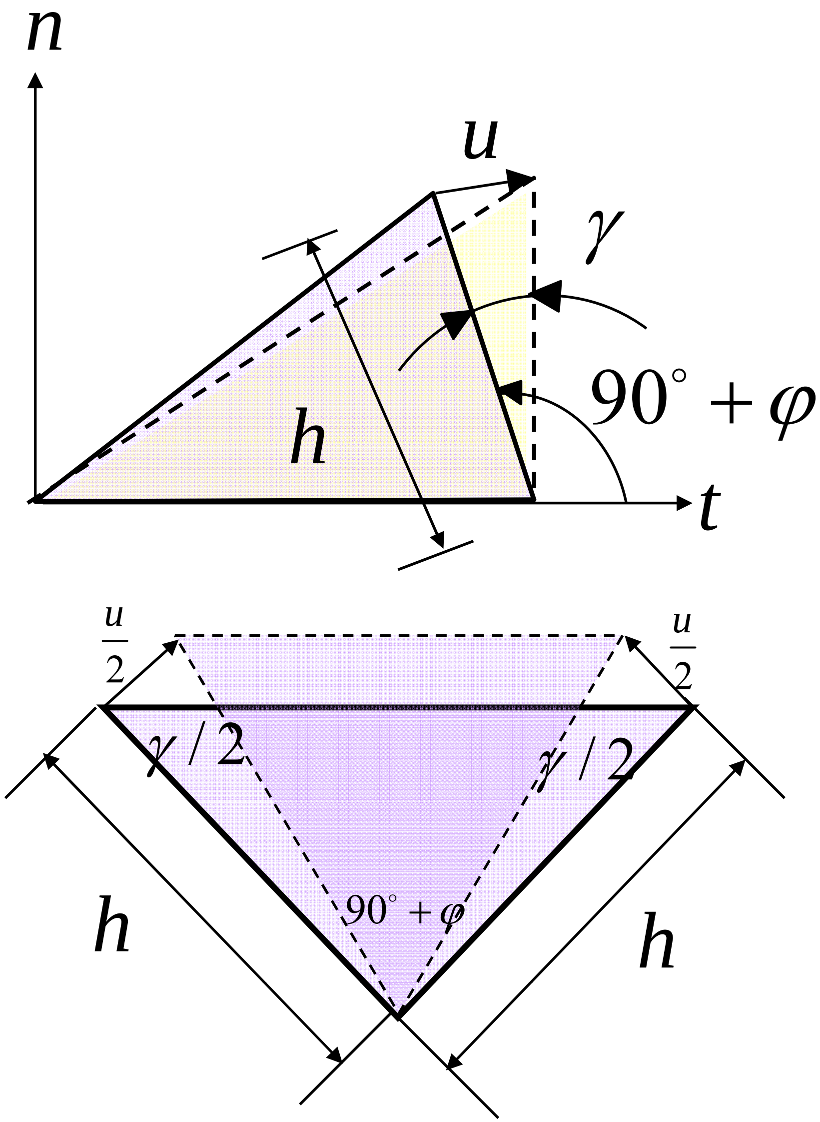


$$\varepsilon_n = \frac{u\tan\varphi}{h} = \gamma\tan\varphi$$

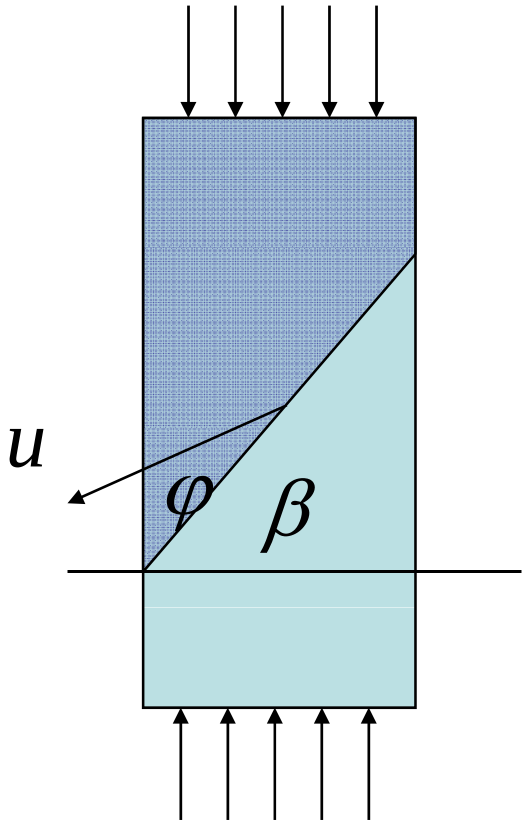
$$\varepsilon_t = 0 \qquad \gamma_{nt} = \gamma = \frac{u}{h}$$

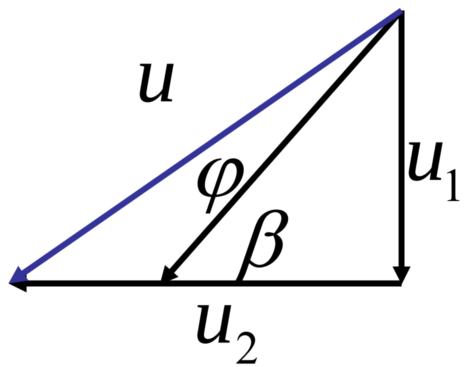


Triangular Zone

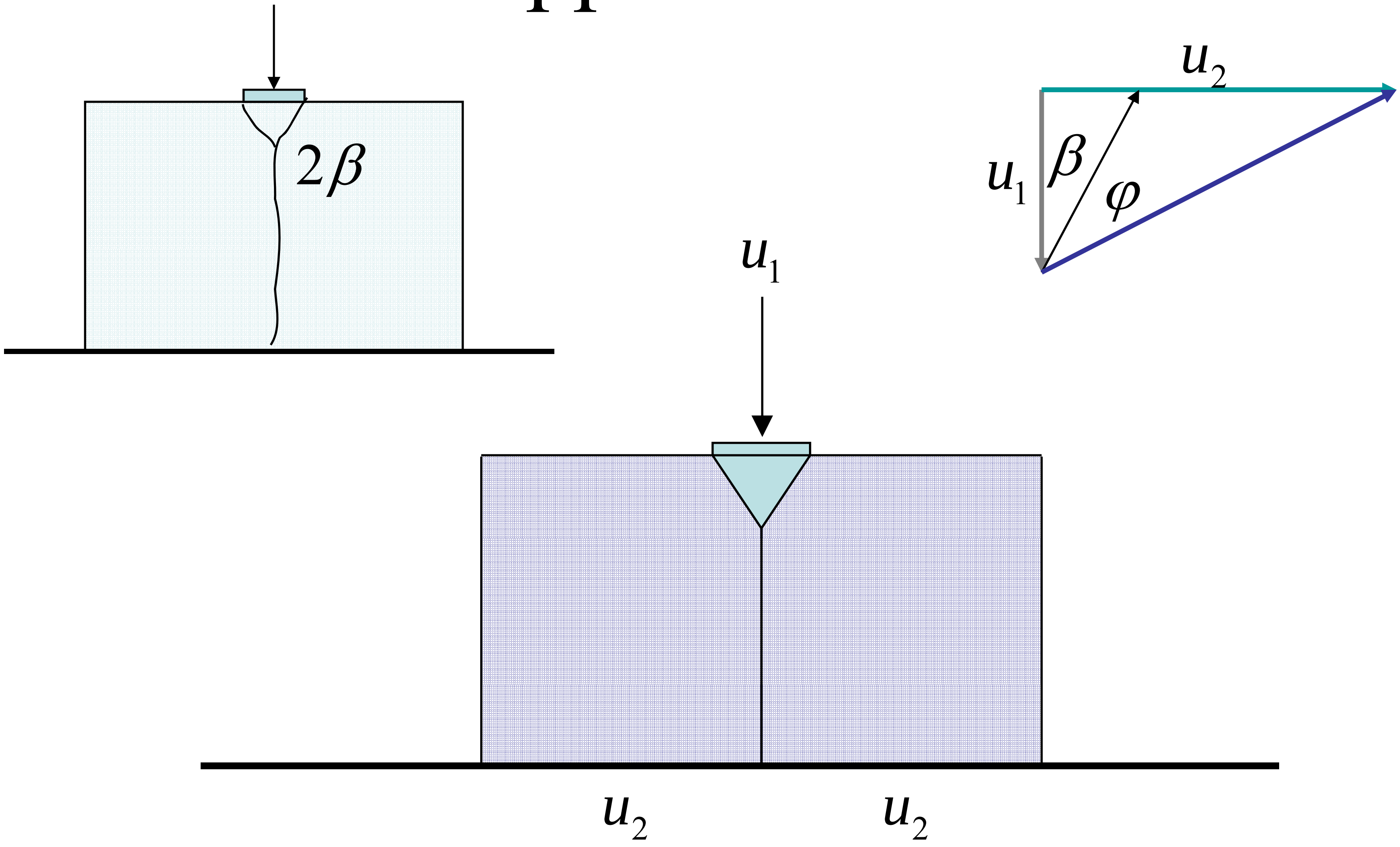


Applications

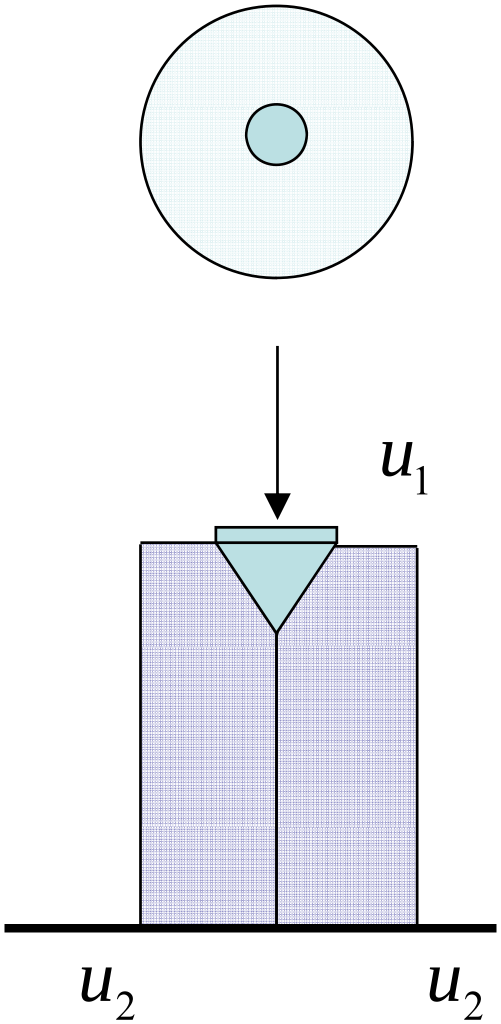


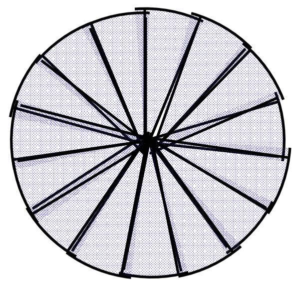


Applications

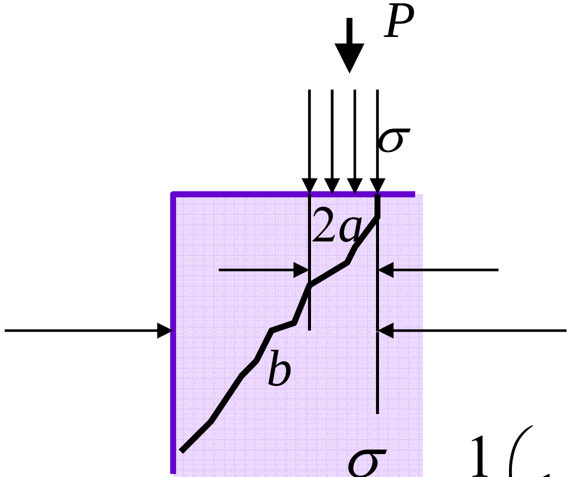


Applications

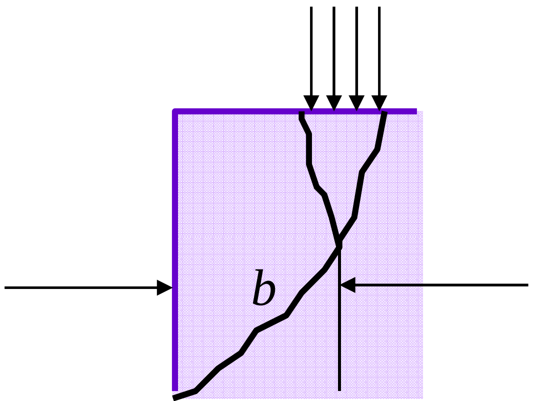




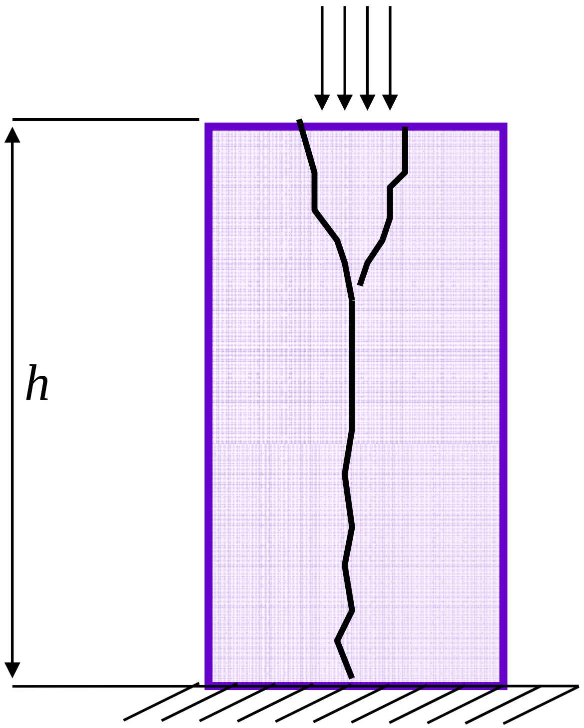
Design for concentrated loading



$$\frac{\sigma}{f_c} = \frac{1}{2}\left(1 + \frac{b}{a}\right)$$



$$\frac{\sigma}{f_c} = 1.5 + K_1\left(\frac{b}{a} - 1\right)\frac{f_t}{f_c}, K_1 \cong 3.6$$

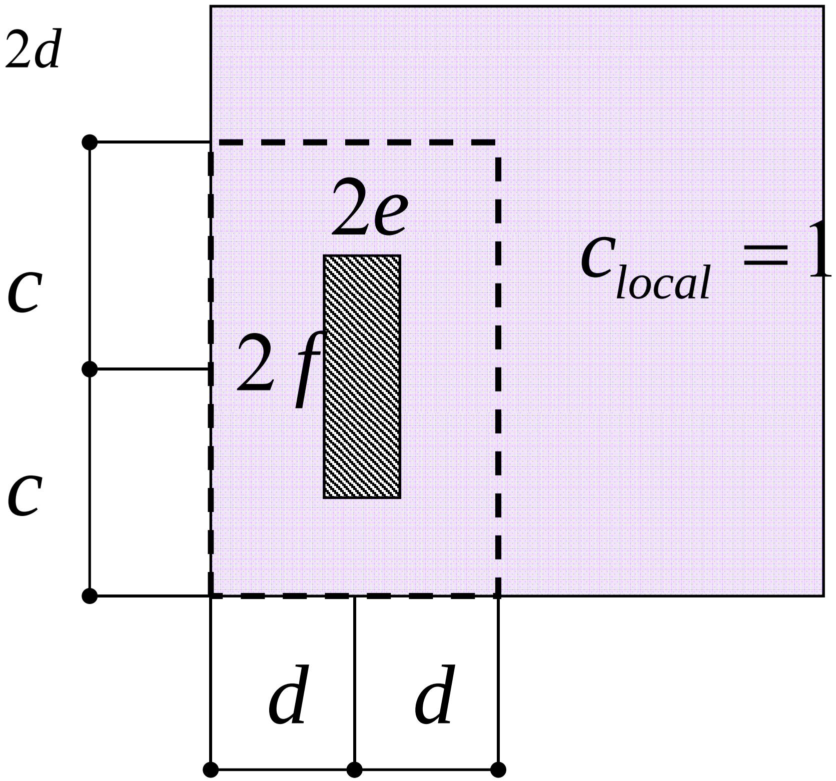


$$\frac{\sigma}{f_c} = 1.0 + K_2\left(\frac{h}{2a} - 1\right)\frac{f_t}{f_c}, K_2 \cong 2.48$$

Semi-empirical formula

$$A_o = 2e \bullet 2f$$

$$A = 2c \bullet 2d$$



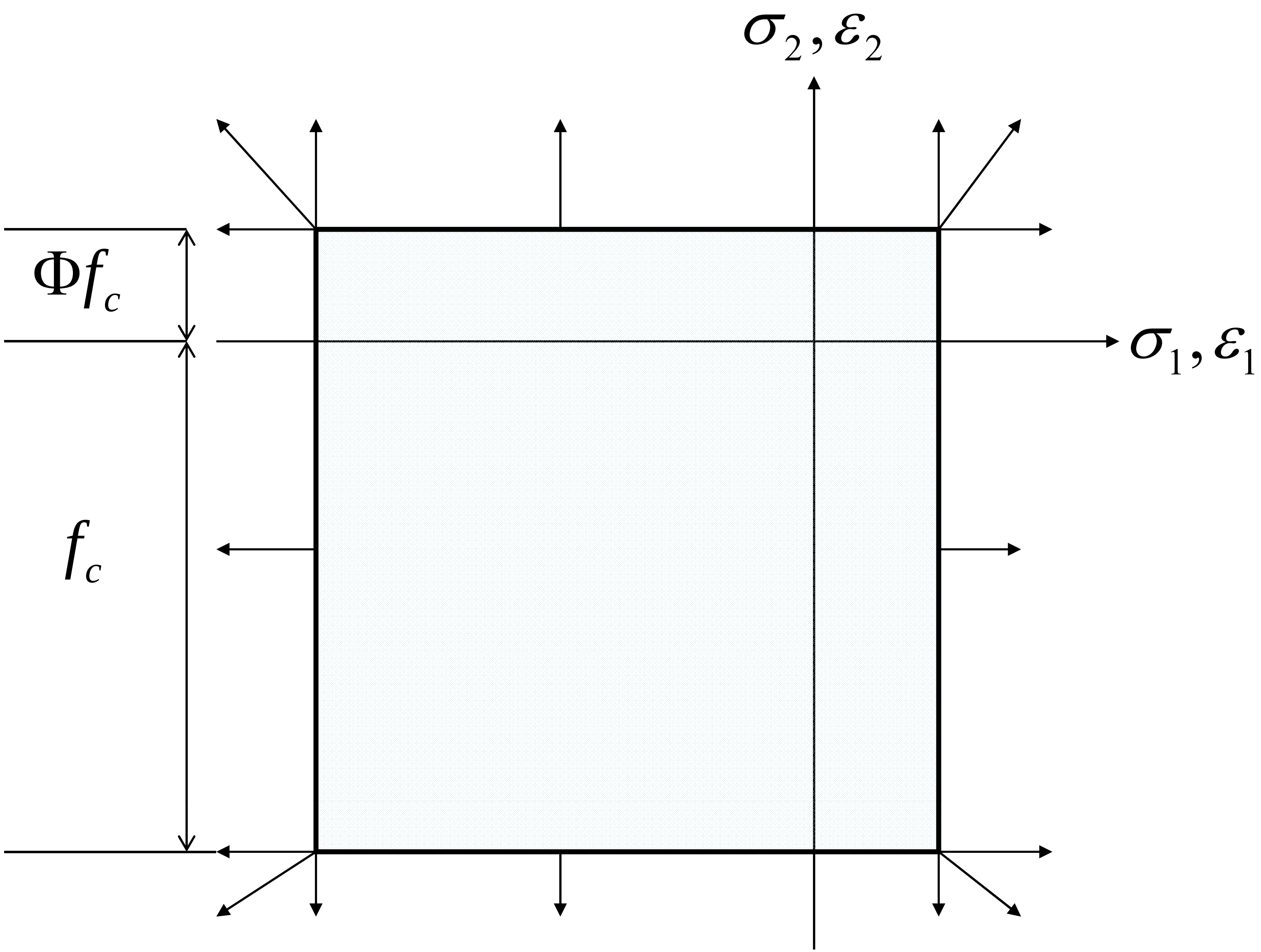
$$c_{local} = 1 + K_4 \left(\sqrt{\frac{A}{A_0}} - 1 \right) \frac{f_t}{f_c}$$

$$a = \frac{1}{2}\sqrt{A_0}, b = \frac{1}{2}\sqrt{A}$$

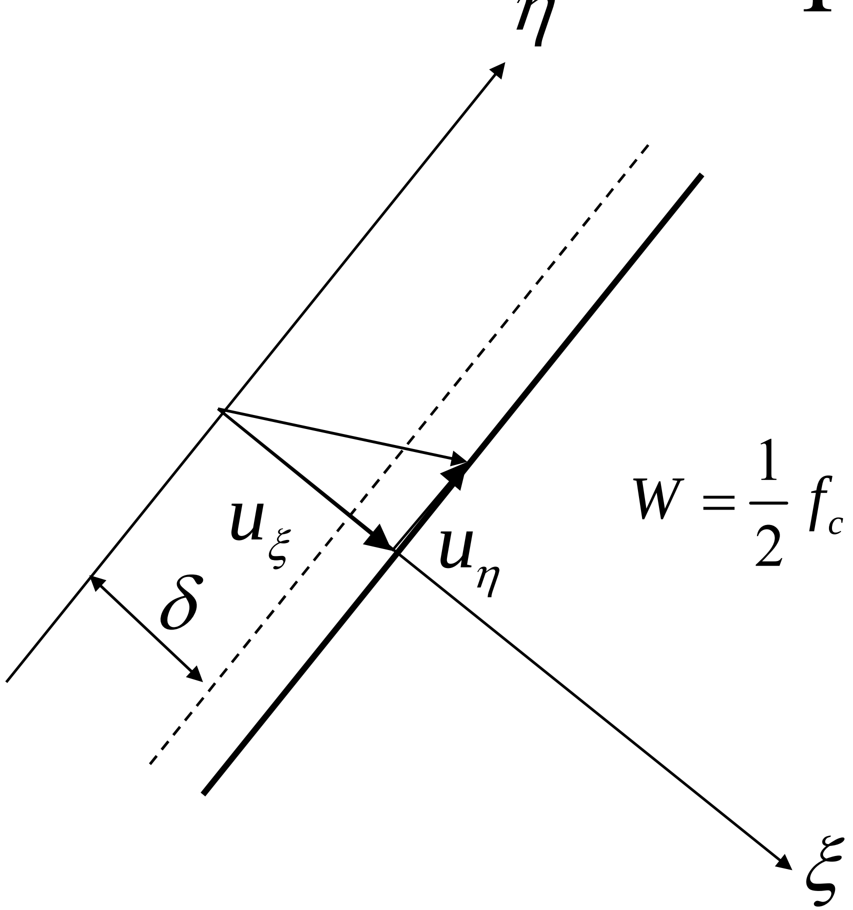
Disks

- Constitutive Equations
- Dissipation Works
- Yield Zones
- Applications
- Strength Reduction due to Initial Cracks
- Strut and Tie Models
- Shear Walls

Disk Element

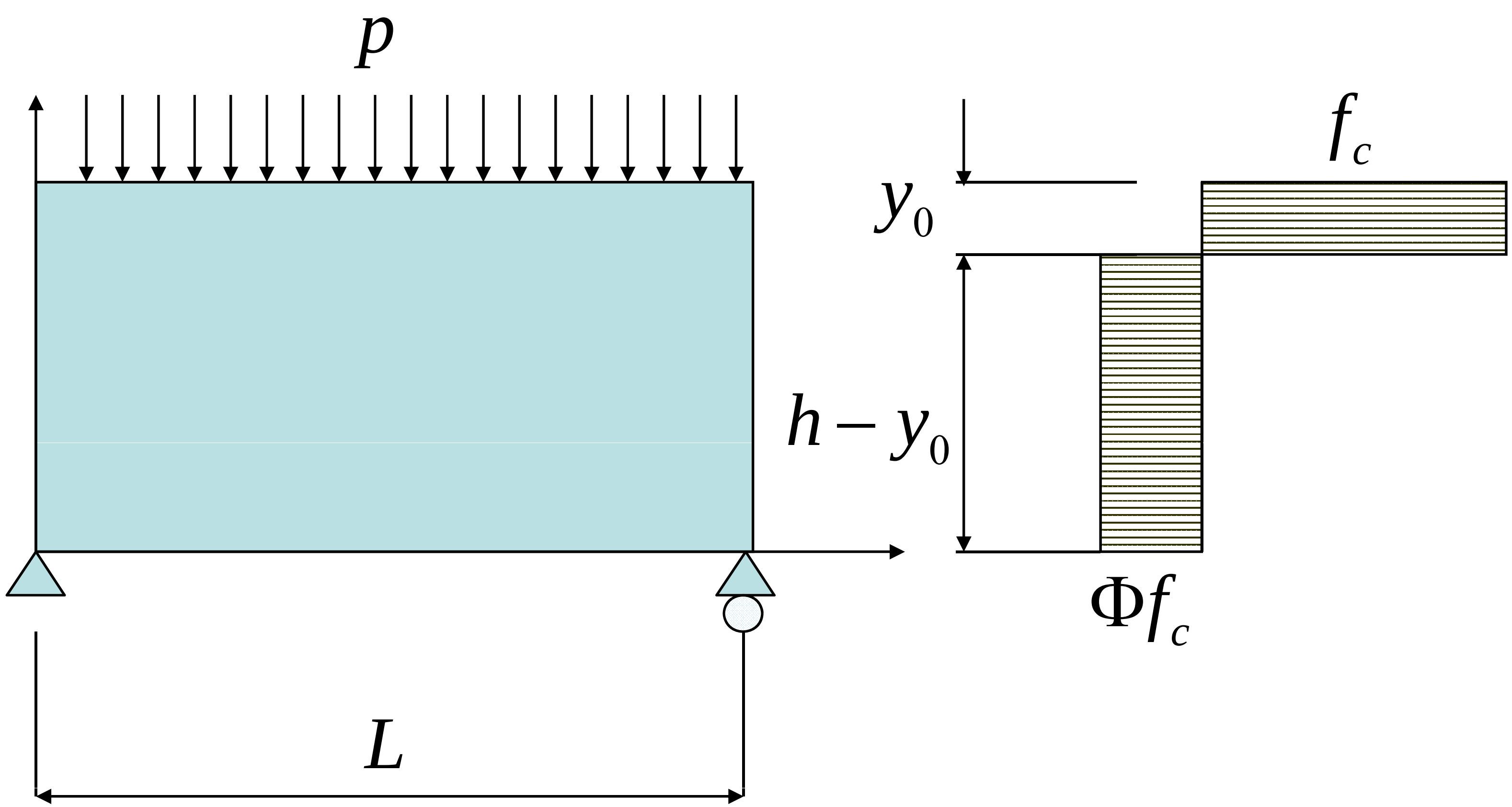


Dissipation



$$W = \frac{1}{2} f_c \left[\left(1+\Phi\right) \sqrt{\left(\frac{u_\xi}{\delta}\right)^2 + \left(\frac{u_\eta}{\delta}\right)^2} - \left(1-\Phi\right)\frac{u_\xi}{\delta} \right]$$

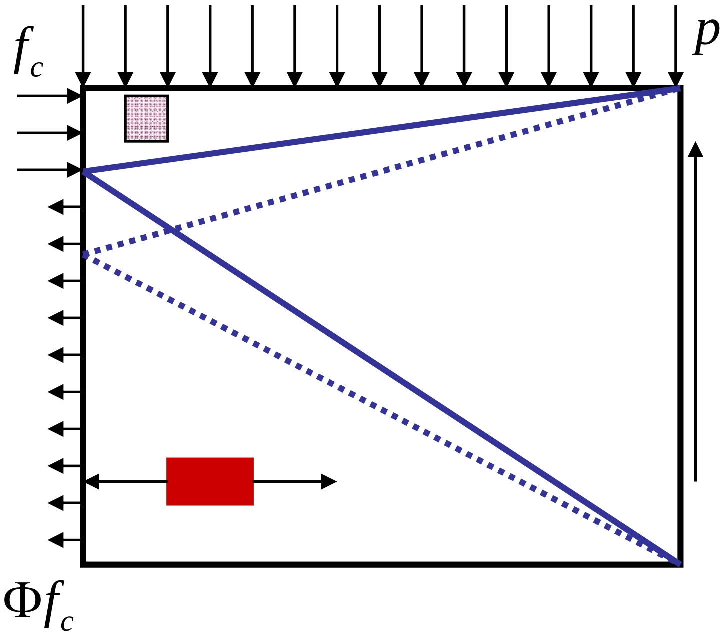
Known Solutions

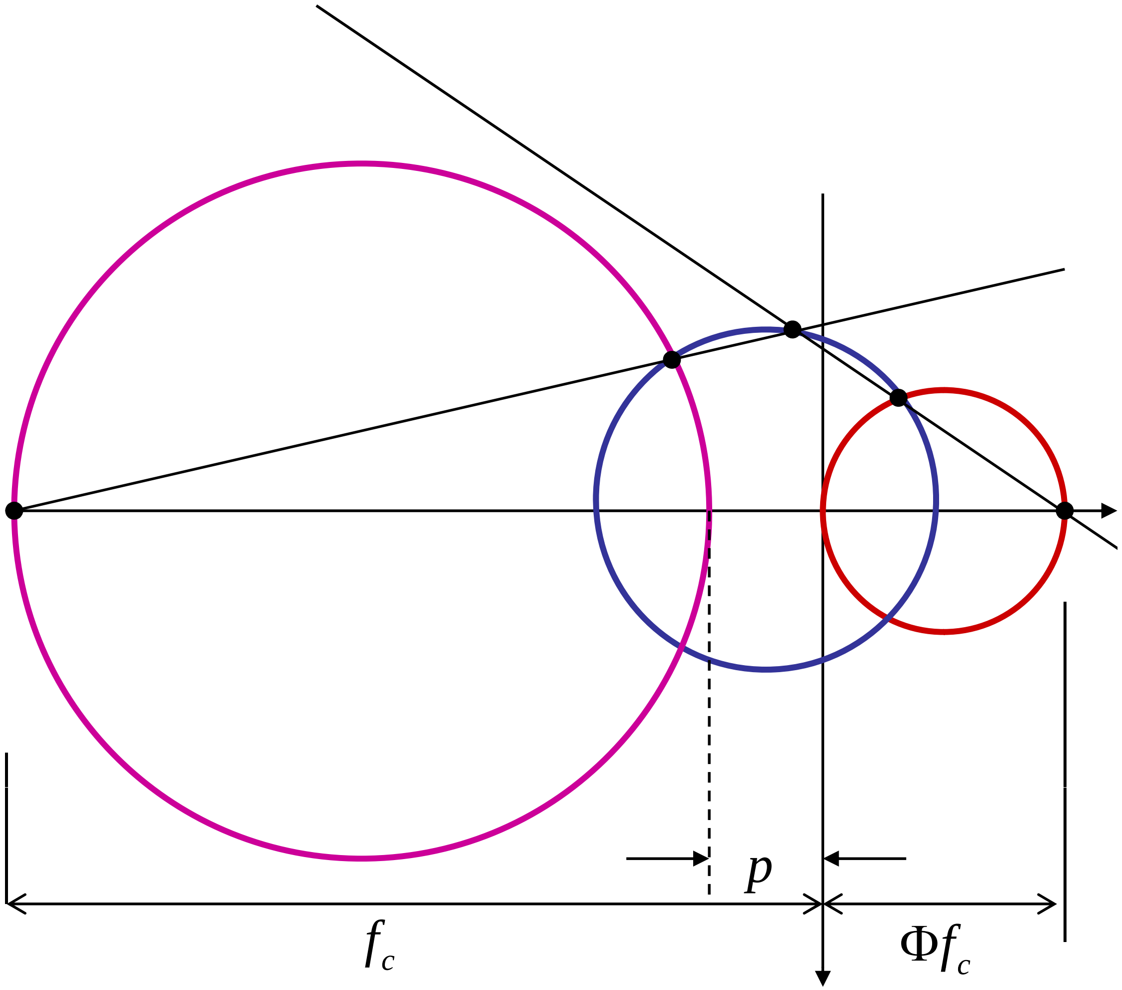


$$M_p = \frac{1}{2}\frac{\Phi}{1+\Phi} t h^2 f_c$$

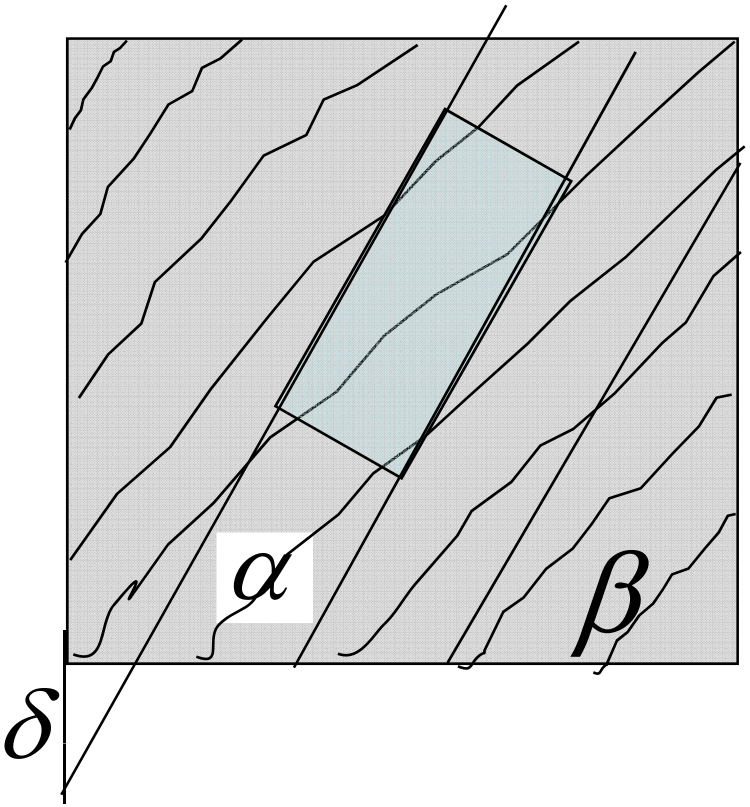
$$p = \frac{4\Phi h^2 f_c}{\left(1+\Phi\right) L^2}$$

Discontinuous Stress Fields





Strength Reduction due to Sliding in Initial Cracks



$$\frac{f_{cs}}{\nu_o f_c} = \frac{\frac{1}{4}\nu_s}{\left|\sin\gamma\cos\gamma\right| - 0.75\sin^2\gamma} \le 1$$

Strut and Tie Models

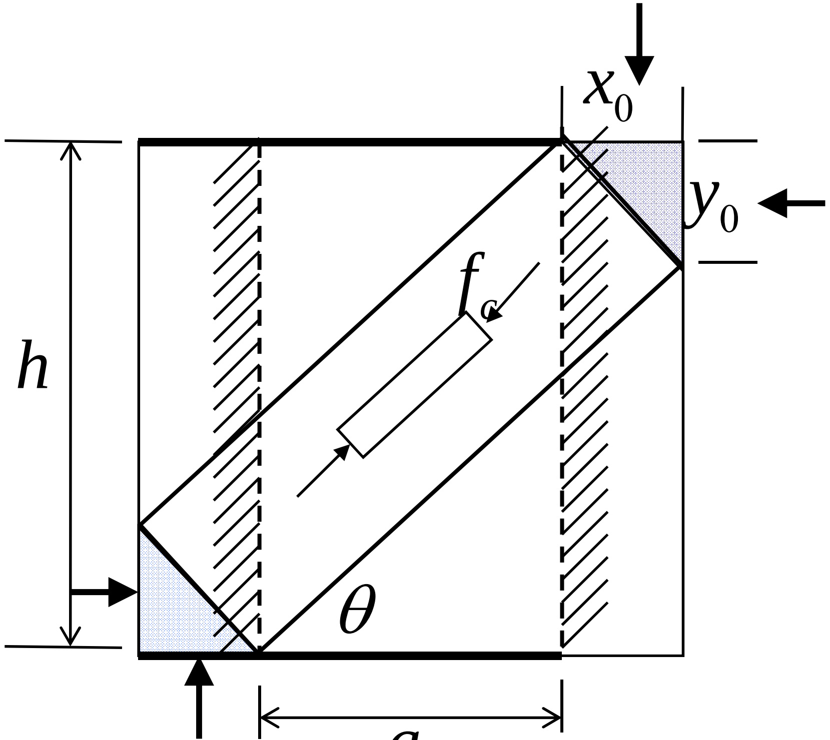
- Strut
- Strut and Tie Systems
- Fans

\- Non-concentric fans

\- Concentric fans

\- Fans with Bond

Single Strut



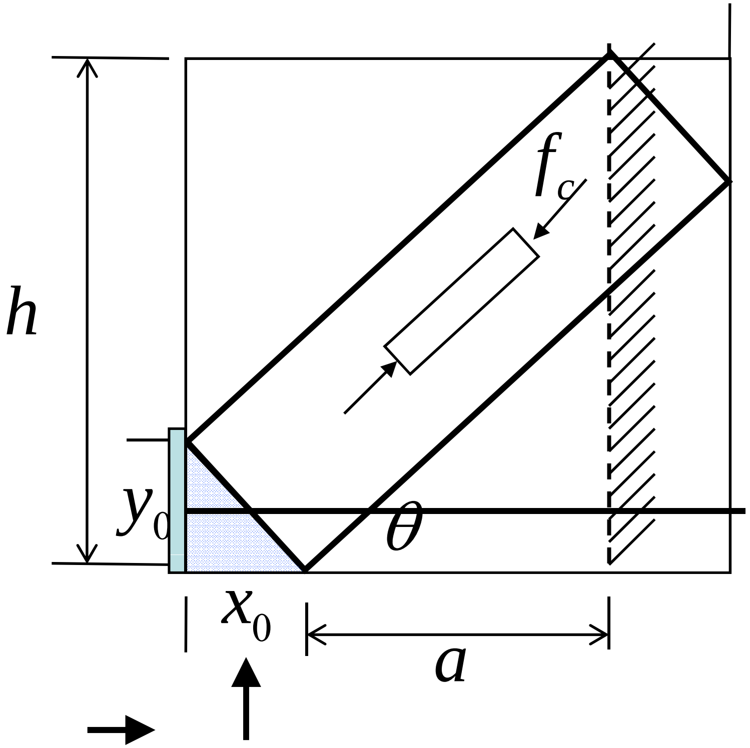
$$\tan\theta = \frac{x_0}{y_0} = \frac{h - y_0}{a + x_0}$$

$$\frac{x_0}{h} = \frac{1}{2}\left[\sqrt{1 + \left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

$$\frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1 + \left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

$$\frac{x}{h} = \frac{1}{2}\left[\sqrt{4\frac{y_0}{h}\left(1 - \frac{y_0}{h}\right) + \left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

Single Strut



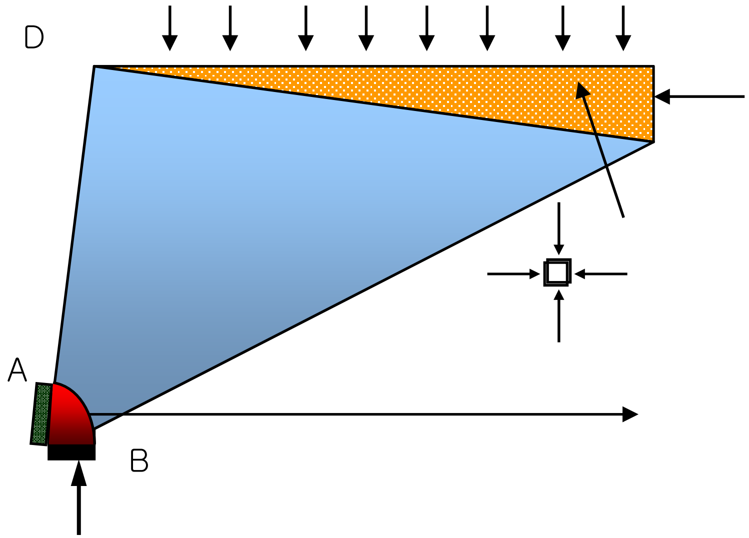
$$A_s f_Y = y_0 t f_c$$

$$\Phi = \frac{A_s f_Y}{thf_c} = \frac{y_0}{h} \leq \frac{1}{2}$$

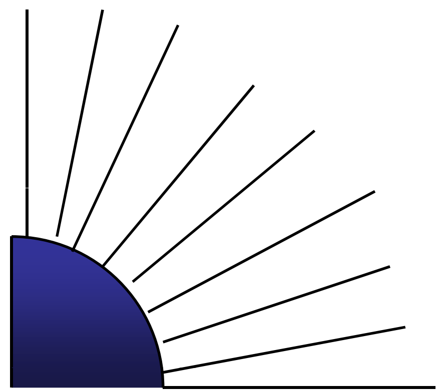
$$\frac{\tau}{f_c} = \frac{P}{thf_c} = \frac{x_0}{h} = \frac{1}{2}\left[\sqrt{4\Phi\left(1-\Phi\right)+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right]$$

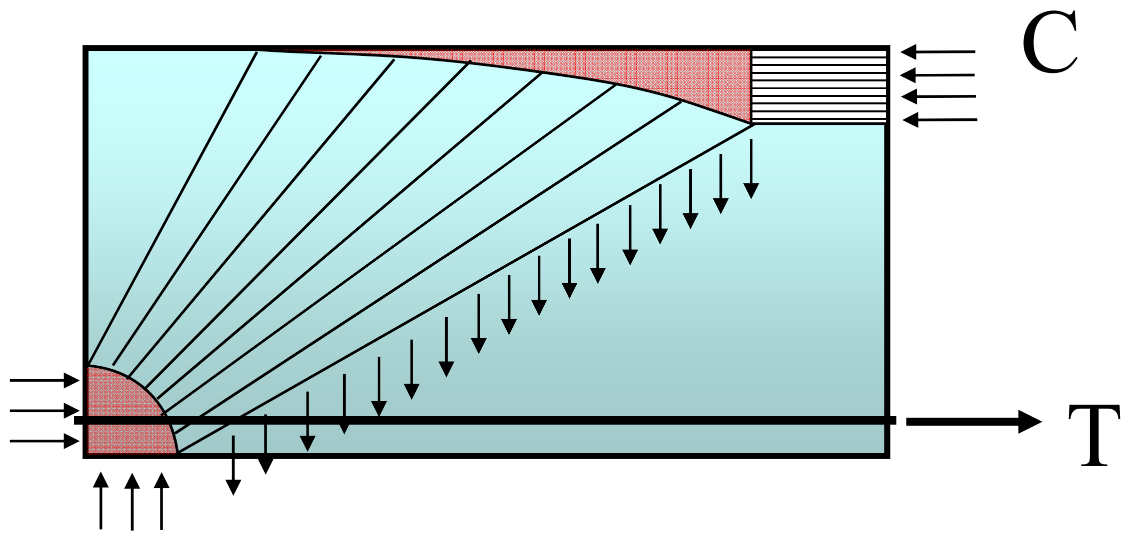
Fan-shaped stress fields

- Non-concentric fan
- Uniform Normal stress type
- Uniform Shear stress type

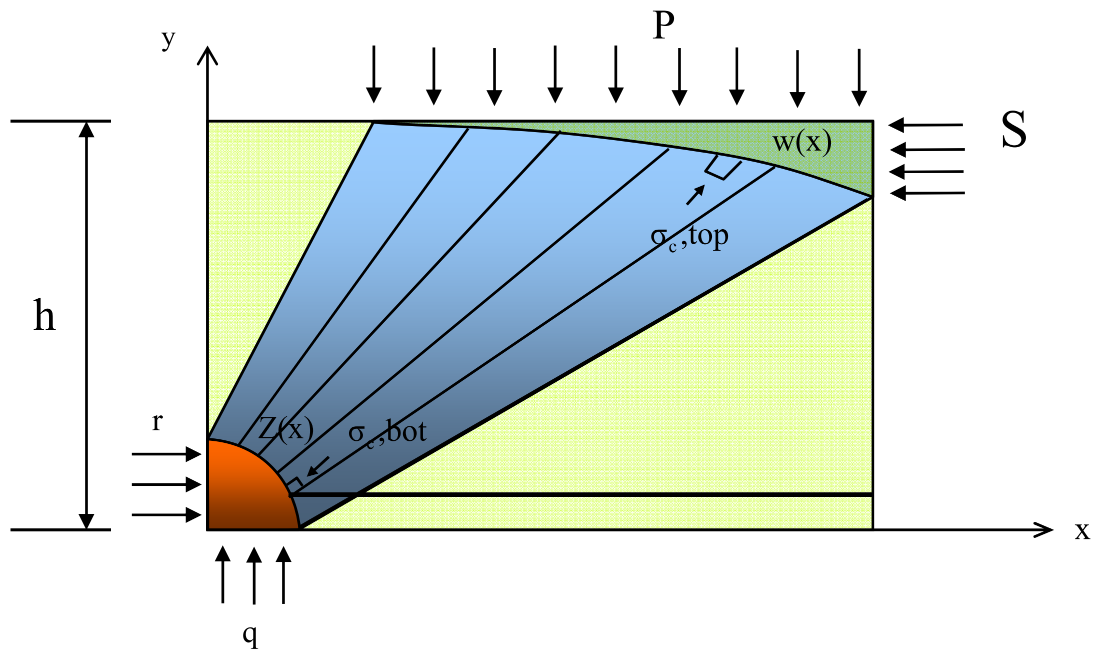


- Concentric Fan



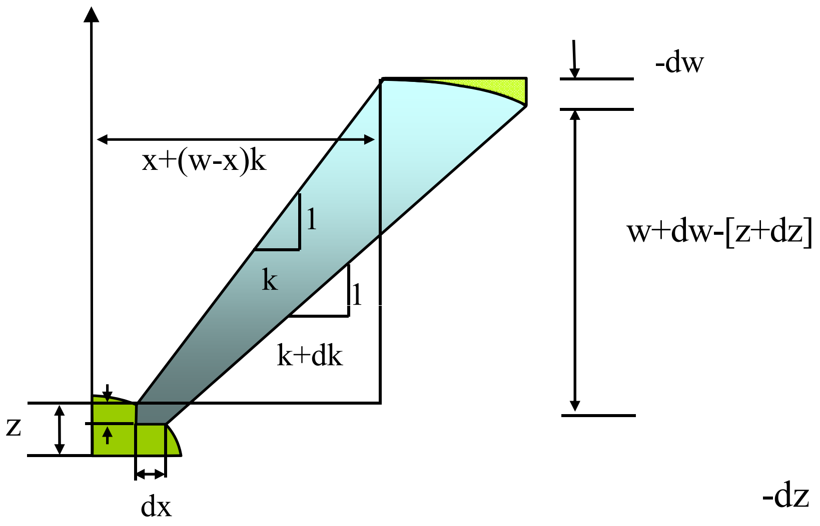


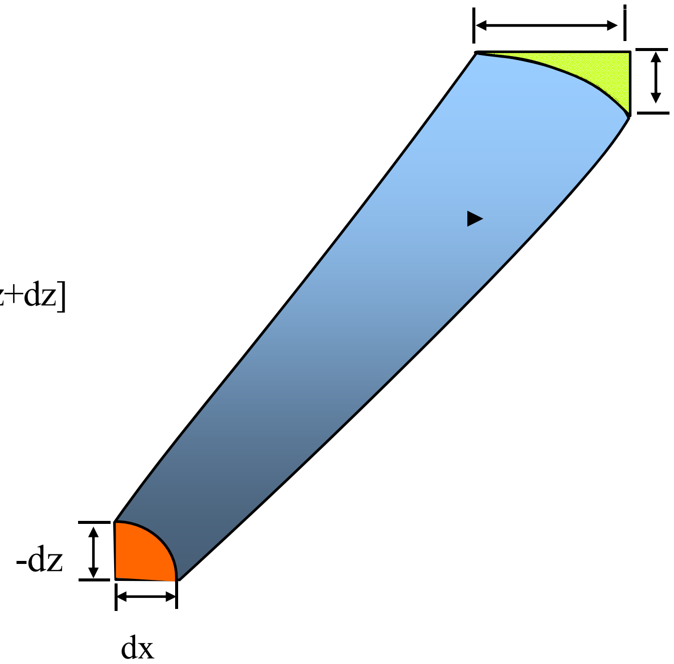
Boundary Stress on
Non-concentric fan-shaped stress field



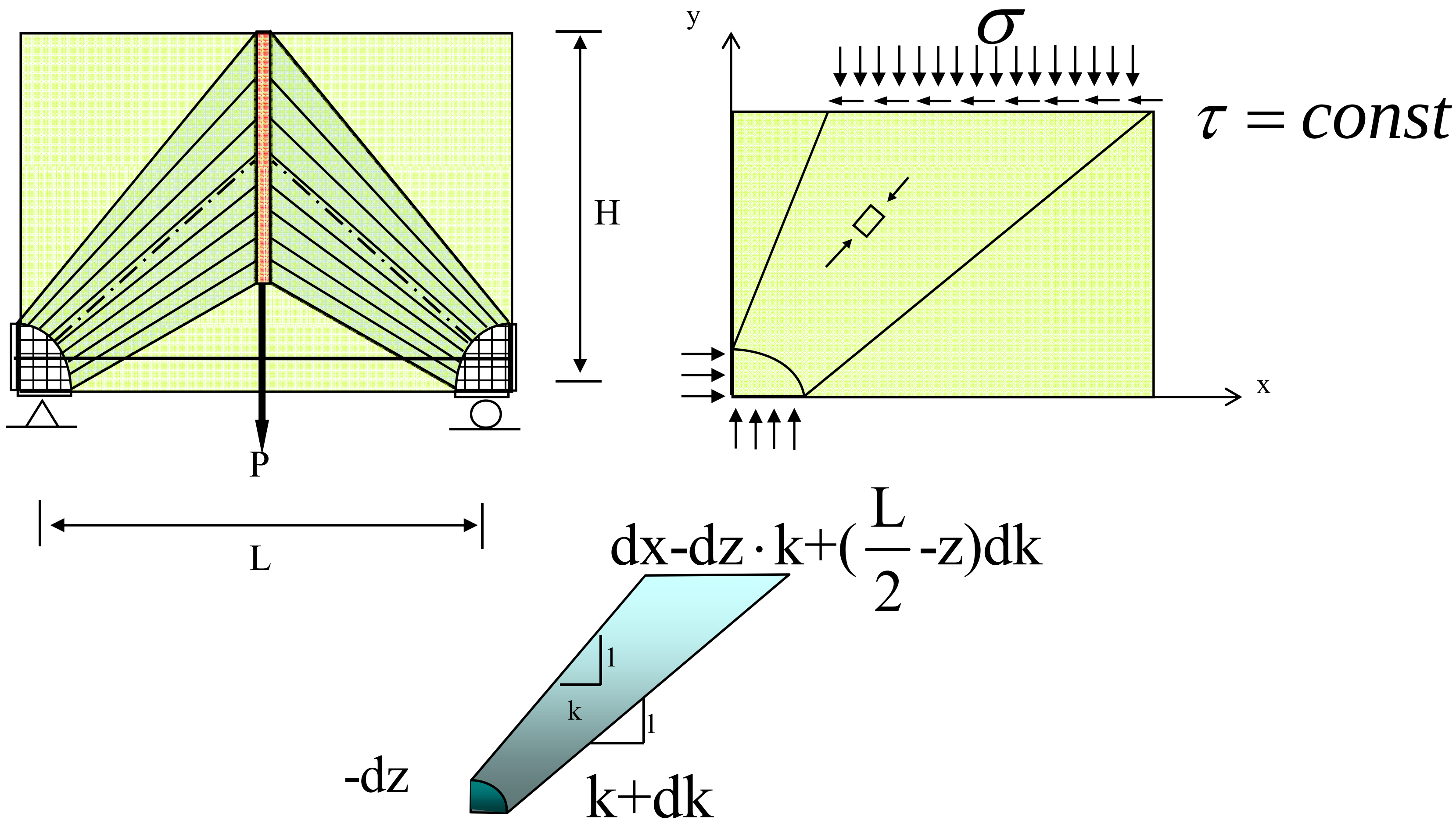
Uniform normal stress on boundary of Non-concentric fan-shaped stress field

$$d\square = dx + (dw - dz)k + (w - z)dk$$

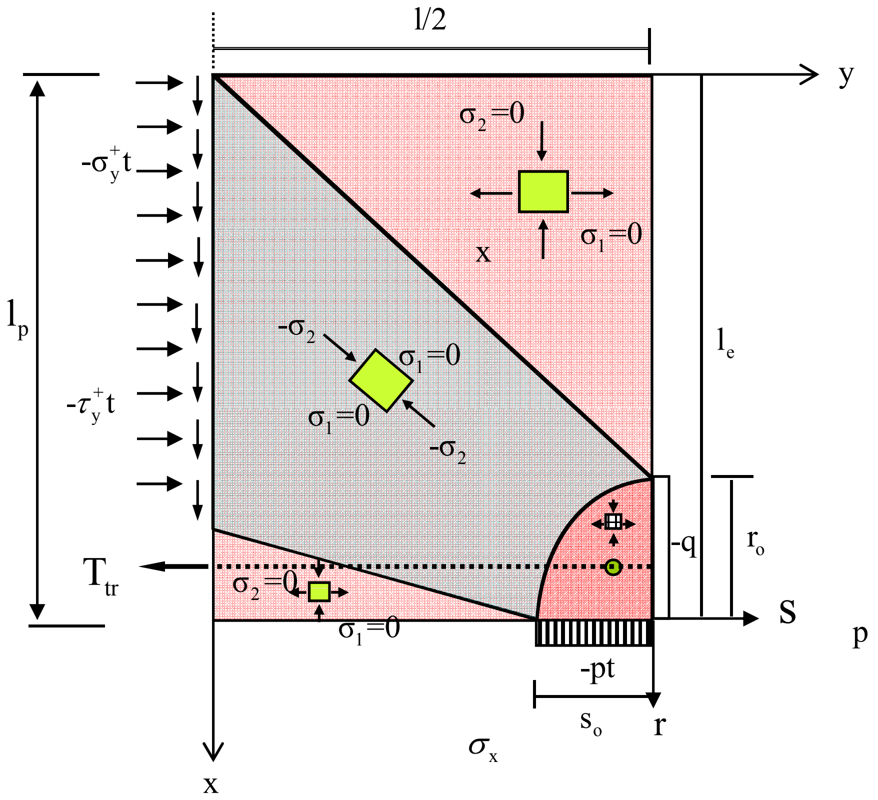




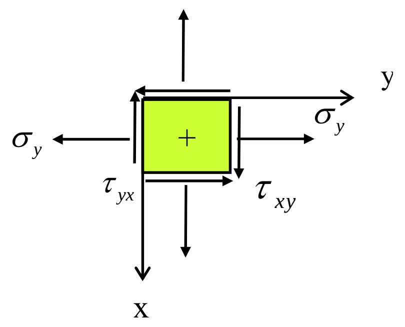
Geometry for Infinitesimal Elements of Non-concentric Fan-shaped stress field (Uniform Normal stress type)



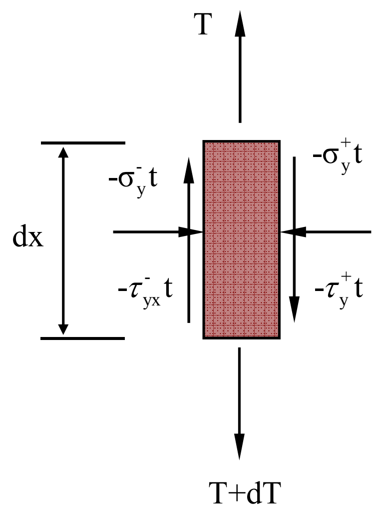
Uniform shear stress on boundary of Non-concentric fan-shaped stress field



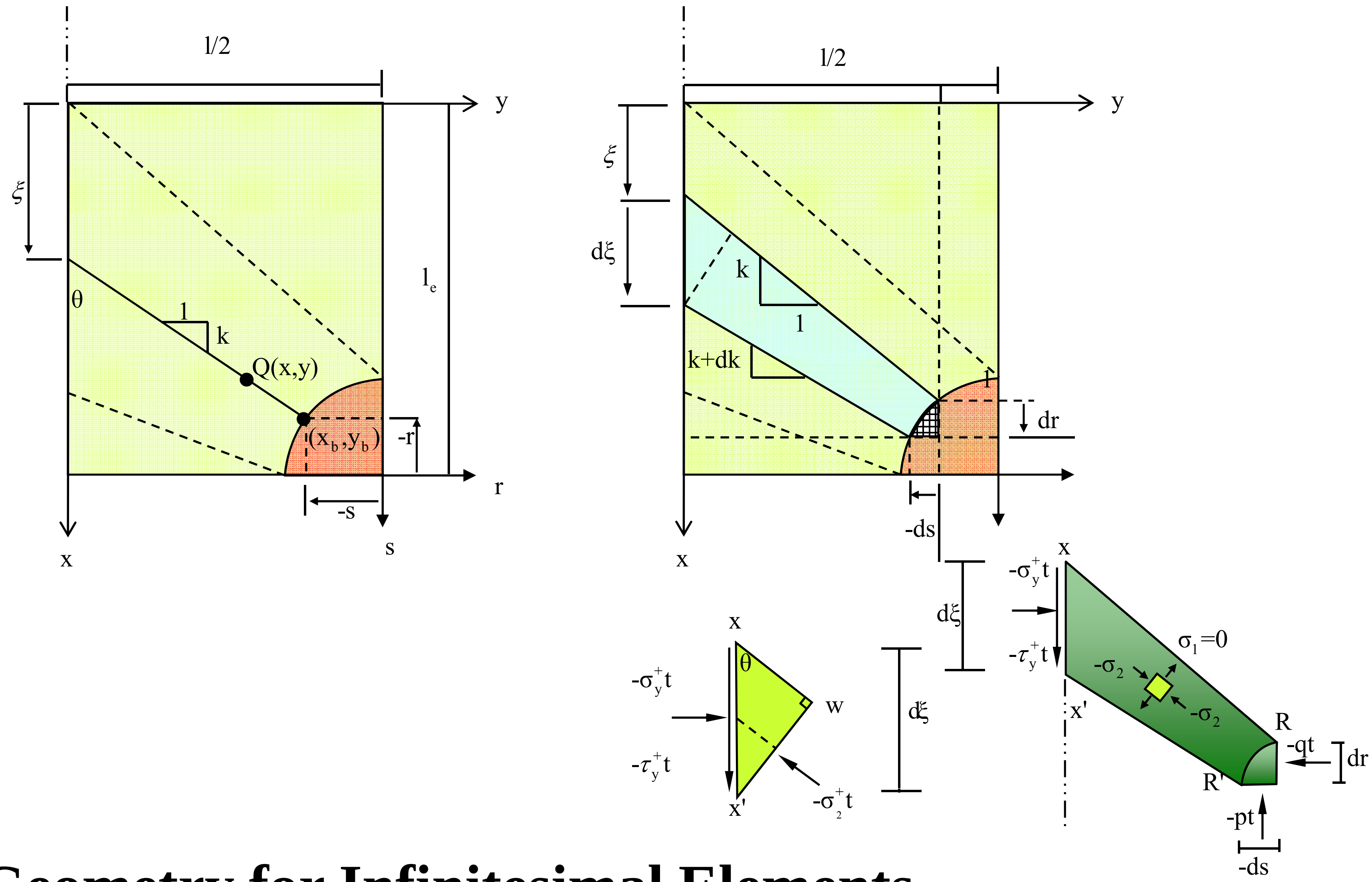
l/2
y
l_p
$-\sigma_y^+ t$
$-\tau_y^+ t$
$\sigma_2=0$
$\sigma_1=0$
x
$-\sigma_2$
$\sigma_1=0$
$\sigma_1=0$
$-\sigma_2$
l_e
-q
r_o
T_{tr}
$\sigma_2=0$
$\sigma_1=0$
S
p
-pt
s_o
r
σ_x
x



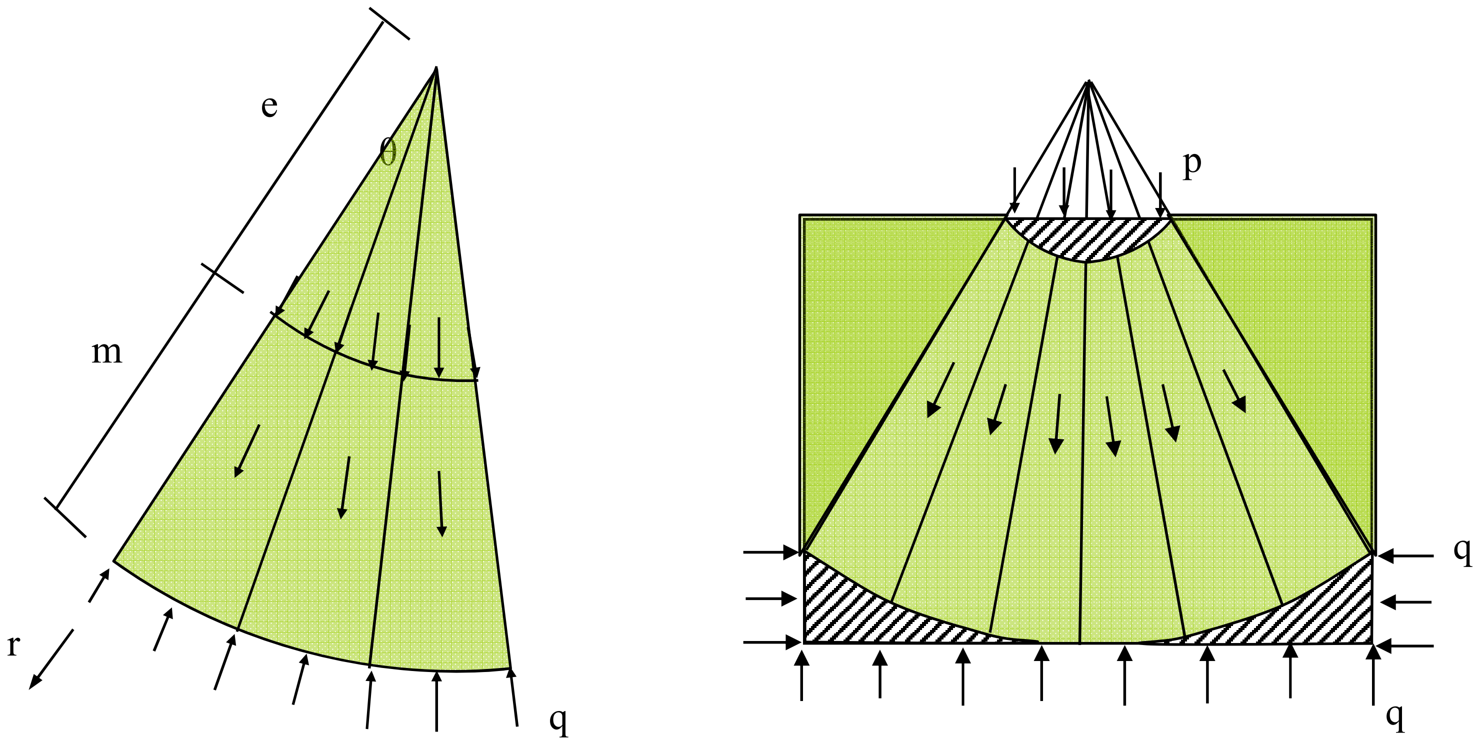
y
σ_y
σ_y
τ_{yx}
τ_{xy}
x



T
dx
$-\sigma_y^- t$
$-\sigma_y^+ t$
$-\tau_{yx}^- t$
$-\tau_y^+ t$
T+dT



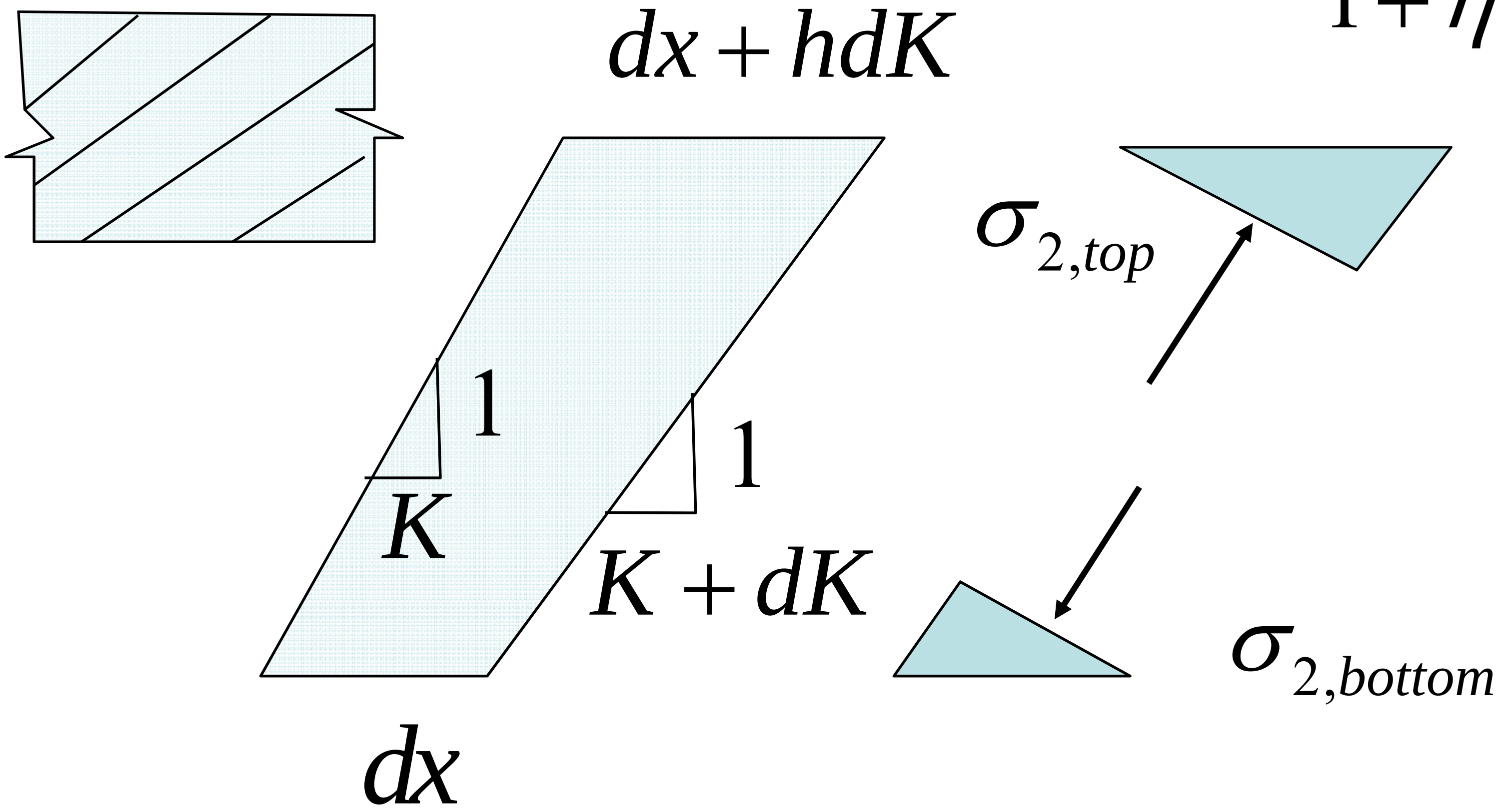
Geometry for Infinitesimal Elements of Non-concentric Fan-shaped stress field (Uniform Shear stress type)



Concentric Fan-shaped stress field

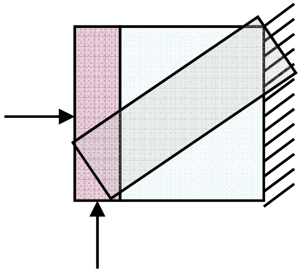
dx

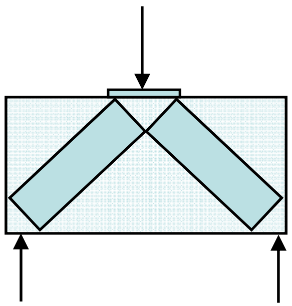
$$\sigma_{2,top} = \frac{\sigma_{2,bottom}}{1 + \eta \dfrac{dK}{dx}}$$

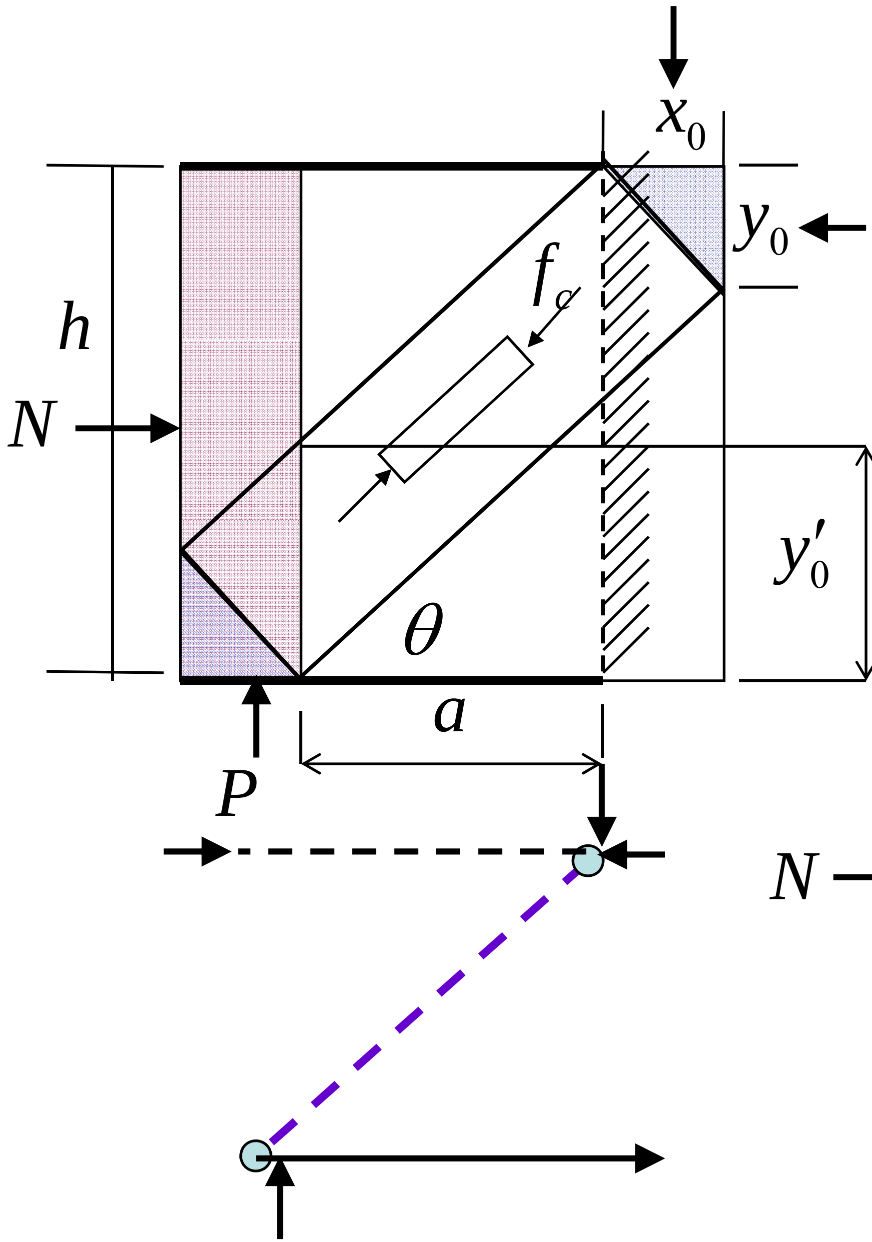


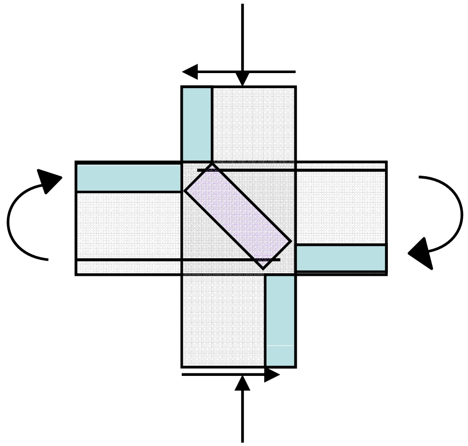
Stress State in Diagonal Compression Stress Fields

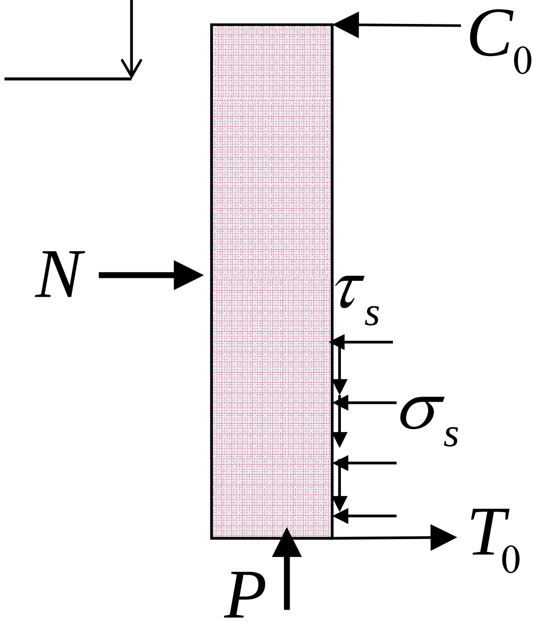
Single Strut action



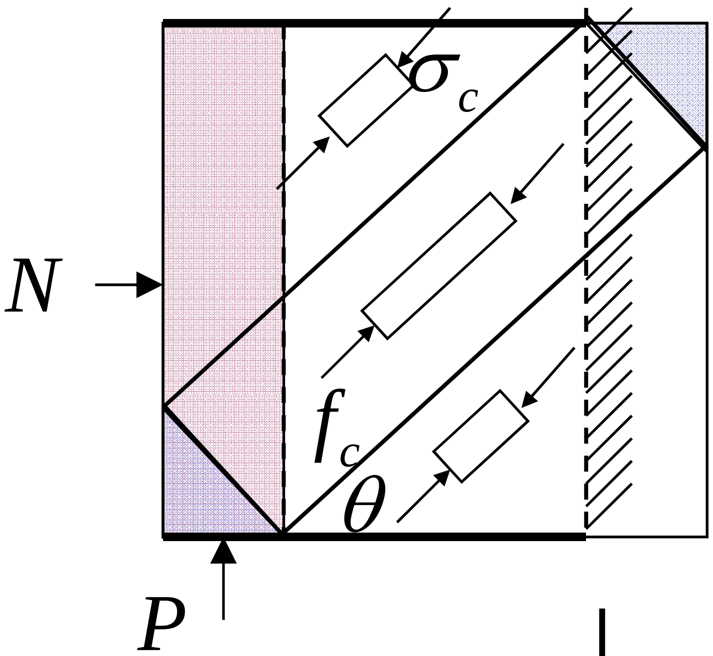


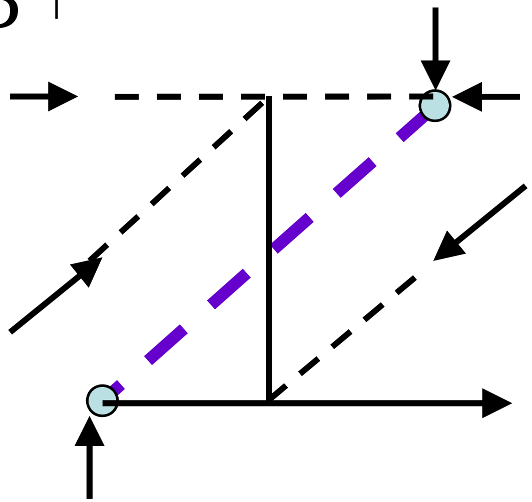


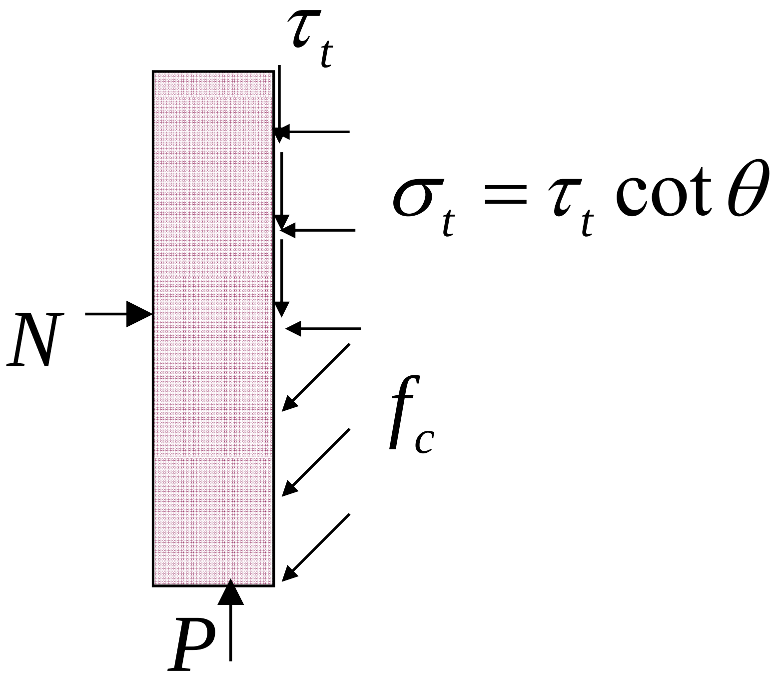




Strut with Diagonal Compression Field



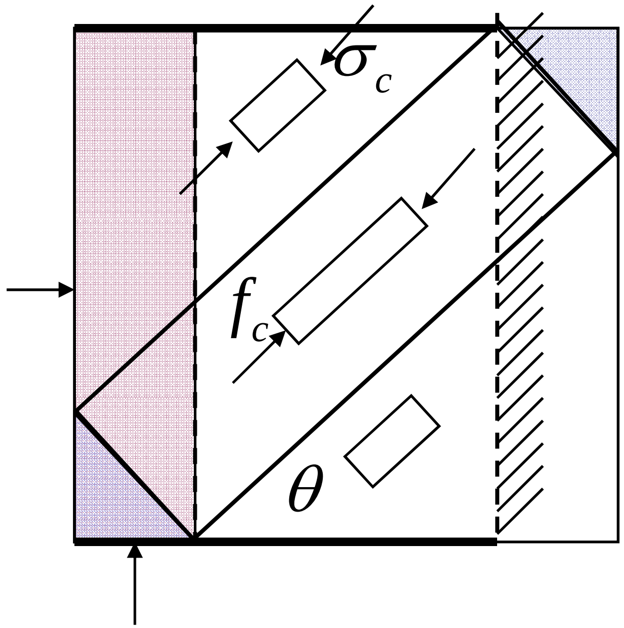


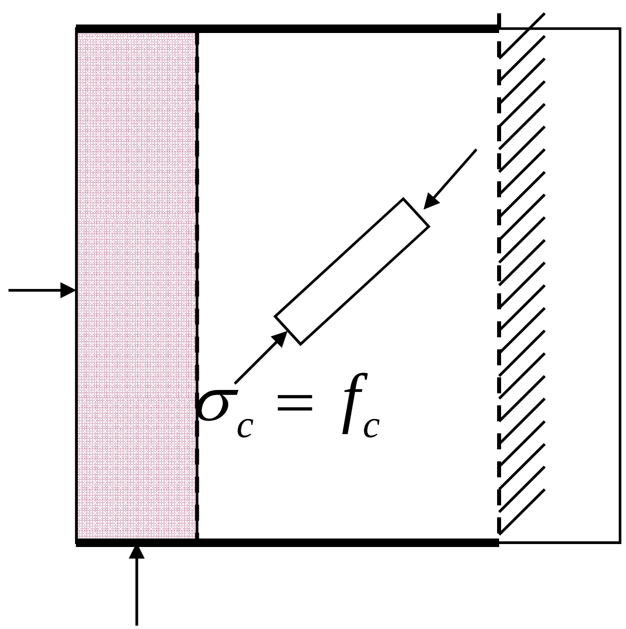


$$P_s = \frac{1}{2} t h f_c \left[\sqrt{1 + \left(\frac{a}{h}\right)^2} - \frac{a}{h} \right]$$

$$\tau_t = \frac{P - P_s}{t\left(h - y_0'\right)}$$

Diagonal Compression Field





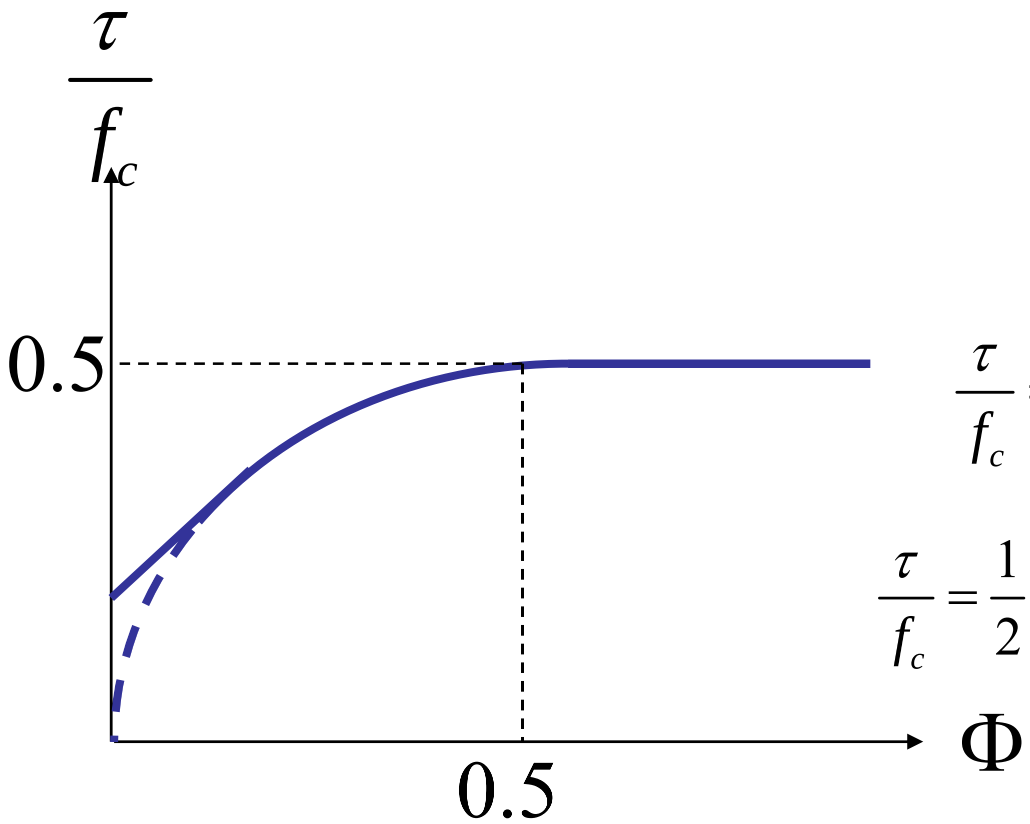
$$\sin 2\theta = \frac{2\tau}{f_c}$$

$$rf_{Yy} = \tau \tan\theta$$

$$\frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right] + \Phi_y \frac{a}{h}$$

$$\frac{\tau}{f_c} = \sqrt{\Phi_y\left(1-\Phi_y\right)}$$

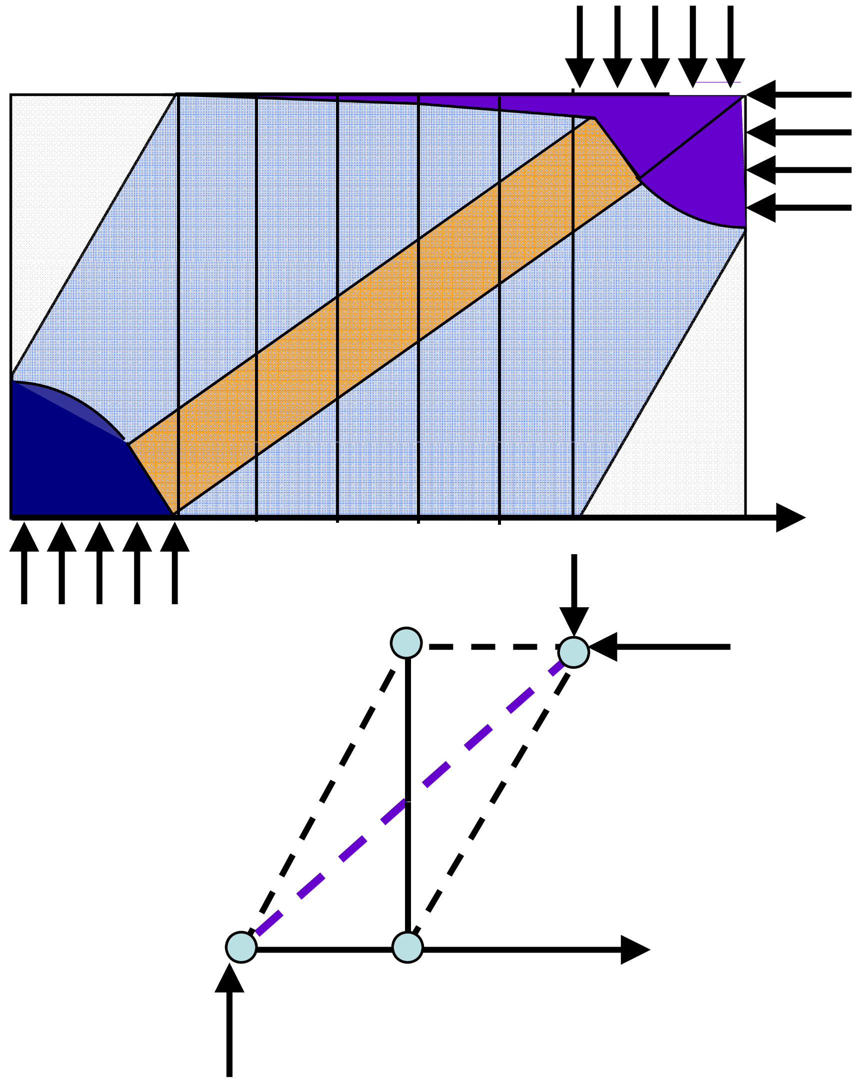
Maximum Shear Strength



$$\frac{\tau}{f_c} = \sqrt{\Phi_y\left(1-\Phi_y\right)}$$

$$\frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right] + \Phi_y\frac{a}{h}$$

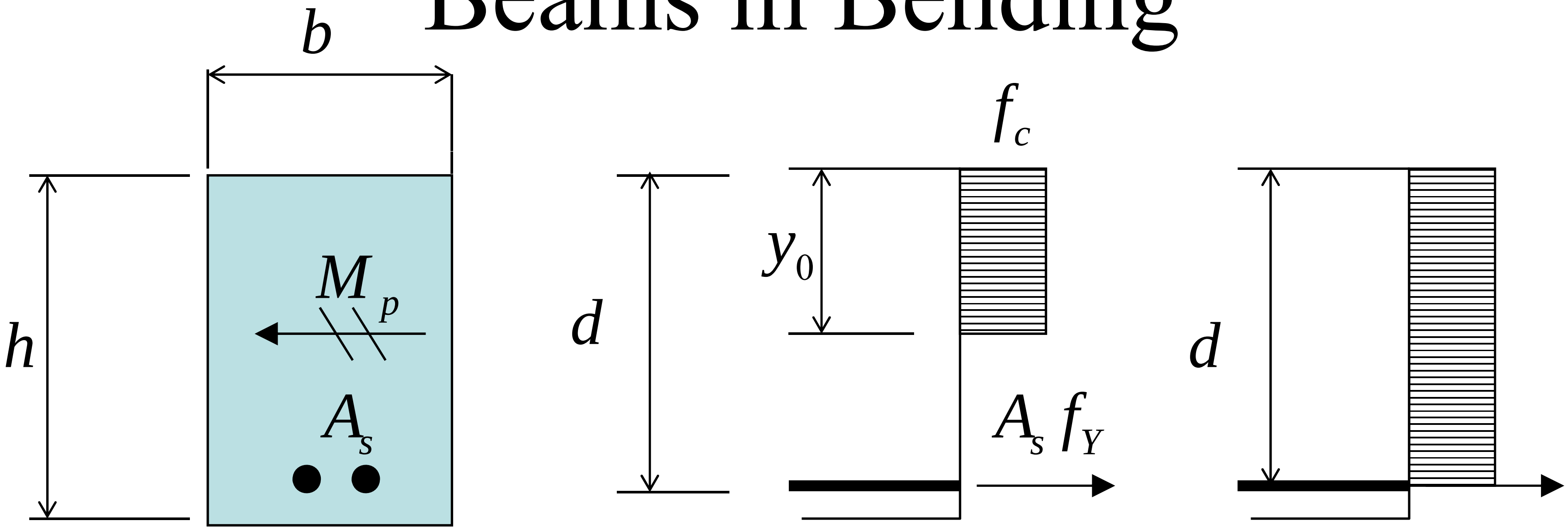
Strut with Fan-Shaped Stress Field



Chap. 5 Beams

- Beams in Bending
- Beams in Shear
- Transverse reinforcement
- Without transverse reinforcement
- Effective Concrete Compressive Strength
- Arch Actions
- Design
- With Normal Forces
- Beams in Torsion

Beams in Bending



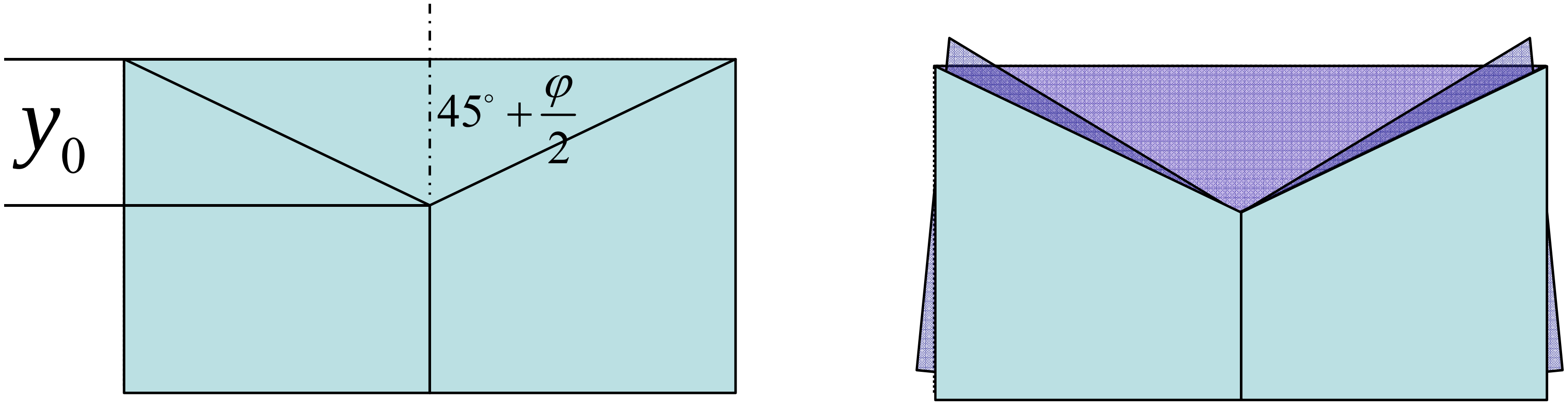
$$A_s f_Y = y_0 b f_c$$

$$\frac{y_0}{d} = \Phi$$

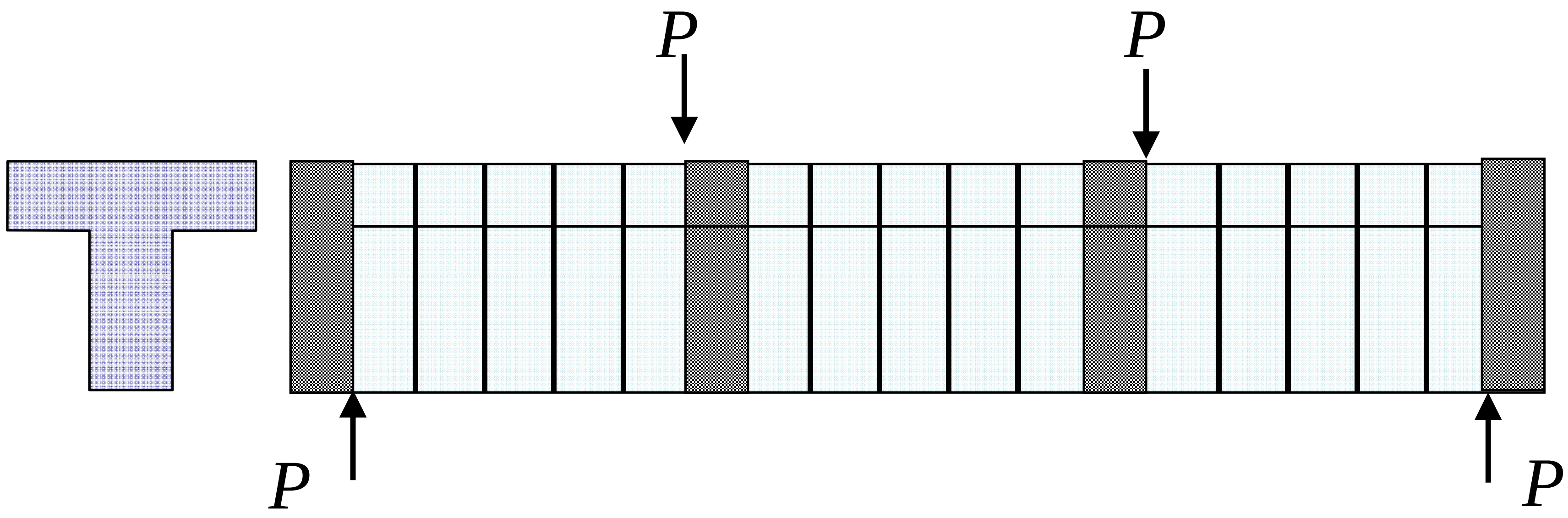
$$M_p = \left(1 - \frac{1}{2}\Phi\right)\Phi b d^2 f_c$$

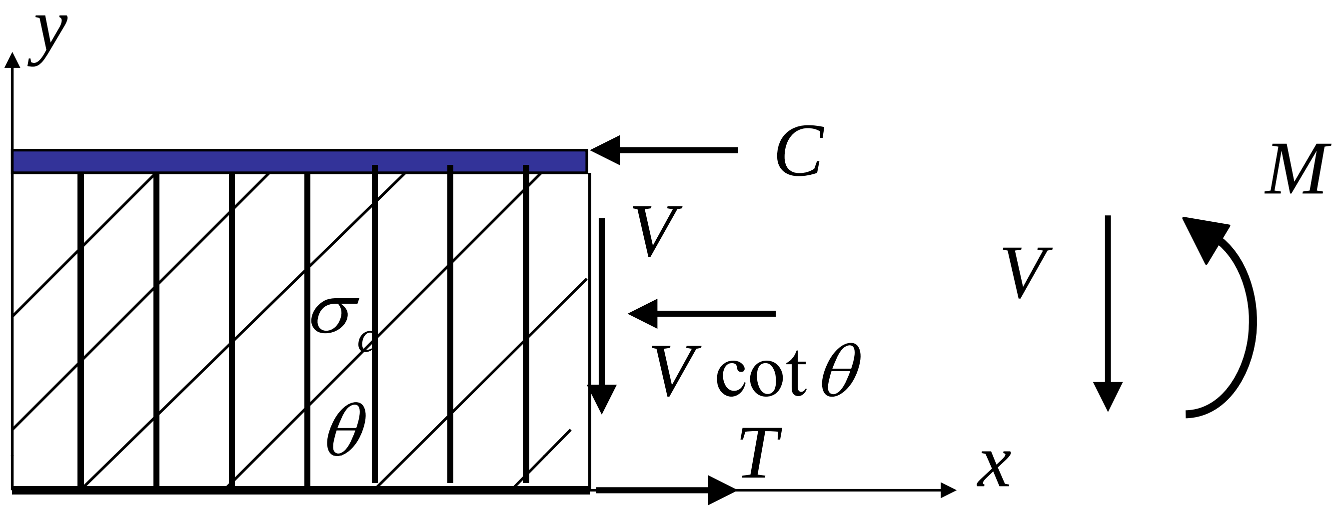
$$M_p = b d^2 f_c$$

Failure Mechanism in Bending

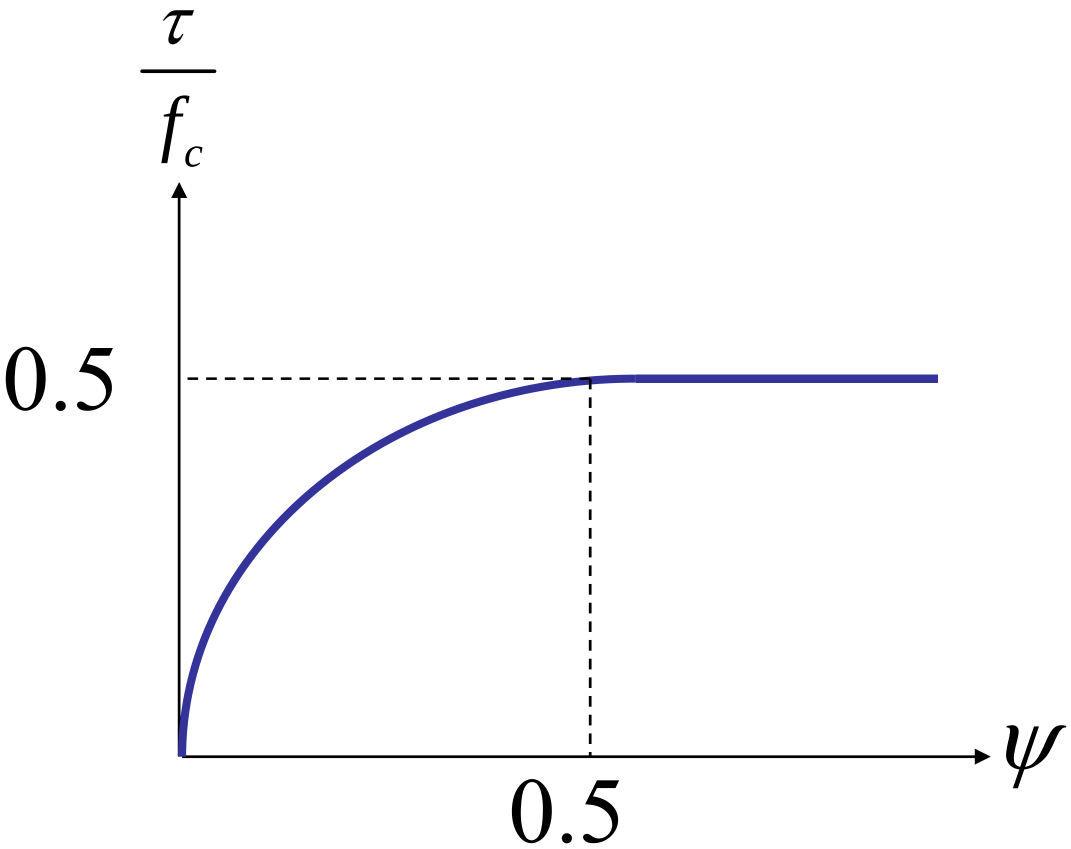


Beams in Shear



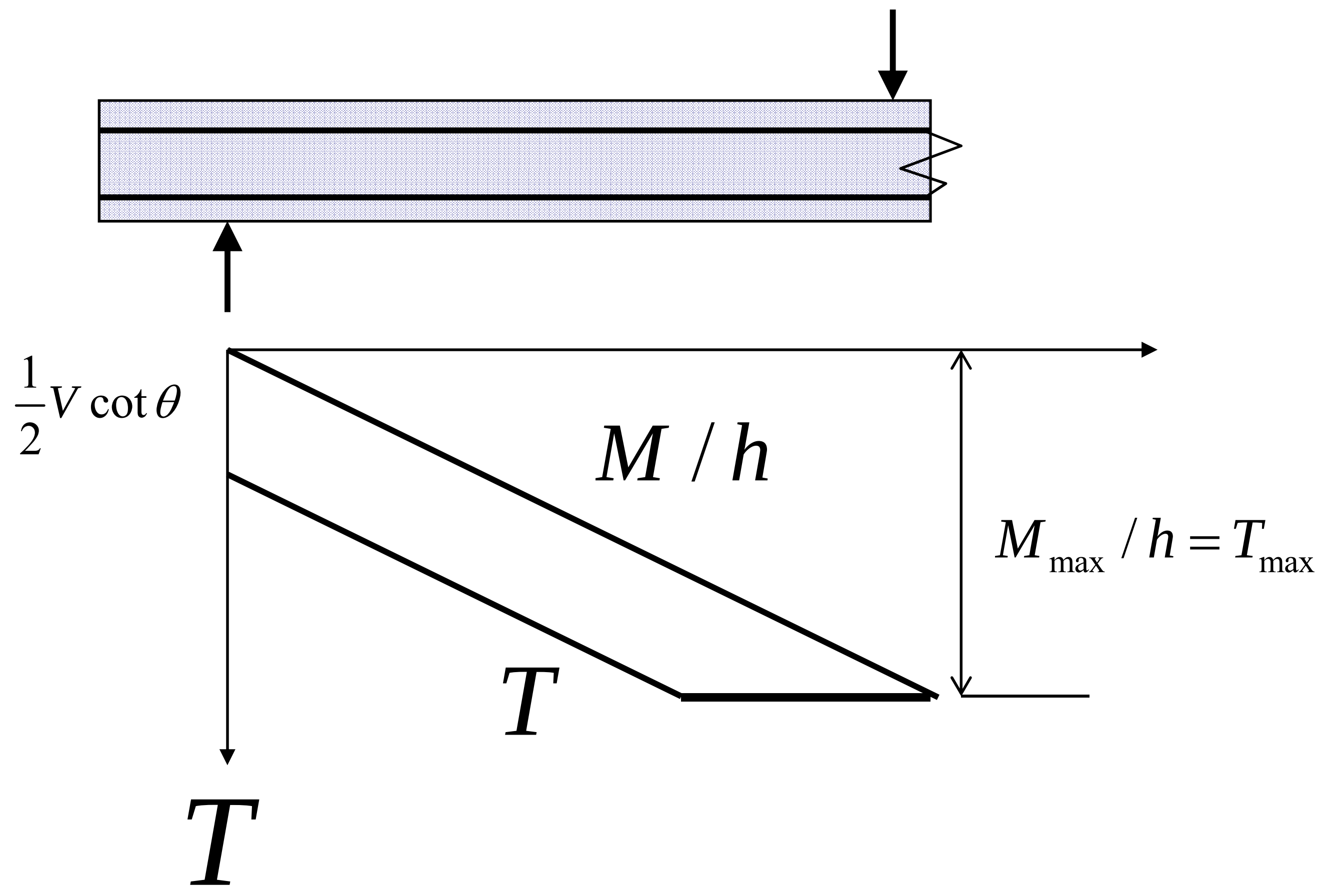


Lower Bound Solution

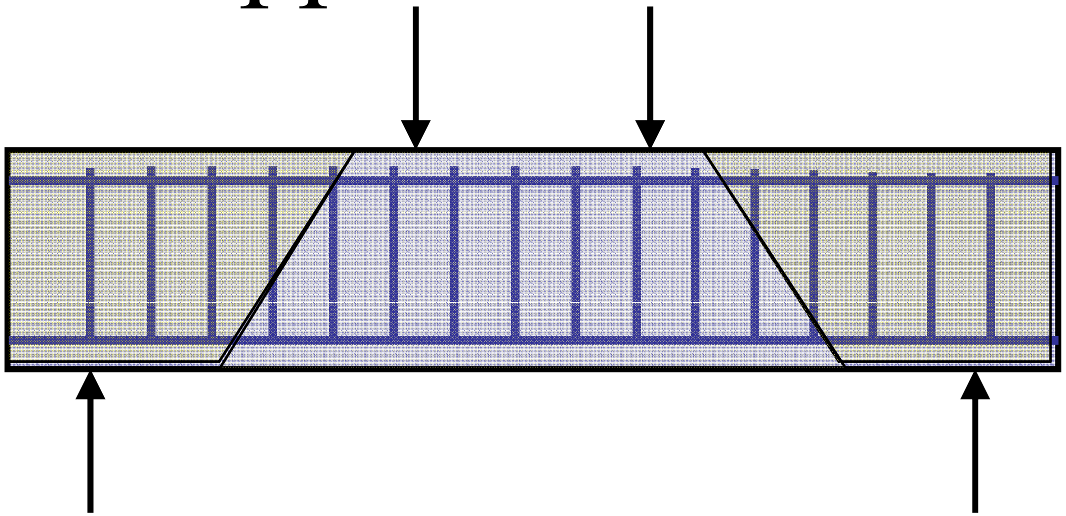


$$\frac{\tau}{f_c} = \begin{cases} \sqrt{\psi\left(1-\psi\right)} \\ \dfrac{1}{2} \end{cases}$$

Shift in Tension Force in Stringer



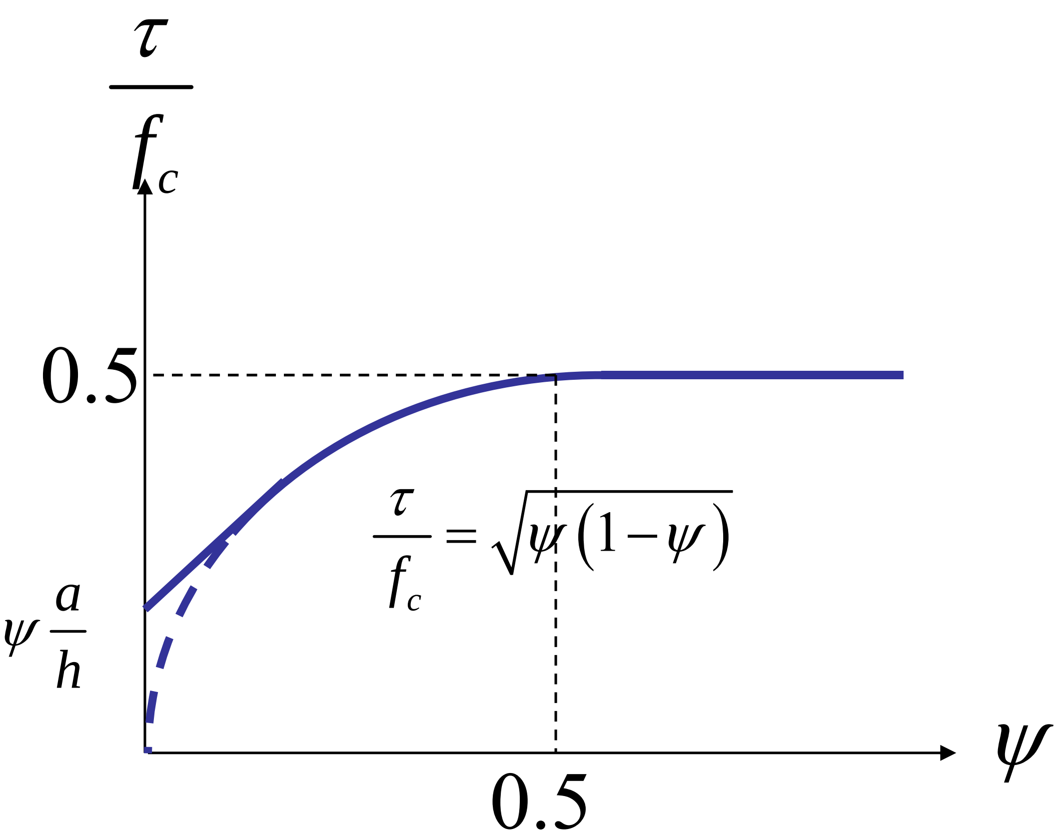
Upper Bound Solution



$$\beta = 2\theta \qquad \tan\theta = \sqrt{\frac{\psi}{1-\psi}}$$

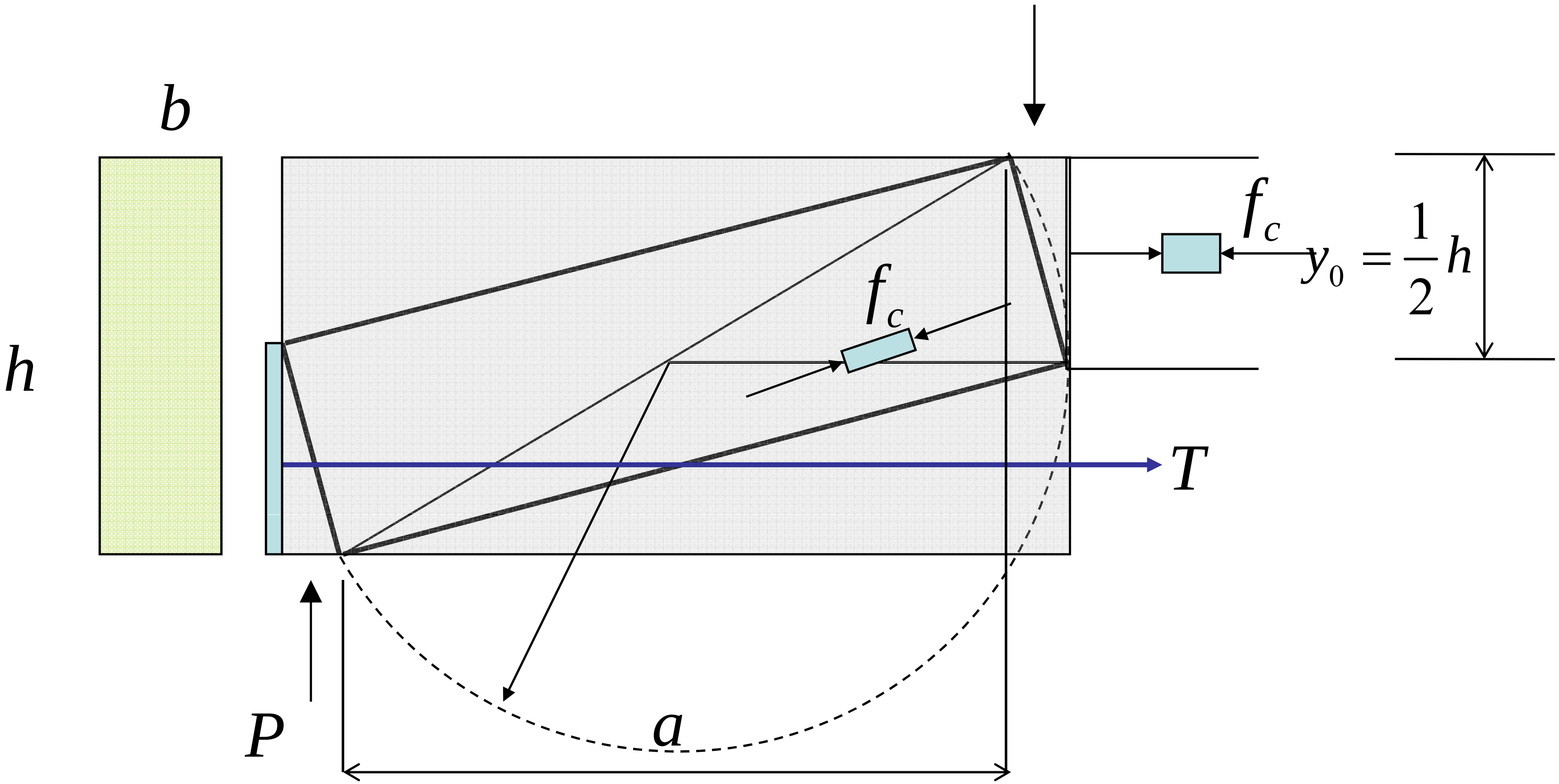
$$\frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right] + \psi\frac{a}{h}$$

Upper Bound Solutions

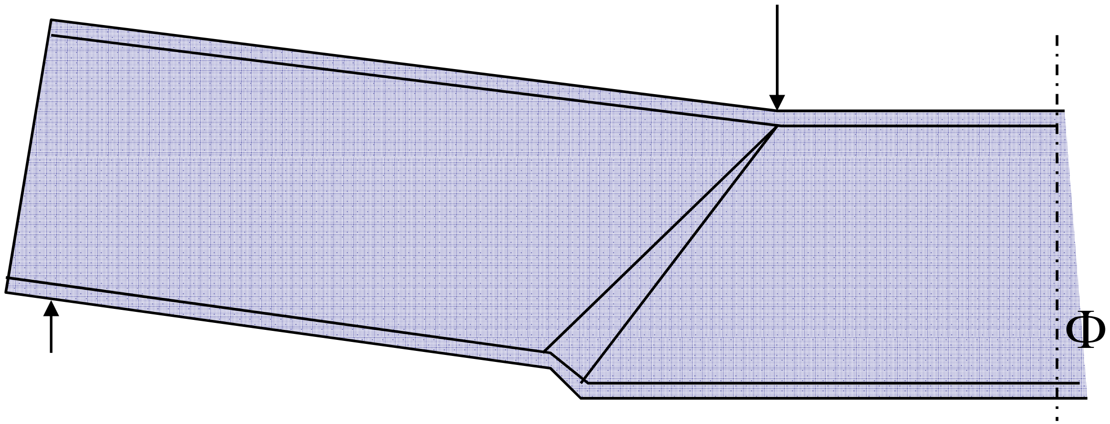


$$\frac{\tau}{f_c} = \frac{1}{2}\left[\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right] + \psi\frac{a}{h}$$

Shear Capacity without Shear Reinforcement



Influence of Longitudinal Reinforcement on Shear Capacity



$$\Phi = \frac{A_{sl} f_{Yl}}{bhf_c}$$

$$T_Y = \frac{Pa}{h} + \frac{1}{2} P \cot\theta$$

$$\tau = \frac{P}{bh} = r f_{Yw} \cot\theta$$

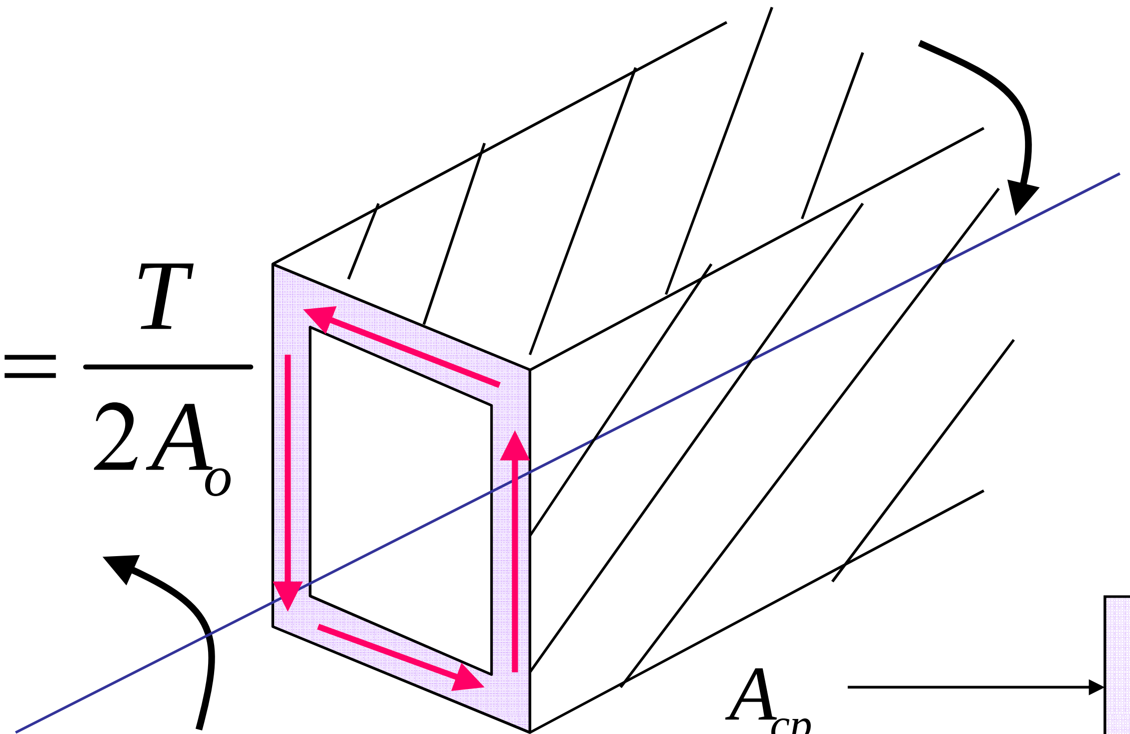
$$\frac{\tau}{f_c} = \psi \left[\sqrt{\frac{2\Phi}{\psi} + \left(\frac{a}{h}\right)^2} - \frac{a}{h} \right]$$

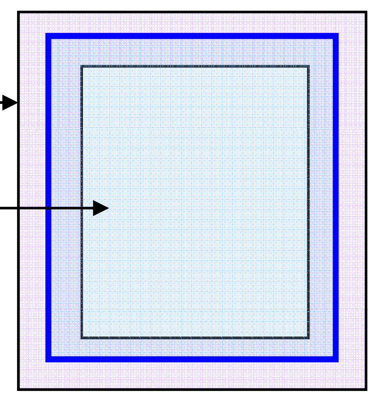
Beams in Torsion

- Space truss models
- Closed thin walled sections
- Extended application of beam models in shear

Torsional Strength Model

$$q = \frac{T}{2A_o}$$



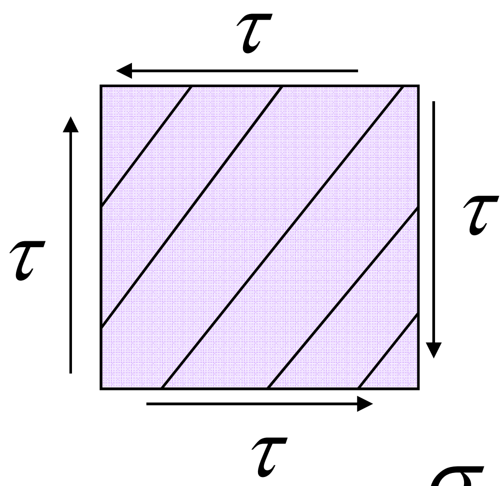


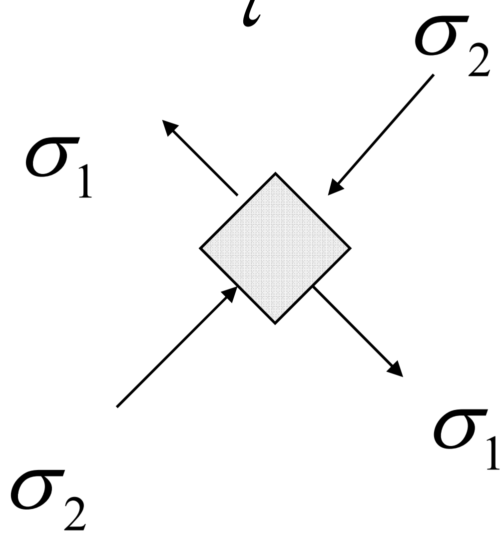
Cracking Torsional Limit

$$\sigma_1 = \tau = \frac{T}{2A_0 t}$$

$$t = \frac{3A_{cp}}{4p_{cp}}, A_0 = \frac{2}{3}A_{cp}$$

$$\sigma_1 = \tau = \frac{Tp_{cp}}{A_{cp}^2}$$





Cracking Torsion

$$\sigma_1 = \frac{1}{3}\sqrt{f_{ck}}$$

압축-인장 파괴 강도: 균열 비틀림

$$\sigma_1 = 0.7\sqrt{f_{ck}}$$

휨인장 파괴 강도: 균열 휨강도 산정

$$\sigma_1 = \frac{1}{6}\sqrt{f_{ck}}$$

압축-인장 파괴 강도: 전단 강도산정

$$T_{cr} = \frac{1}{6}\sqrt{f_{ck}}\left(\frac{A_{cp}^2}{p_{cp}}\right)$$

비틀림 효과 무시

$$T = \frac{1}{4}T_{cr}$$

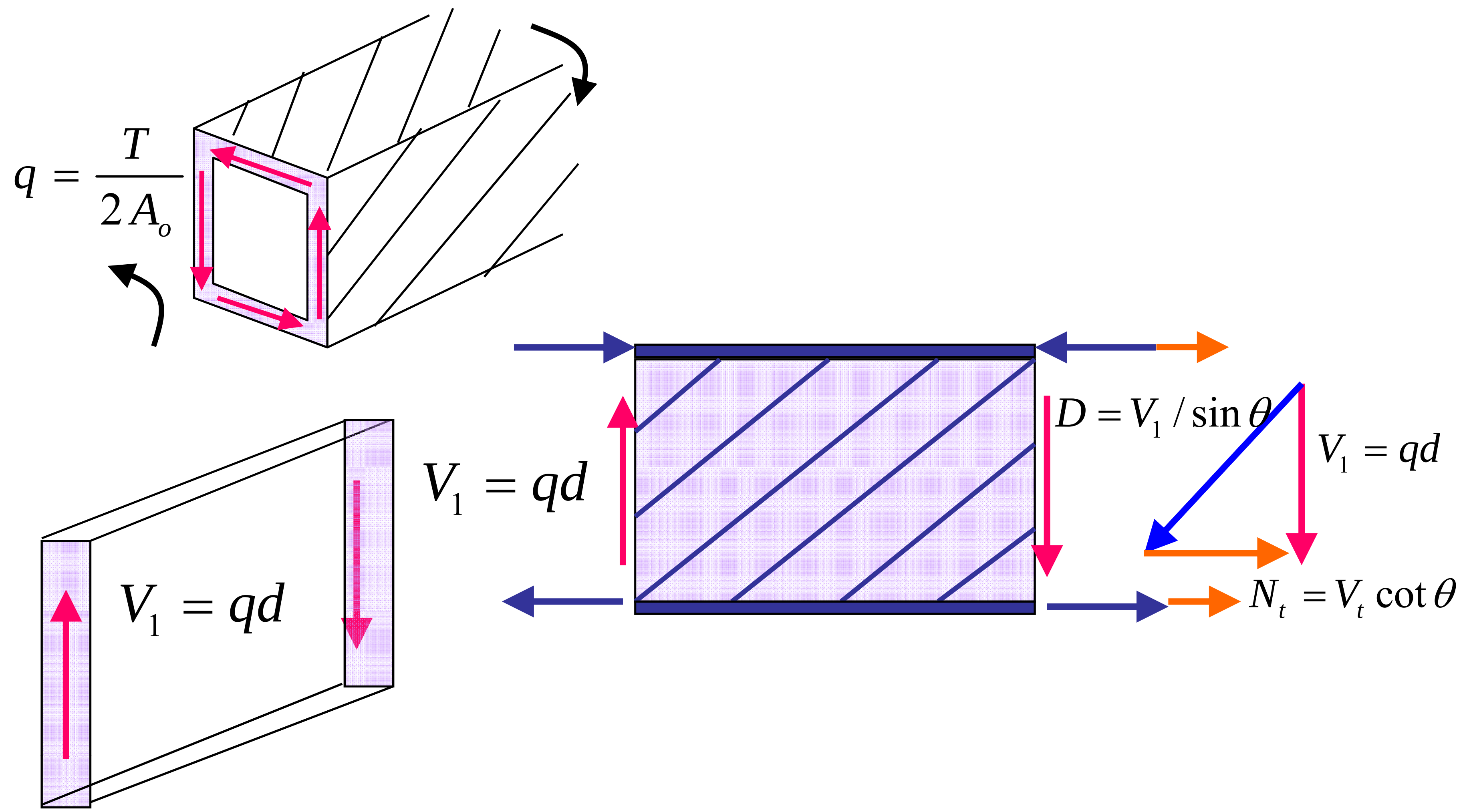
전단-비틀림 상관

$$\left(\frac{T}{T_{cr}}\right)^2 + \left(\frac{V}{V_{cr}}\right)^2 = 1$$

$$V = 0.97V_{cr}$$

$$T_{th} = \phi\frac{\sqrt{f_{ck}}}{12}\left(\frac{A_{cp}^2}{p_{cp}}\right)$$

비틀림 강도



비틀림 강도

$$T = V_1 y_o + V_2 x_o$$

$$V_2 = \frac{A_t f_{yv} y_o}{s} \cot\theta$$

$$T_n = \frac{2 A_o A_t f_{yv}}{s} \cot\theta$$

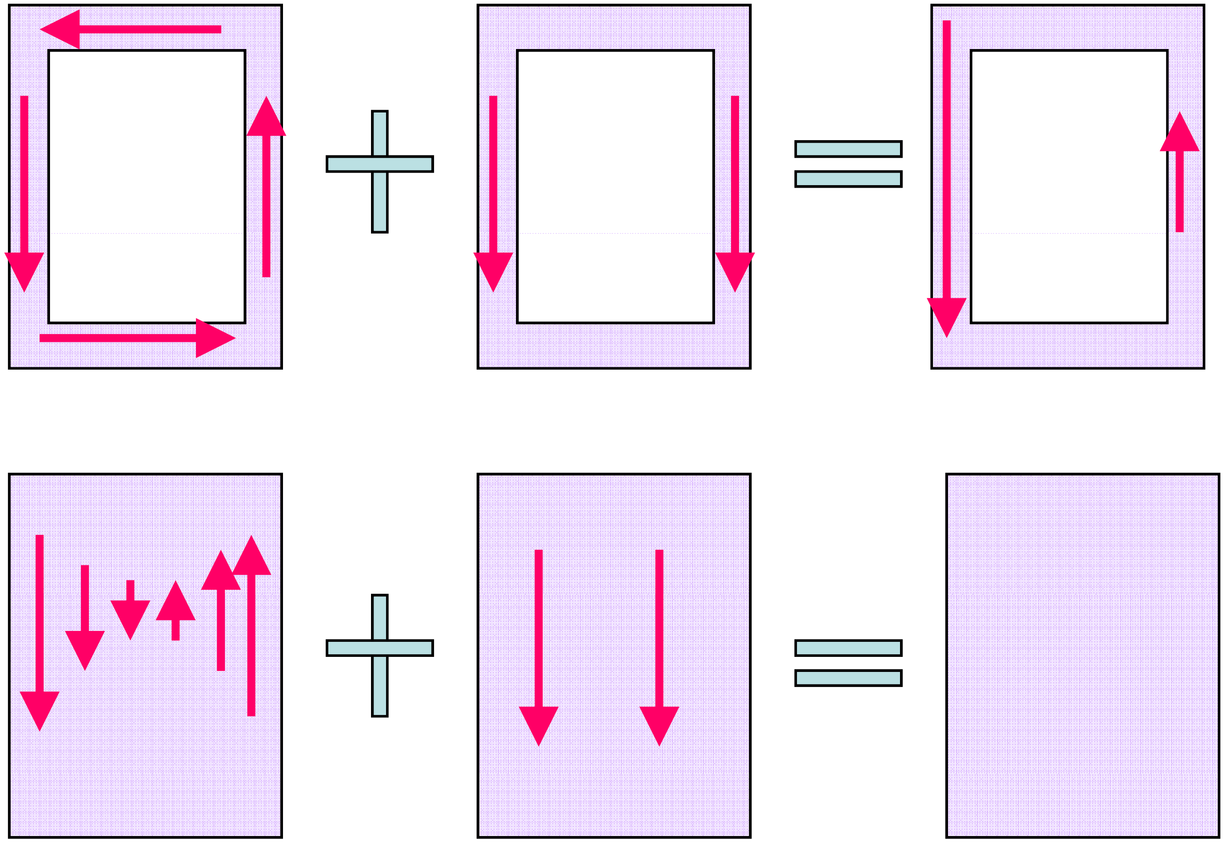
길이방향 철근

$$N = \frac{T_n}{2A_o} 2\left(x_o + y_o\right)\cot\theta$$

$$A_l f_{yl} = N$$

$$A_l = \frac{T_n p_h}{2A_o f_{fl}} \cot\theta$$

전단-비틀림 조합



Direct sum

Root-square sum

압축장의 압축파괴 한계

비틀림으로 유발된 압축응력

$$f_{cd} = \frac{V_2}{ty_o \cos\theta \sin\theta} = \frac{T_u p_h}{1.7 A_{oh}^2 \cos\theta \sin\theta}$$

전단력으로 유발된 압축응력

$$f_{cd} = \frac{V_u}{b_w d \cos\theta \sin\theta}$$

최소철근

$$A_v + 2A_t = \frac{1}{16}\sqrt{f_{ck}}\frac{b_w s}{f_{yv}} \geq \frac{b_w s}{3f_{yv}}$$

$$A_{l,\min} = \frac{5\sqrt{f_{ck}}}{12 f_{yl}}A_{cp} - \left(\frac{A_t}{s}\right)p_h\left(\frac{f_{yv}}{f_{yl}}\right)$$

Slabs

- Statical Conditions
- Geometrical Conditions
- Constitutive Equations
- Yield Zones
- Upper Bound Solutions
- Lower Bound Solutions

Upper bound Solutions

- Assume yield lines
- Work equations
- Minimize the upper bound solutions

Axes of rotations
Lines of plastic hinges
Fan mechanisms
Corner levers

Lower Bound Solutions

$$\frac{\partial^2 m_x}{\partial x^2} - 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} = -p$$

$$m_x = a + bx + cx^2$$

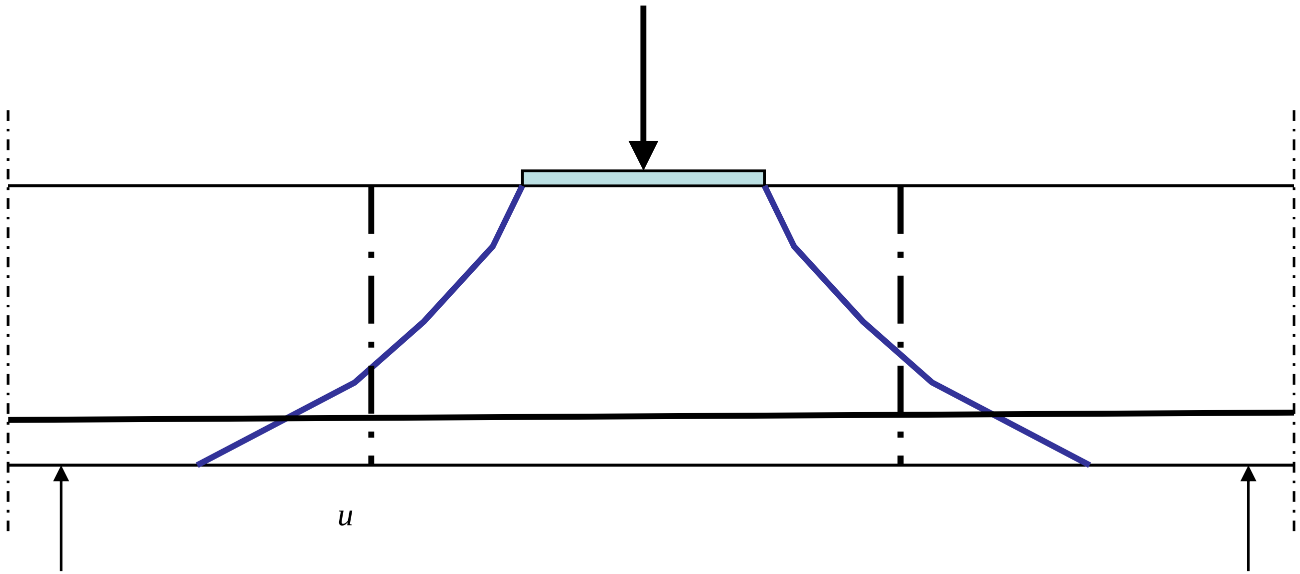
$$m_y = d + ey + fy^2$$

$$m_{xy} = g + hx + my + ixy$$

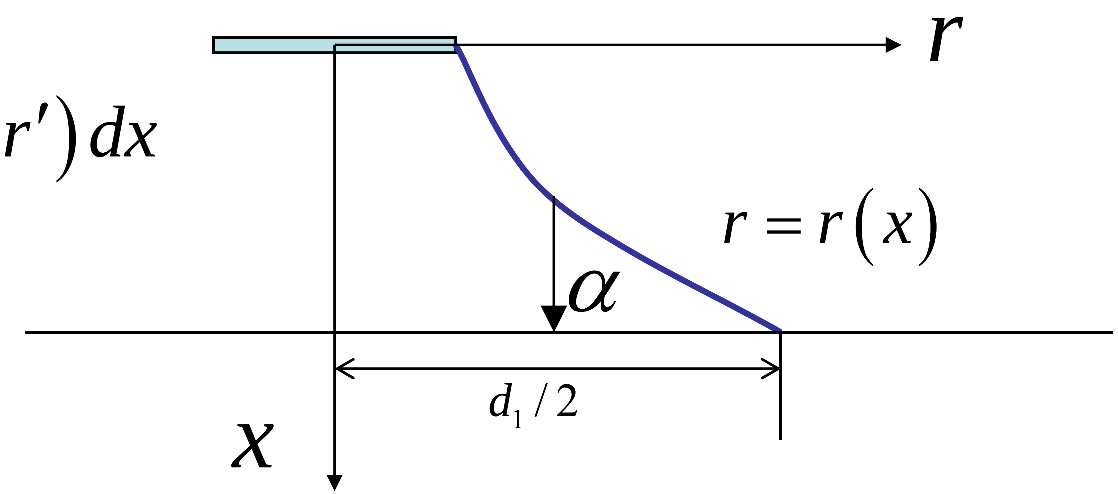
Punching Shear

- Upper Bound Solutions
- Practical Applications
- Eccentric Loading
- Effect if Counter-pressure
- Edge and Corner Loads

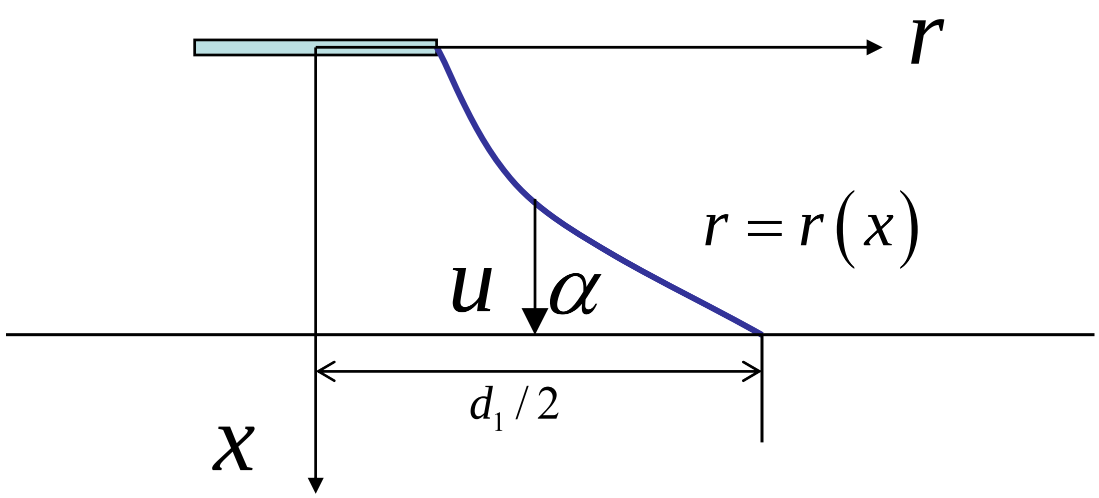
Upper Bound Solution



$$P = \pi f_c \int_0^h F\left(r, r'\right) dx$$



Shape of failure surface



$$P = \pi f_c \int_0^h F\left(r, r'\right) dx$$

Truncated cone with half-angle α_0

$$r = \frac{d}{2} + x \tan \alpha_0$$

$$P = \pi f_c \frac{h}{2} \frac{\left(d \cos \alpha_0 + h \sin \alpha_0\right)\left(l - m \sin \alpha_0\right)}{\cos^2 \alpha_0}$$

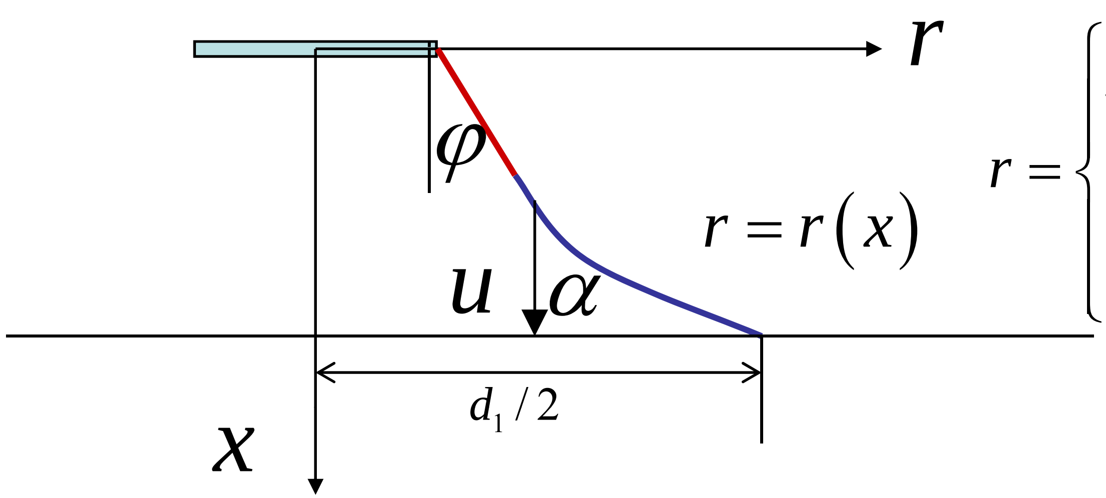
Use of Euler Equation

- Minimization of functional

$$P = \pi f_c \int_0^h F\left(r, r'\right) dx$$

- Complete solution

$$r = a\cosh\frac{x}{c} + b\sinh\frac{x}{c}$$



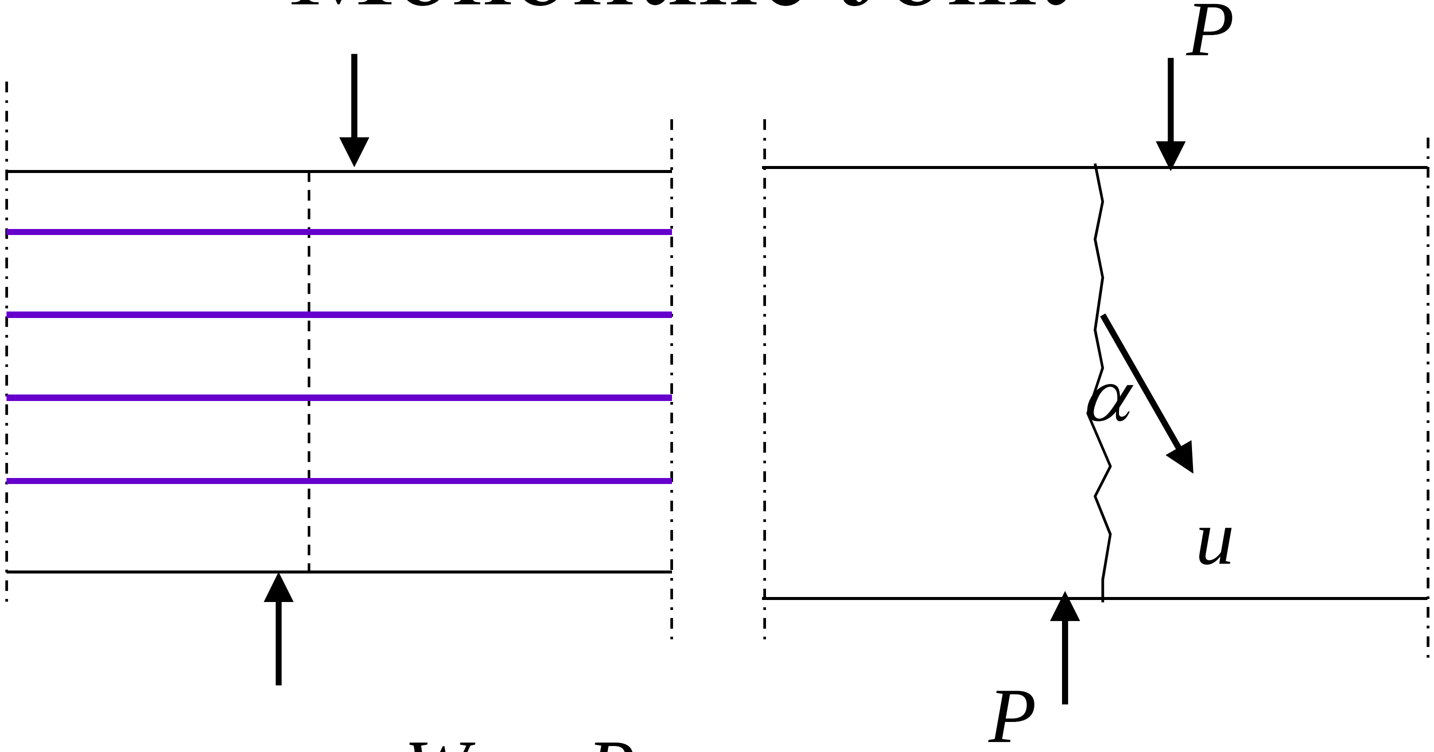
$$r = \begin{cases} \dfrac{d}{2} + x\tan\varphi \\ a\cosh\dfrac{x-h_0}{c} + b\sinh\dfrac{x-h_0}{c} \end{cases}$$

Shear in Joints

- Monolithic Concrete
- Strength of Different Types of Joints

- Crack as a Joint
- Construction Joint
- Butt Joints
- Keyed Joints

Monolithic Joint

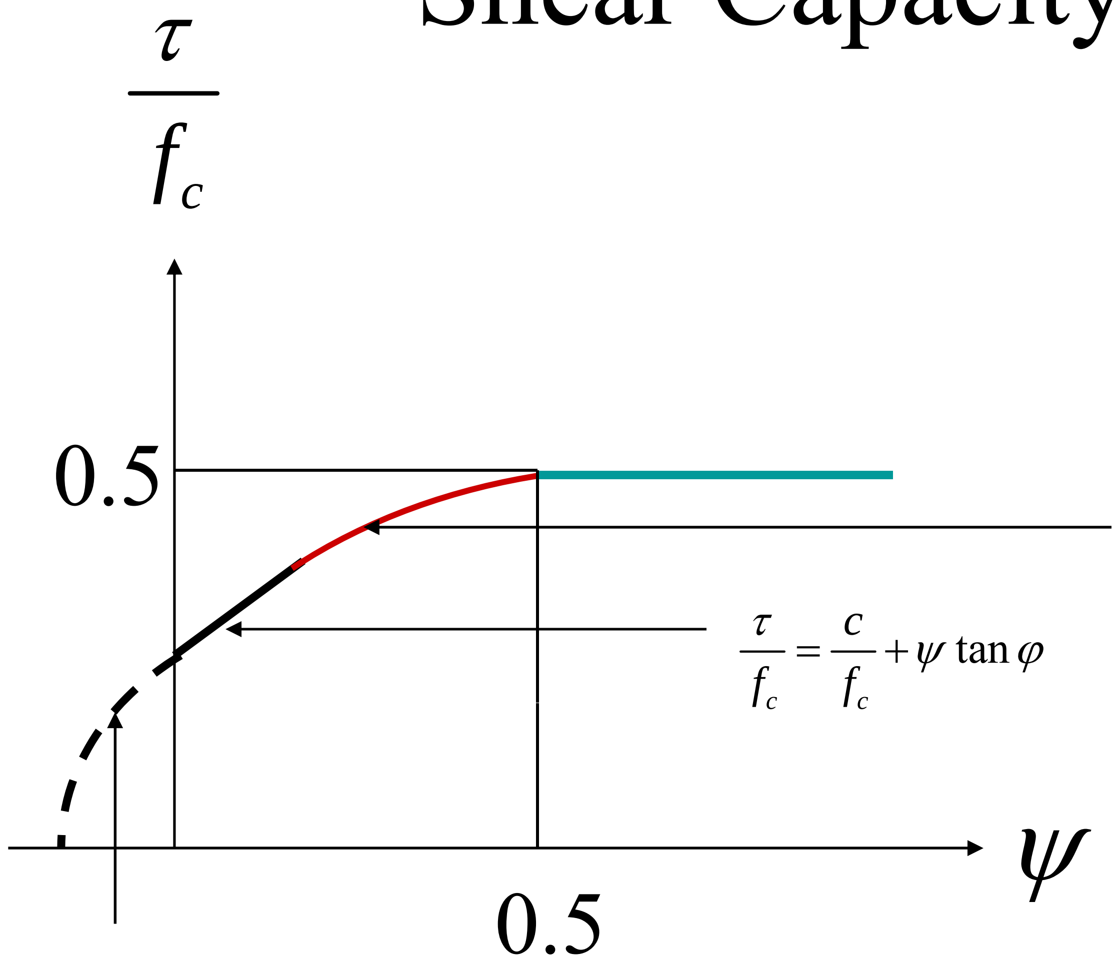


$$W_E = Pu\cos\alpha$$

$$D_R = A_s f_Y u \sin\alpha$$

$$D_c = \frac{1}{2} f_c u\left(1 - \sin\alpha\right)$$

Shear Capacity

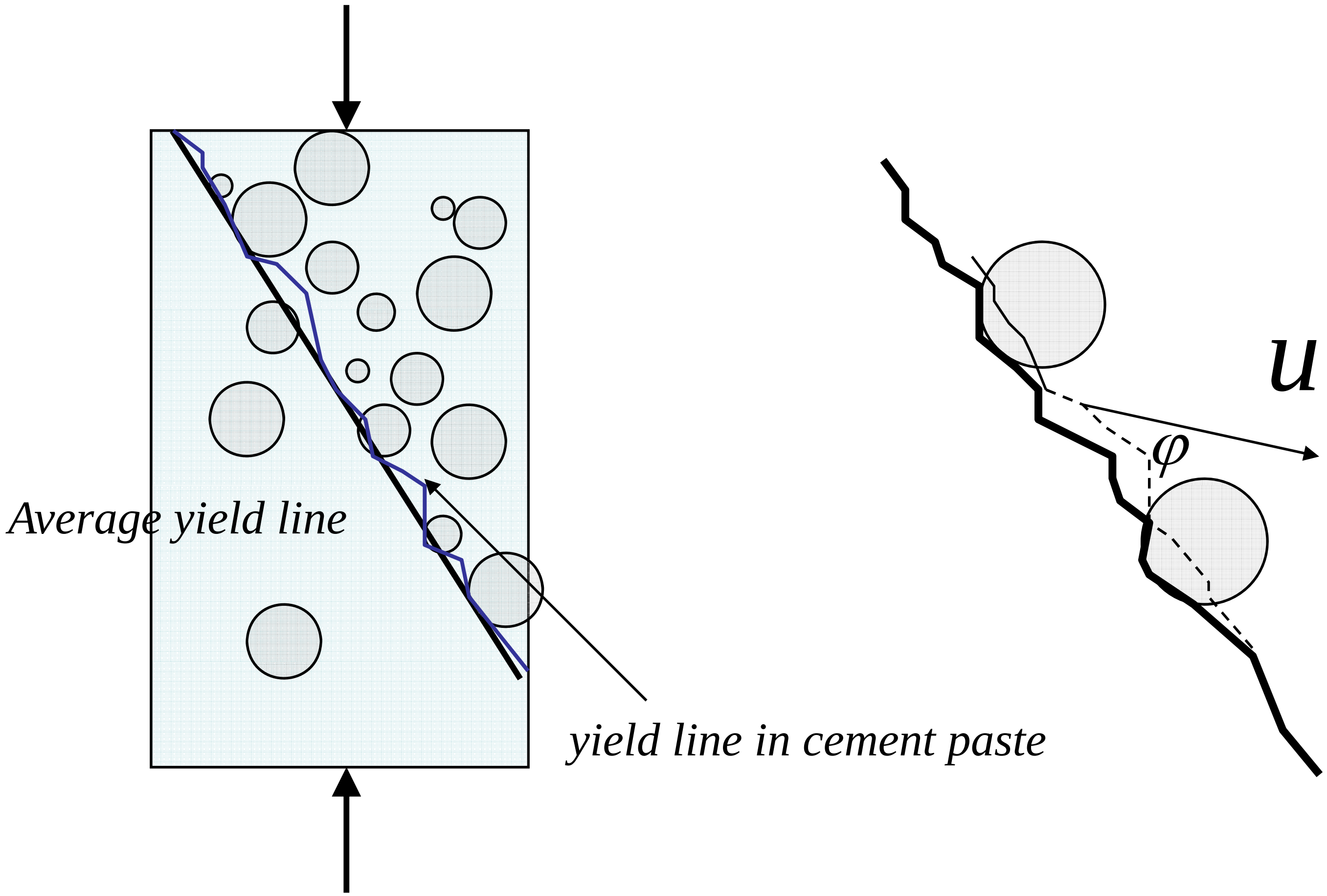


$$\frac{\tau}{f_c} = \sqrt{\psi\left(1-\psi\right)}$$

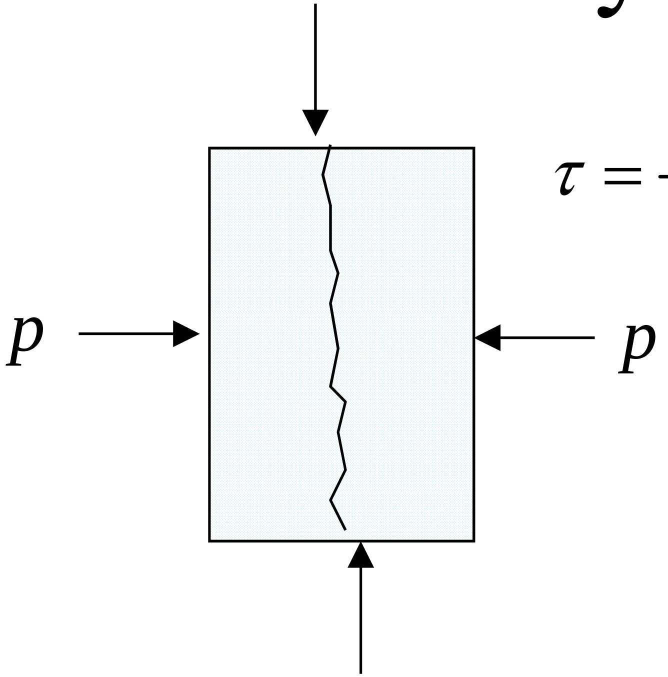
$$\frac{\tau}{f_c} = \frac{c}{f_c} + \psi \tan\varphi$$

$$\frac{\tau}{f_c} = \sqrt{\left(\psi + \frac{f_t}{f_c}\right)\left[1 - 2\frac{f_t}{f_c}\frac{\sin\varphi}{1-\sin\varphi} - \left(\psi + \frac{f_t}{f_c}\right)\right]}$$

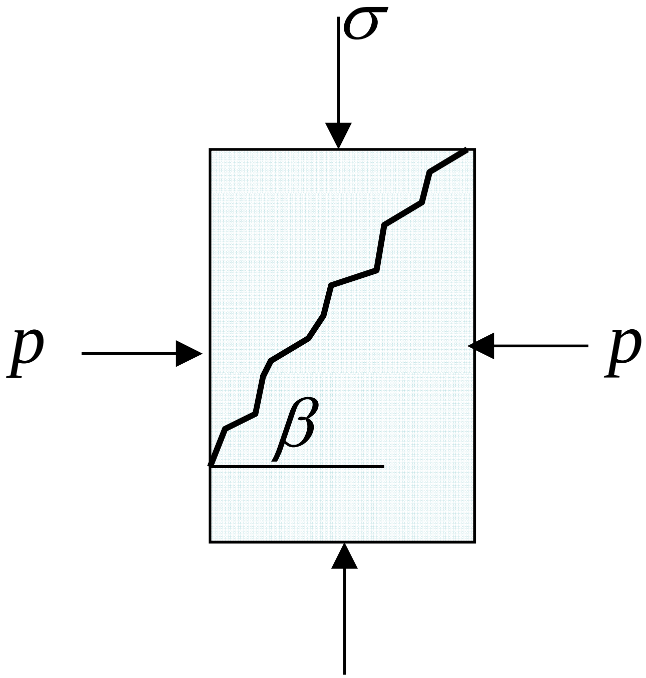
Crack as a Joint



Load carrying capacity by friction



$$\tau = \frac{p\left(\tan\alpha + \mu\right)}{1 - \mu\tan\alpha}$$

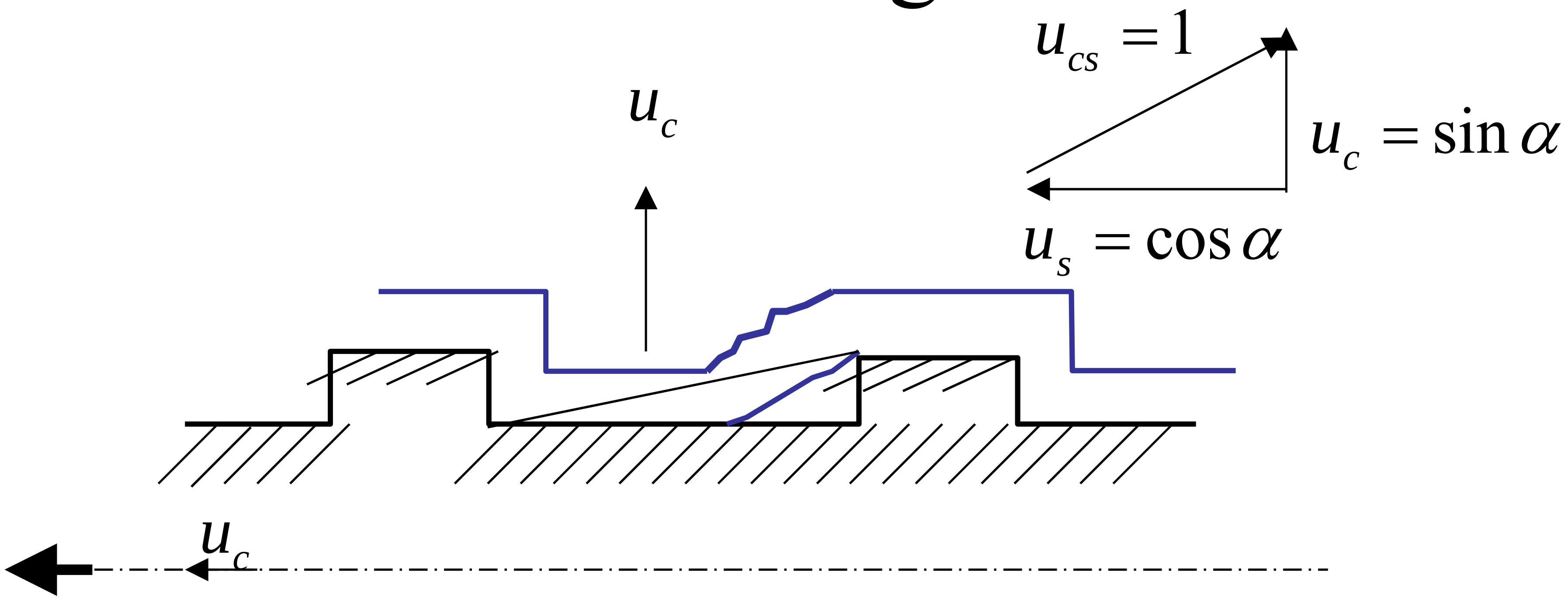


$$\sigma = p\frac{\tan\beta + \mu\tan\beta\tan\left(\beta - \alpha\right)}{\tan\left(\beta - \alpha\right) - \mu}$$

Bond Strength

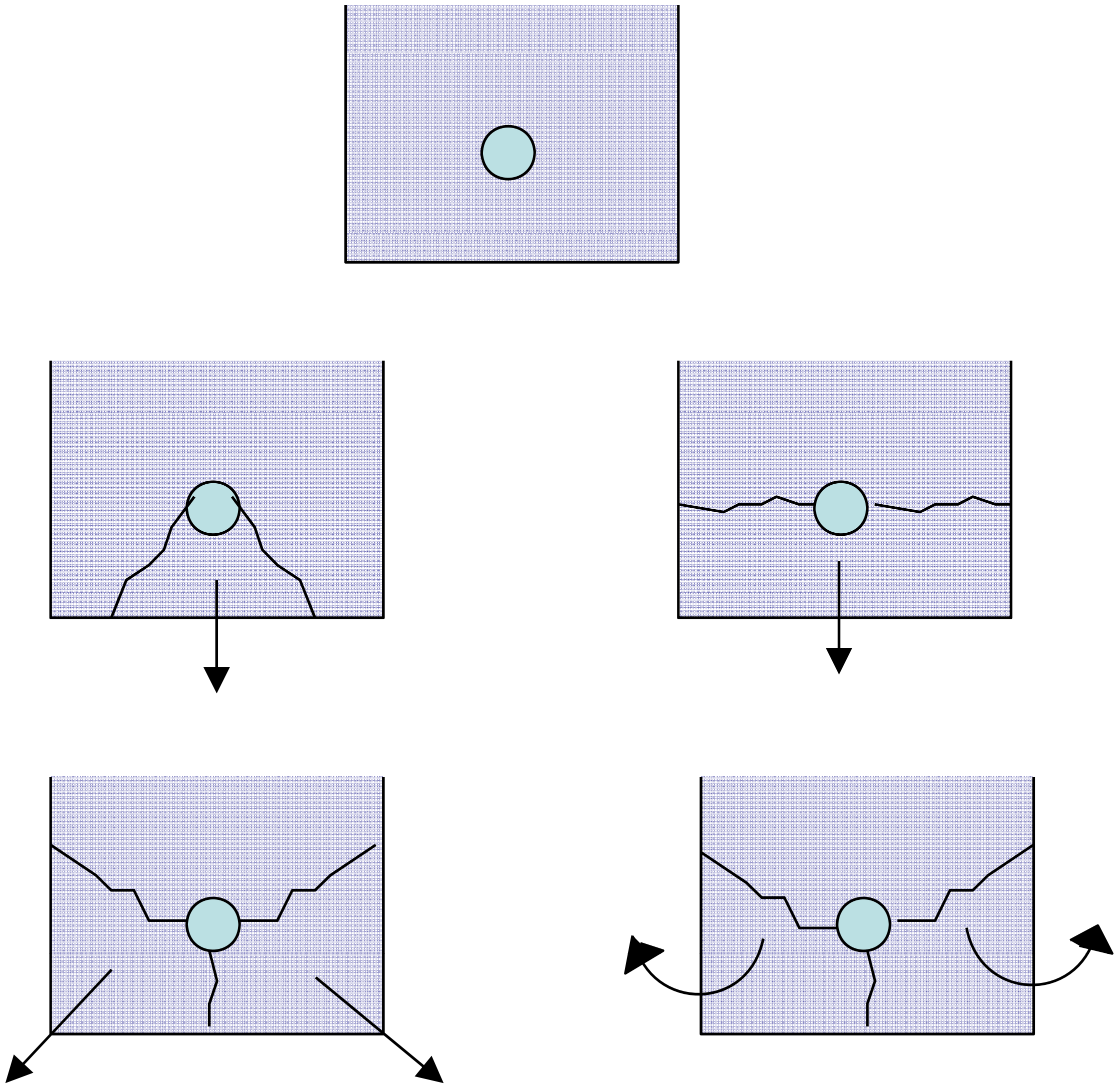
- Local Bond Strength
- Failure Mechanisms in Sections
- Effect of Transverse Reinforcement

Local Bond Strength Models



$$\frac{\tau}{f_c} = \frac{P}{\pi d l f_c} = \frac{L}{\pi d l f_c \cos\alpha} + \frac{S}{\pi n d l f_c \cos\alpha} + \frac{B}{\pi n d l f_c \cos\alpha}$$

Failure Mechanisms



Closing Remarks and Future Topics

- Compatibility based Upper bound Solutions
- Ultimate Deformation Estimation Models
- Application to Composite structures
- How to handle Brittle Failures : Fracture mechanics
- Bond Strength applications