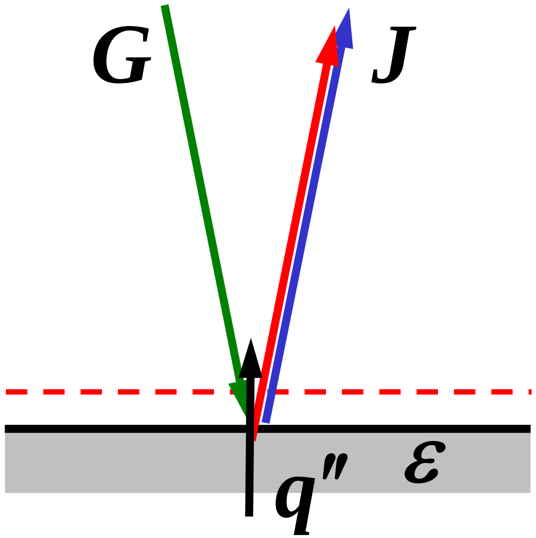
RADIATION EXCHANGE IN AN ENCLOSURE WITH DIFFUSE-GRAY SURFACES

- **Net radiation method**
- **Simplified zone analysis**

  Electric network analogy
- **Generalized zone analysis**

  Methods for solving integral equations

Net Radiation Method



diffuse-gray surface at T

net radiative heat flux

$$q'' = J - G$$

irradiation

$$G = \int_0^\infty \int_\cap i_{\lambda,i} \cos\theta_i d\omega_i d\lambda$$

radiosity $J = q''_e + q''_r$

$$= \int_0^\infty \int_\cap i_{\lambda,e} \cos\theta_e d\omega_e d\lambda + \int_0^\infty \int_\cap i_{\lambda,r} \cos\theta_r d\omega_r d\lambda$$

$$q''_e = \int_0^\infty \int_\cap i_{\lambda,e} \cos\theta_e d\omega_e d\lambda$$

$$= \int_0^\infty \int_\cap \varepsilon'_\lambda i_{\lambda b,e} \cos\theta_e d\omega_e d\lambda$$

$$= \int_0^\infty e_{\lambda b} \left[\frac{1}{\pi} \int_\cap \varepsilon'_\lambda \cos\theta_e d\omega_e \right] d\lambda$$

$$= \int_0^\infty \varepsilon_\lambda e_{\lambda b} d\lambda$$

$$= \sigma T^4 \frac{\int_0^\infty \varepsilon_\lambda e_{\lambda b} d\lambda}{\sigma T^4}$$

$$= \varepsilon \sigma T^4$$

$$q_r'' = \int_0^\infty \int_\cap i_{\lambda,r} \cos\theta_r d\omega_r d\lambda$$

$$\rho_\lambda = \frac{\int_{\cap_r} i_{\lambda,r}(\hat{\Omega}_r)\cos\theta_r d\omega_r}{\int_{\cap_i} i_{\lambda,i}(\hat{\Omega}_i)\cos\theta_i d\omega_i}$$

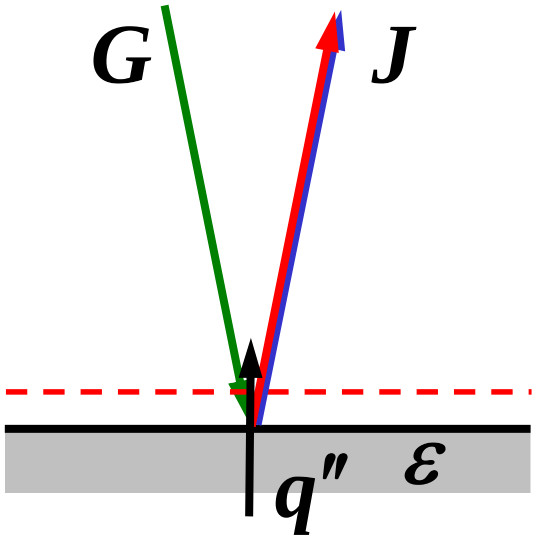
$$\int_{\cap_r} i_{\lambda,r}(\hat{\Omega}_r)\cos\theta_r d\omega_r = \rho_\lambda \int_{\cap_i} i_{\lambda,i}(\hat{\Omega}_i)\cos\theta_i d\omega_i = \rho_\lambda G_\lambda$$

$$q_r'' = \int_0^\infty \rho_\lambda G_\lambda d\lambda$$

$$\rho = \frac{\int_0^\infty \rho_\lambda G_\lambda d\lambda}{G}$$

$$q_r'' = \rho G$$

$$J = q_e'' + q_r'' = \varepsilon\sigma T^4 + \rho G$$



diffuse-gray surface at T

$$q'' = J - G$$

$$J = \varepsilon\sigma T^4 + \rho G$$

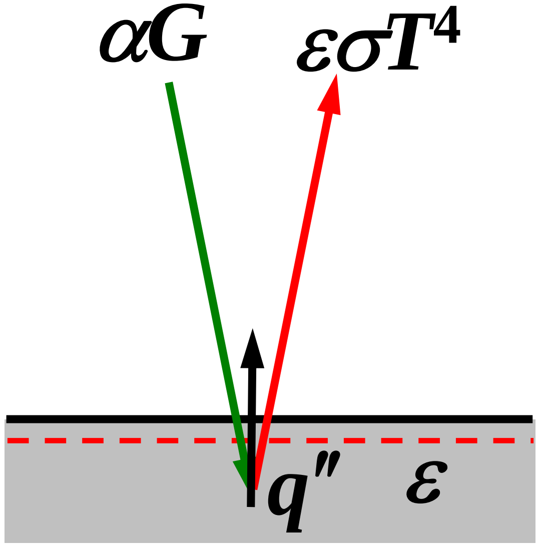
$$q'' = \varepsilon\sigma T^4 + \rho G - G$$

$$= \varepsilon\sigma T^4 - (1-\rho)G$$

$$= \varepsilon(\sigma T^4 - G)$$

$$q'' = J - \frac{1}{\rho}(J - \varepsilon\sigma T^4) = \frac{1}{\rho}(\rho J - J + \varepsilon\sigma T^4)$$

$$= \frac{1}{\rho}(\varepsilon\sigma T^4 - \varepsilon J) = \frac{\varepsilon}{1-\varepsilon}(\sigma T^4 - J)$$



diffuse-gray surface at T

$$q'' = \varepsilon\sigma T^4 - \alpha G$$

$$= \varepsilon\sigma T^4 - \varepsilon G$$

$$= \varepsilon\left(\sigma T^4 - G\right)$$

$$J = \varepsilon\sigma T^4 + \rho G$$

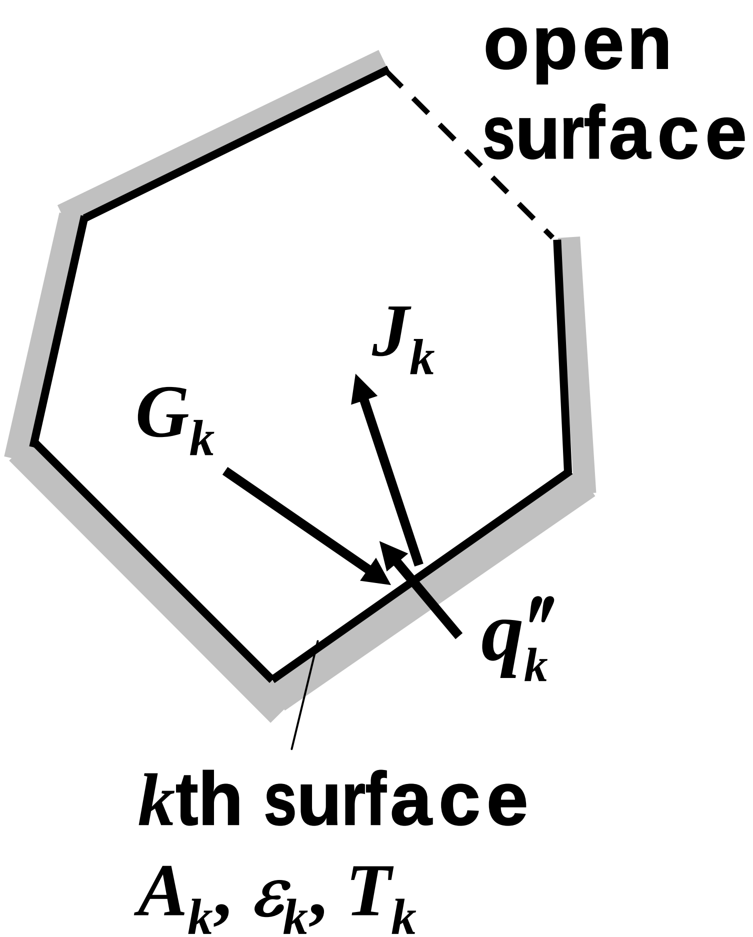
$$G = \frac{1}{\rho}\left(J - \varepsilon\sigma T^4\right)$$

$$q'' = \varepsilon\sigma T^4 - \frac{\alpha}{\rho}\left(J - \varepsilon\sigma T^4\right) = \varepsilon\sigma T^4 - \frac{\varepsilon}{1-\varepsilon}\left(J - \varepsilon\sigma T^4\right)$$

$$= \frac{\varepsilon}{1-\varepsilon}\left(\sigma T^4 - J\right)$$

Simplified Zone Analysis

Enclosure with n surfaces

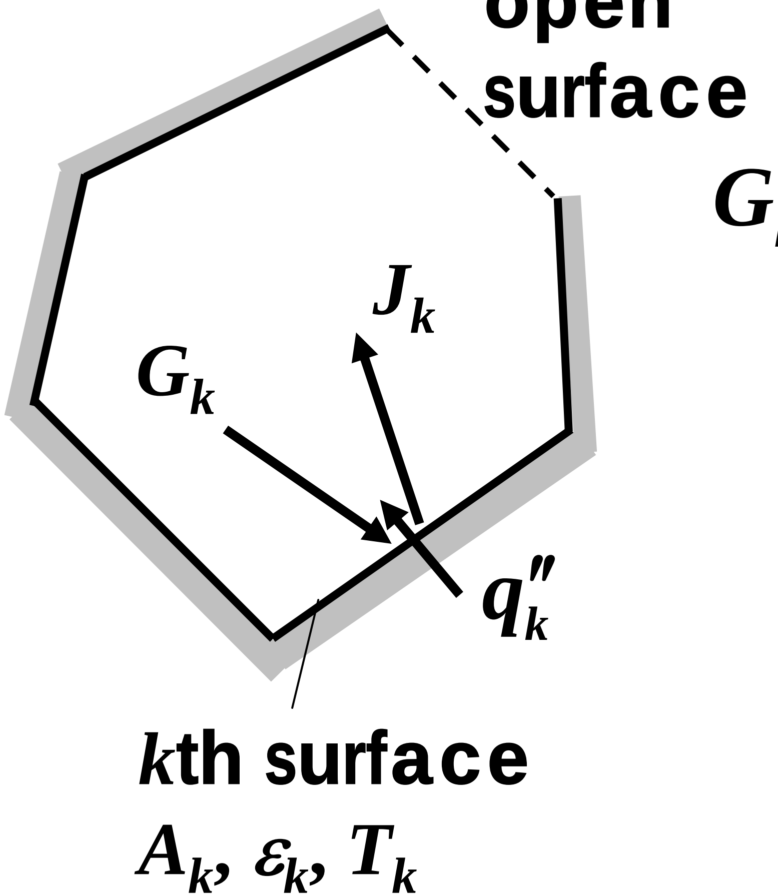


temperature, properties, radiosity, irradiation: uniform over each surface

$$q''_k = J_k - G_k$$

$$J_k = \varepsilon_k \sigma T_k^4 + (1 - \varepsilon_k) G_k$$

$$q''_k = \frac{\varepsilon_k}{1 - \varepsilon_k} \left(\sigma T_k^4 - J_k \right)$$



all irradiation from n surfaces

$$G_k A_k = J_1 A_1 F_{1k} + J_2 A_2 F_{2k} + \cdots + J_n A_n F_{nk}$$

$$= J_1 A_k F_{k1} + J_2 A_k F_{k2} + \cdots + J_n A_k F_{kn}$$

$$= \sum_{i=1}^{n} J_i A_k F_{ki} = A_k \sum_{i=1}^{n} J_i F_{ki}$$

$$G_k = \sum_{i=1}^{n} J_i F_{ki}$$

Summary

$$q''_k = \frac{\varepsilon_k}{1-\varepsilon_k}\left(\sigma T_k^4 - J_k\right)$$

$$J_k = \varepsilon_k \sigma T_k^4 + \left(1-\varepsilon_k\right)\sum_{i=1}^{n} J_i F_{ki}$$

$$k = 1, 2, 3, \cdots, n$$

When T_k or q''_k are specified at the boundary

$2n$ **unknowns: J_k and q''_k or T_k**

When all T_k's are specified, the two equations are decoupled.

n **unknowns: J_k**

Electric Network Analogy

$$q_k'' = \frac{\varepsilon_k}{1-\varepsilon_k}\left(\sigma T_k^4 - J_k\right), \quad G_k = \sum_{i=1}^{n} J_i F_{ki}$$

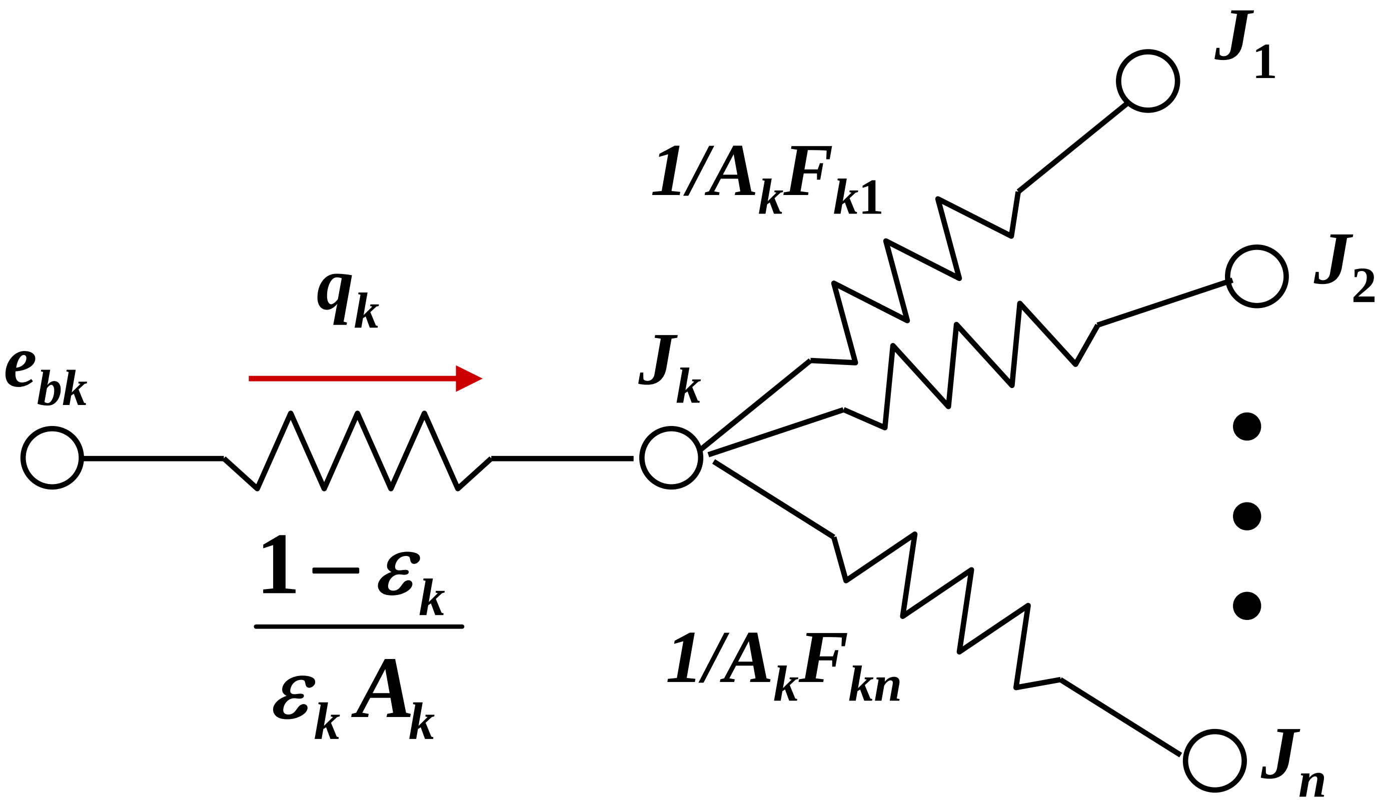
$$q_k = q_k'' A_k = \frac{\sigma T_k^4 - J_k}{\dfrac{1-\varepsilon_k}{\varepsilon_k A_k}} \equiv \frac{e_{bk} - J_k}{R}$$

$$= A_k\left(J_k - G_k\right) = A_k\left(J_k \sum_{i=1}^{n} F_{ki} - \sum_{i=1}^{n} J_i F_{ki}\right)$$

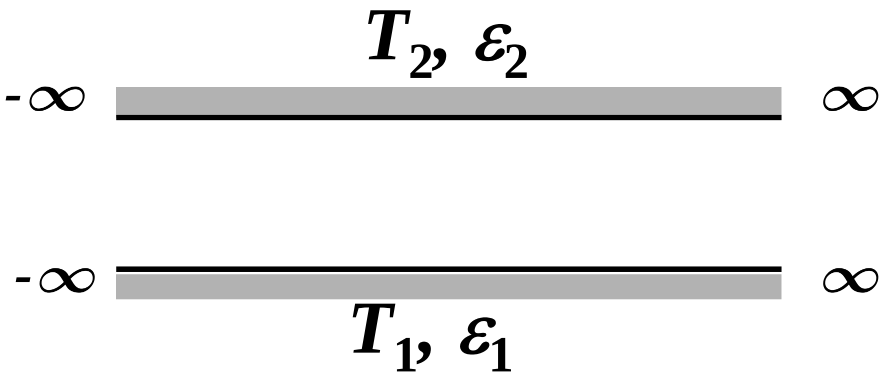
$$= A_k \sum_{i=1}^{n}\left(J_k F_{ki} - J_i F_{ki}\right) = \sum_{i=1}^{n} A_k F_{ki}\left(J_k - J_i\right)$$

$$= \sum_{i=1}^{n} \frac{J_k - J_i}{1/A_k F_{ki}}$$

$$q_k = \frac{e_{bk} - J_k}{\dfrac{1-\varepsilon_k}{\varepsilon_k A_k}} = \sum_{i=1}^{n} \frac{J_k - J_i}{\dfrac{1}{A_k F_{ki}}}$$



Ex 7-6 two infinite parallel plates



$$q_k'' = \frac{\varepsilon_k}{1-\varepsilon_k}\left(\sigma T_k^4 - J_k\right)$$

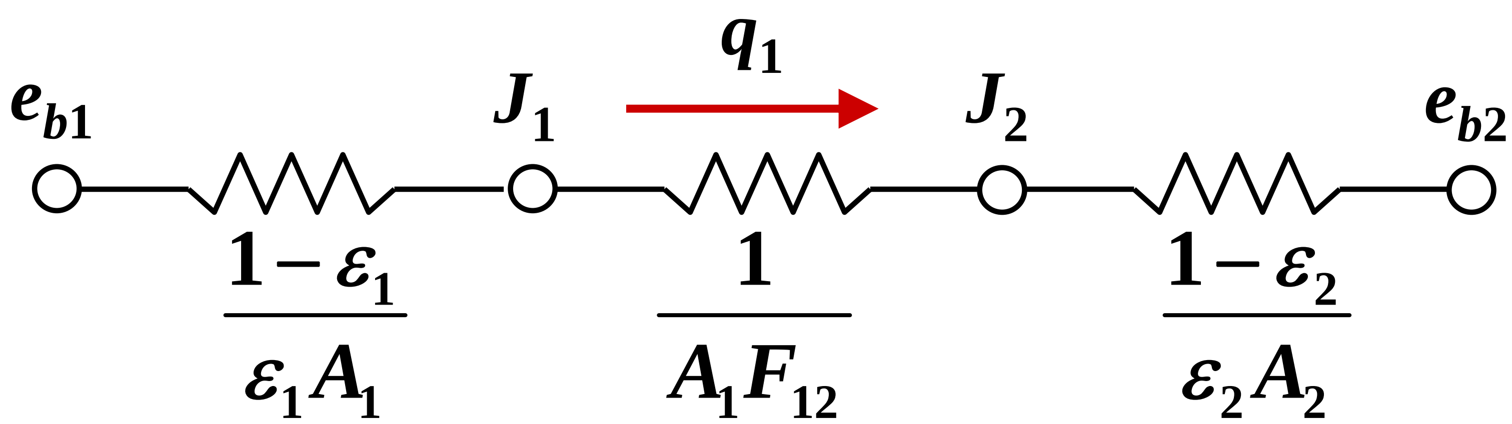
$$J_k = \varepsilon_k \sigma T_k^4 + \left(1-\varepsilon_k\right)\sum_{i=1}^{n} J_i F_{ki}$$

$$q_1'' = \frac{\varepsilon_1}{1-\varepsilon_1}\left(\sigma T_1^4 - J_1\right), \quad q_2'' = \frac{\varepsilon_2}{1-\varepsilon_2}\left(\sigma T_2^4 - J_2\right)$$

$$J_1 = \varepsilon_1 \sigma T_1^4 + \left(1-\varepsilon_1\right) J_2, \quad J_2 = \varepsilon_2 \sigma T_2^4 + \left(1-\varepsilon_2\right) J_1$$

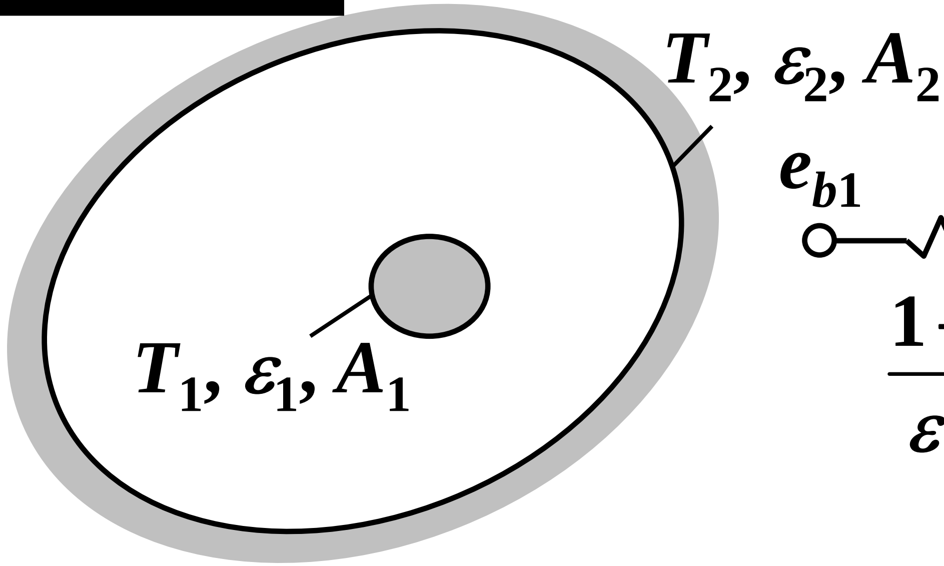
$$q_1'' = -q_2'' = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

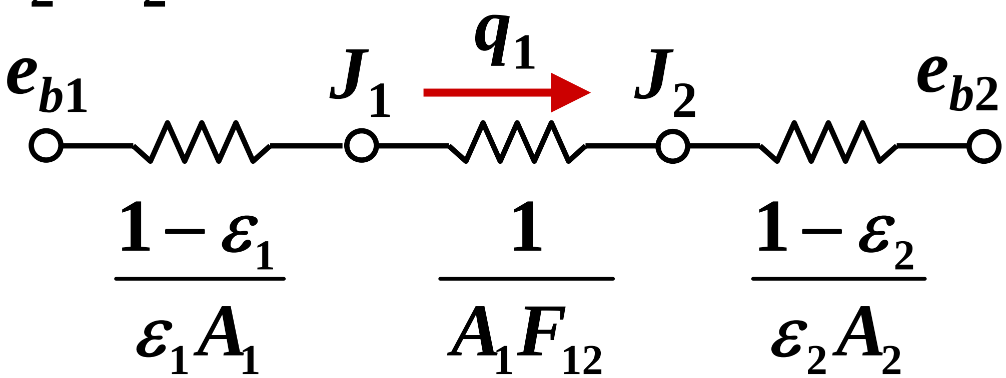
Using network analogy



$$q_1'' = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{\varepsilon_1} + 1 + \frac{1-\varepsilon_2}{\varepsilon_2}} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

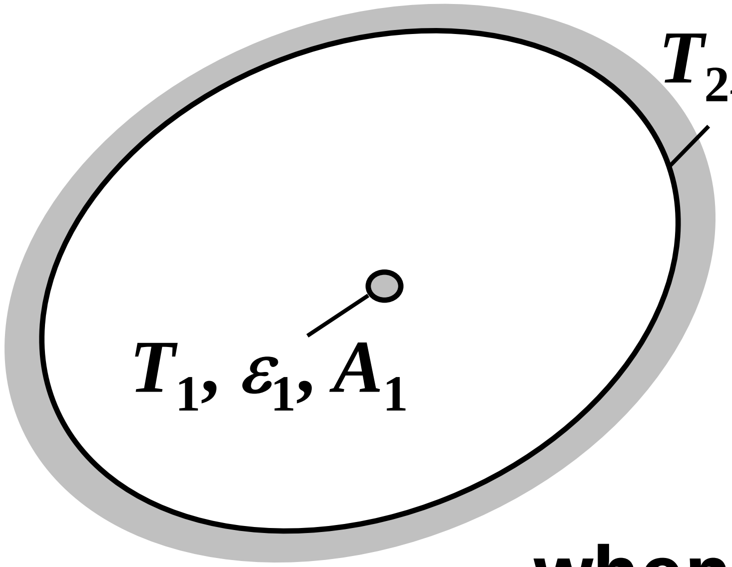
Ex 7-8 a body in an enclosure





$$q_1 = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1}{A_1} + \frac{1-\varepsilon_2}{A_2\varepsilon_2}} = \frac{A_1\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{\varepsilon_1} + 1 + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)}$$

$$= \frac{A_1\sigma\left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)}$$



$$q_1 = \frac{A_1\sigma\left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\varepsilon_2} - 1\right)}$$

when $\frac{A_1}{A_2} \ll 1,\ q_1 = \varepsilon_1\sigma A_1\left(T_1^4 - T_2^4\right)$

The enclosure acts like a black cavity.

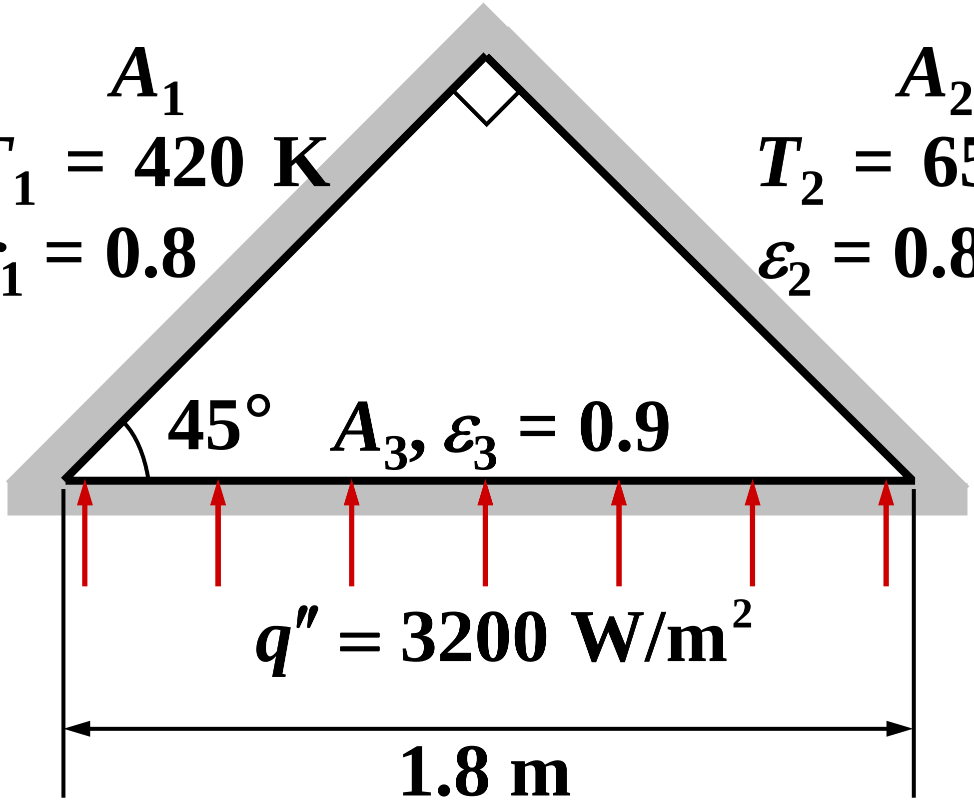
Remark: when A_2 is a black enclosure

$$q_1 = \varepsilon_1\sigma T_1^4 A_1 - \alpha_1 G_1 A_1$$

$$G_1 A_1 = \sigma T_2^4 A_2 F_{21} = \sigma T_2^4 A_1 F_{12} = \sigma T_2^4 A_1$$

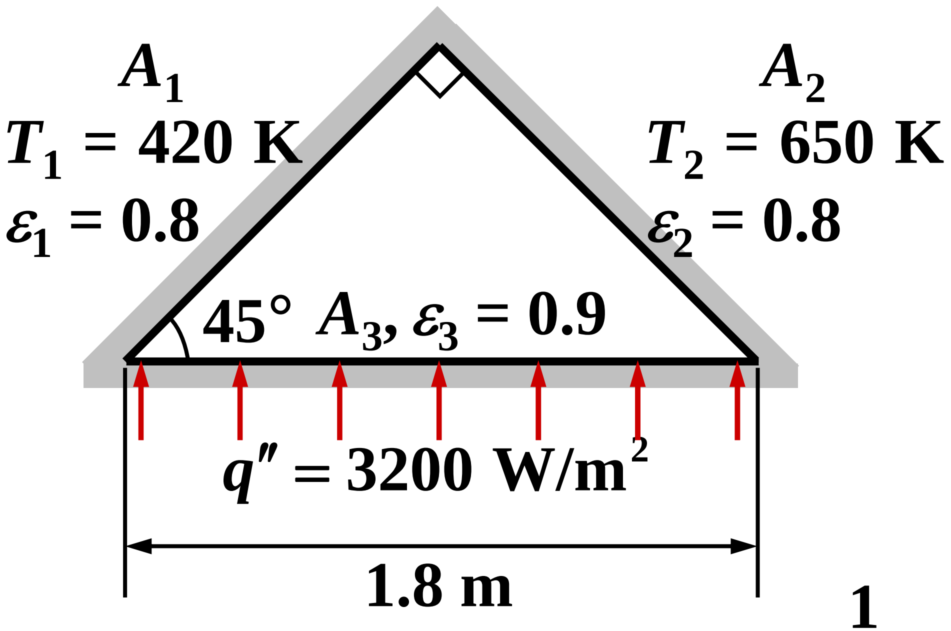
$$q_1 = \varepsilon_1\sigma T_1^4 A_1 - \varepsilon_1\sigma T_2^4 A_1 = \varepsilon_1\sigma A_1\left(T_1^4 - T_2^4\right)$$

Ex 7-14

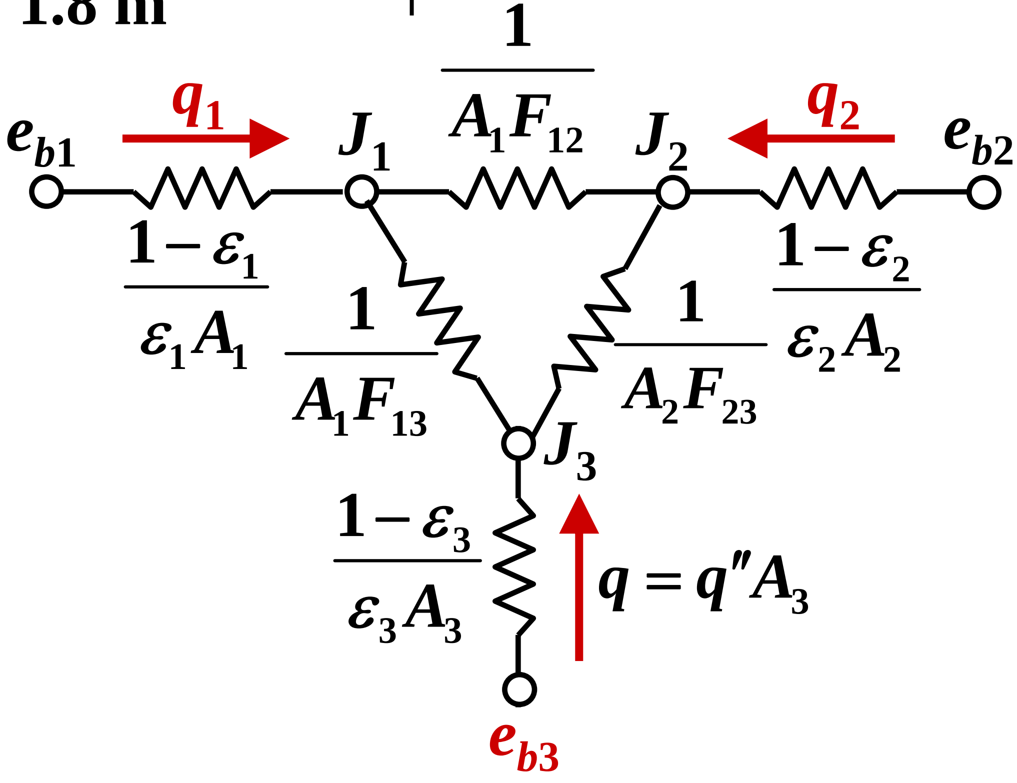


q_1, q_2, T_3 = ?

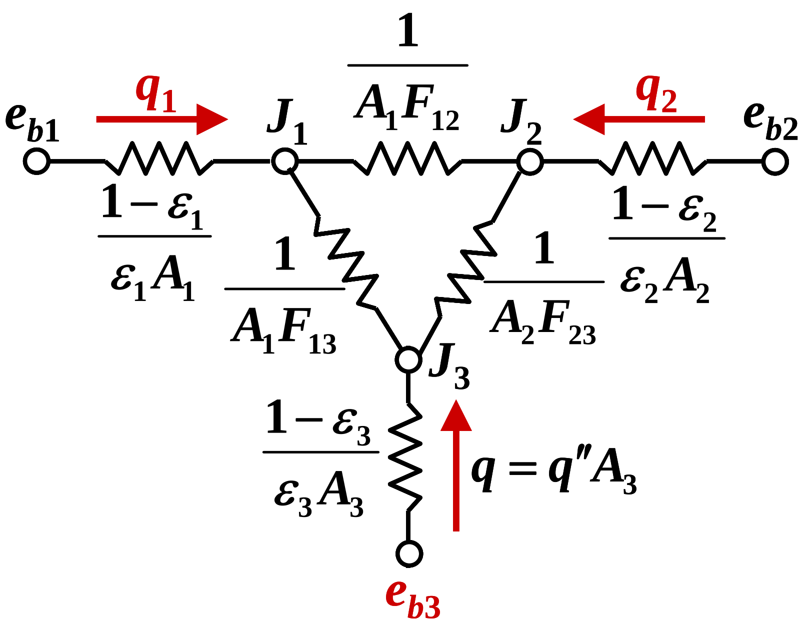
$$q''_k = \frac{\varepsilon_k}{1-\varepsilon_k}\left(\sigma T_k^4 - J_k\right), \quad J_k = \varepsilon_k \sigma T_k^4 + \left(1-\varepsilon_k\right)\sum_{i=1}^{n} J_i F_{ki}$$



A1
T1 = 420 K
ε1 = 0.8
A2
T2 = 650 K
ε2 = 0.8
45°
A3, ε3 = 0.9
q″ = 3200 W/m2
1.8 m



eb1
q1
J1
1
A1F12
J2
q2
eb2
1 − ε1
ε1A1
1
A1F13
1
A2F23
1 − ε2
ε2A2
J3
1 − ε3
ε3A3
q = q″A3
eb3



$$q_1 = \frac{\sigma T_1^4 - J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1}$$

$$q_2 = \frac{\sigma T_2^4 - J_2}{(1-\varepsilon_2)/\varepsilon_2 A_2}$$

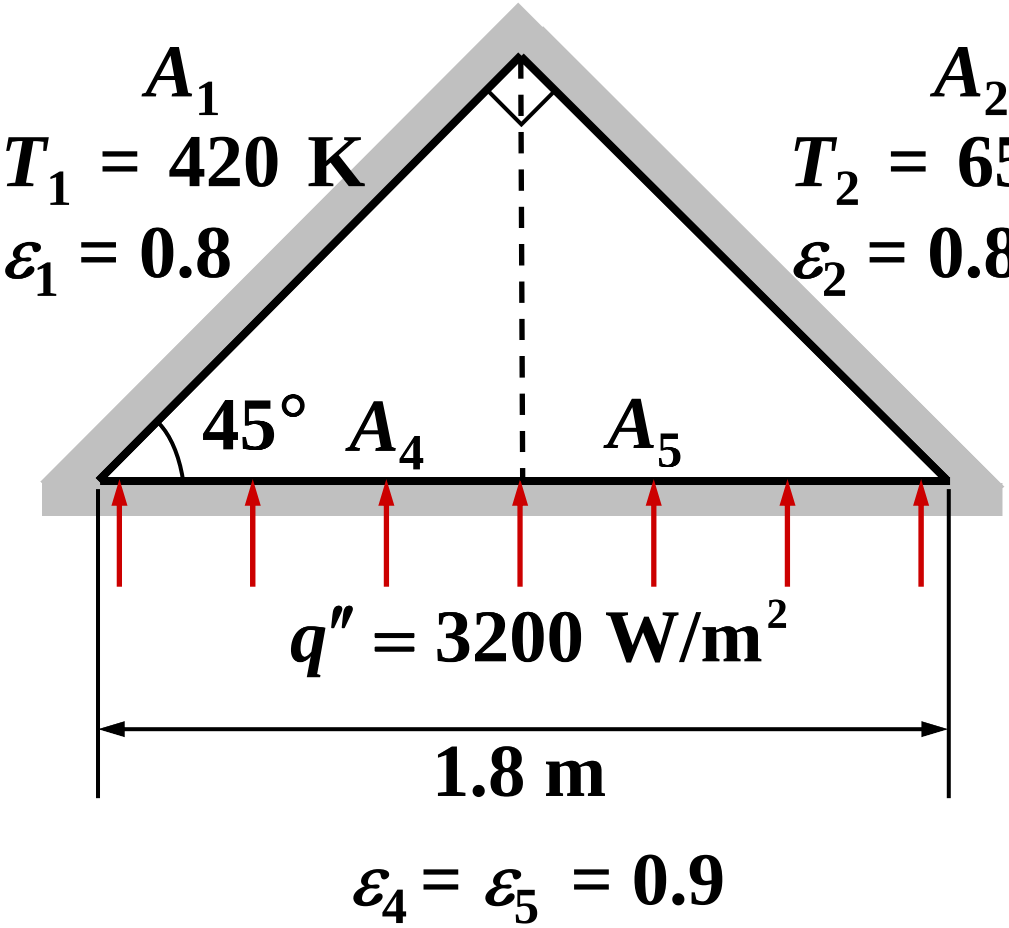
$$q = \frac{\sigma T_3^4 - J_3}{(1-\varepsilon_3)/\varepsilon_3 A_3}$$

$$\frac{\sigma T_1^4 - J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1} = \frac{J_1 - J_2}{1/A_1F_{12}} + \frac{J_1 - J_3}{1/A_1F_{13}}$$

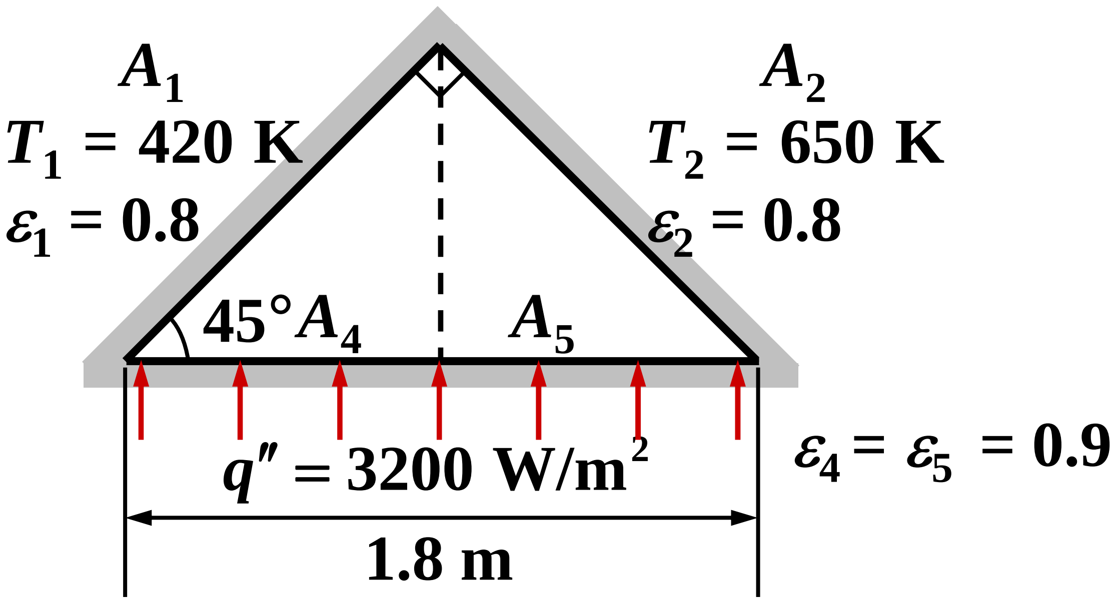
$$\frac{\sigma T_2^4 - J_2}{(1-\varepsilon_2)/\varepsilon_2 A_2} = \frac{J_2 - J_1}{1/A_1F_{12}} + \frac{J_2 - J_3}{1/A_2F_{23}}$$

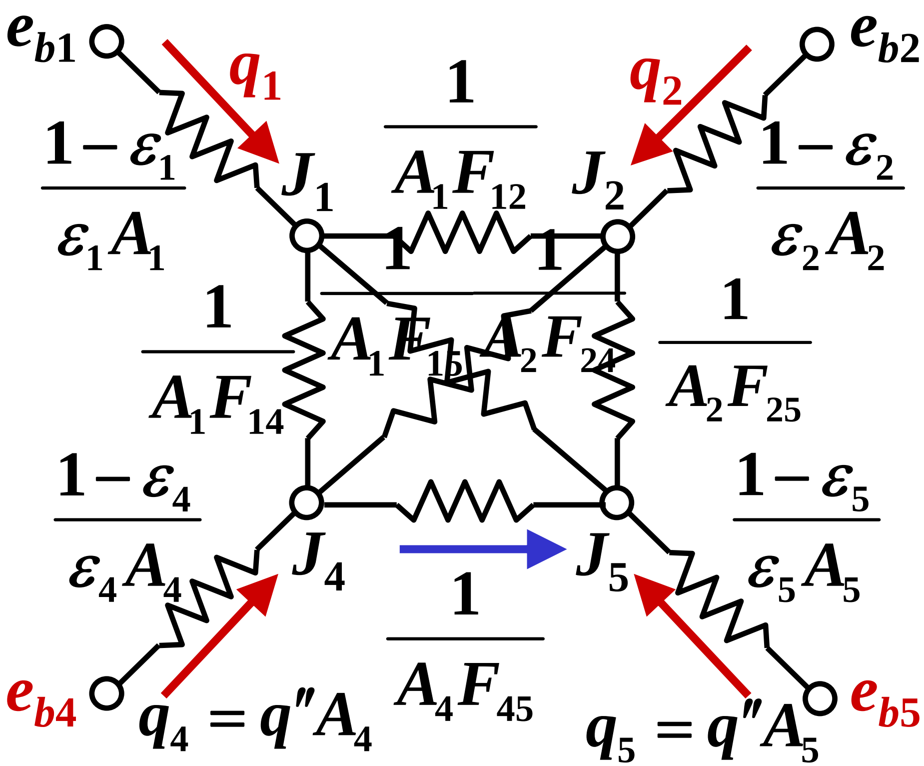
$$q = \frac{J_3 - J_2}{1/A_2F_{23}} + \frac{J_3 - J_1}{1/A_1F_{13}}$$

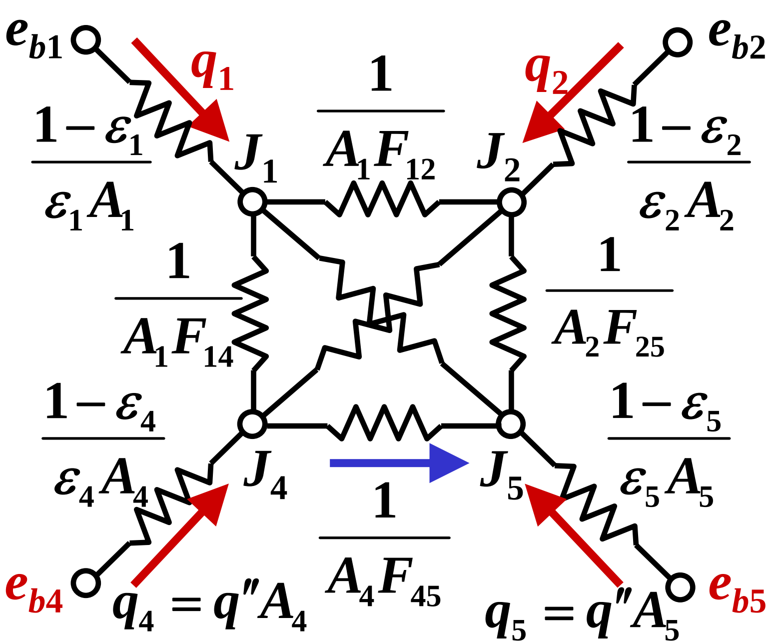
$q_1'' = -6346\ \text{W/m}^2$, $q_2'' = 1820\ \text{W/m}^2$, $T_3 = 649.1\ \text{K}$


A_1
$T_1 = 420$ K
$\varepsilon_1 = 0.8$
A_2
$T_2 = 650$ K
$\varepsilon_2 = 0.8$
45°
A_4
A_5
$q'' = 3200$ W/m^2
1.8 m
$\varepsilon_4 = \varepsilon_5 = 0.9$

$q_1, q_2, T_4, T_5 = ?$


A1
T1 = 420 K
ε1 = 0.8
A2
T2 = 650 K
ε2 = 0.8
45°
A4
A5
q″ = 3200 W/m2
ε4 = ε5 = 0.9
1.8 m


eb1
q1
eb2
q2
1 − ε1 / ε1A1
J1
1 / A1F12
J2
1 − ε2 / ε2A2
1 / A1F14
1 / A1F15
1 / A2F24
1 / A2F25
1 − ε4 / ε4A4
J4
J5
1 − ε5 / ε5A5
1 / A4F45
eb4
q4 = q″A4
q5 = q″A5
eb5



$$q_1 = \frac{\sigma T_1^4 - J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1}$$

$$q_2 = \frac{\sigma T_2^4 - J_2}{(1-\varepsilon_2)/\varepsilon_2 A_2}$$

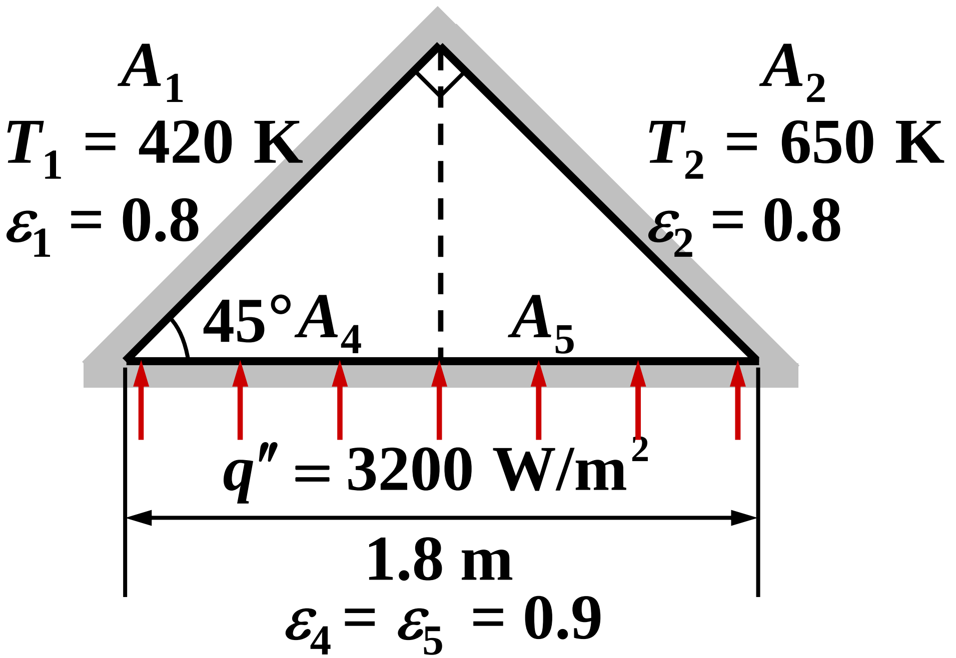
$$q_4 = \frac{\sigma T_4^4 - J_4}{(1-\varepsilon_4)/\varepsilon_4 A_4}$$

$$q_5 = \frac{\sigma T_5^4 - J_5}{(1-\varepsilon_5)/\varepsilon_5 A_5}$$

$$\frac{\sigma T_1^4 - J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1} = \frac{J_1 - J_2}{1/A_1F_{12}} + \frac{J_1 - J_4}{1/A_1F_{14}} + \frac{J_1 - J_5}{1/A_1F_{15}}$$

$$\frac{\sigma T_2^4 - J_2}{(1-\varepsilon_2)/\varepsilon_2 A_2} = \frac{J_2 - J_1}{1/A_1F_{12}} + \frac{J_2 - J_4}{1/A_2F_{24}} + \frac{J_2 - J_5}{1/A_2F_{25}}$$

$$q_4 = \frac{J_4 - J_1}{1/A_1F_{14}} + \frac{J_4 - J_2}{1/A_2F_{24}}, \quad q_5 = \frac{J_5 - J_1}{1/A_1F_{15}} + \frac{J_5 - J_2}{1/A_2F_{25}}$$

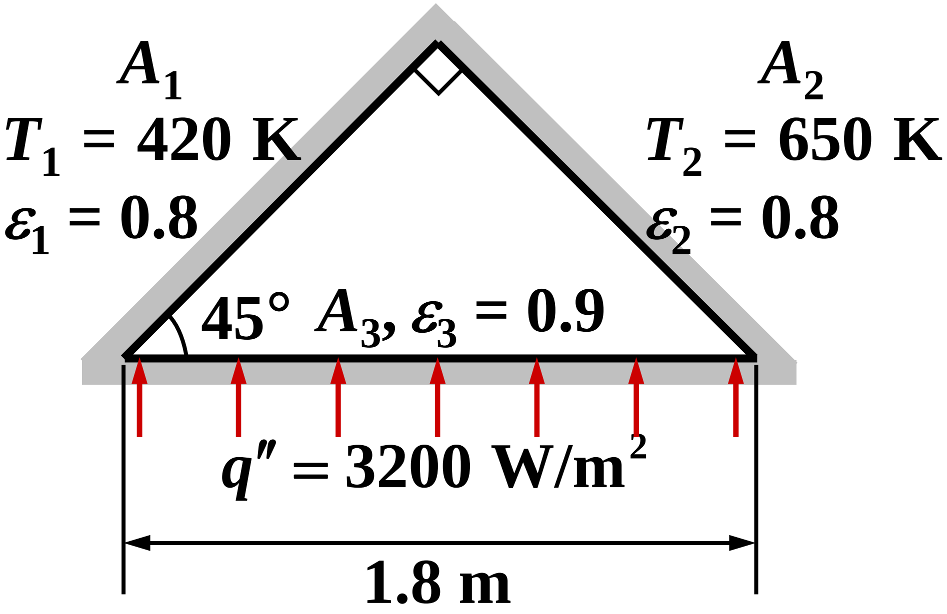


$$q''_1 = -6049\ \mathrm{W/m^2}$$

$$q''_2 = 1524\ \mathrm{W/m^2}$$

$$T_4 = 623.3\ \mathrm{K}$$

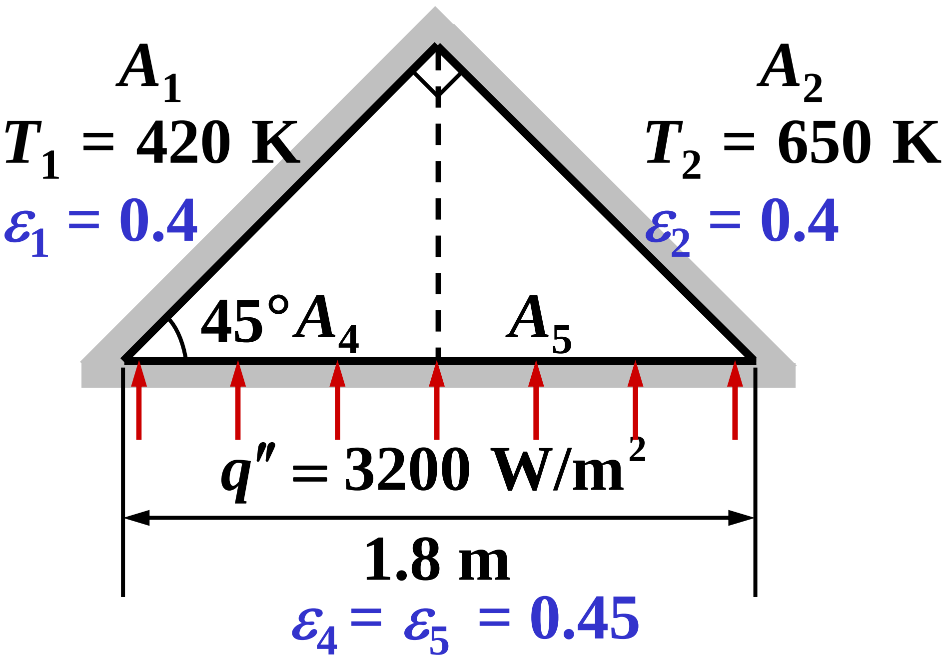
$$T_5 = 669.7\ \mathrm{K}$$



$$q''_1 = -6346\ \mathrm{W/m^2}$$

$$q''_2 = 1820\ \mathrm{W/m^2}$$

$$T_3 = 649.1\ \mathrm{K}$$

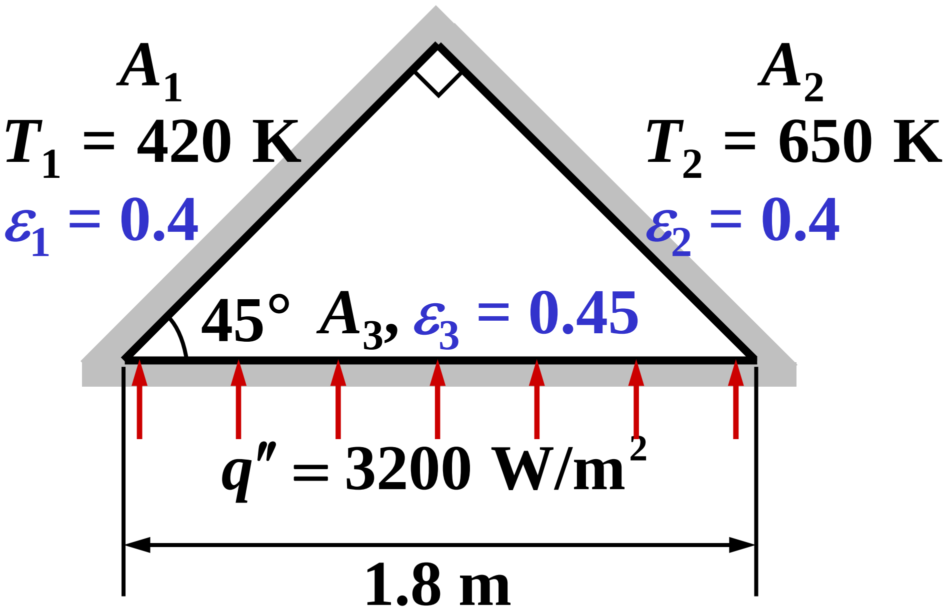


$$q''_1 = -4038 \text{ W/m}^2$$

$$q''_2 = -487 \text{ W/m}^2$$

$$T_4 = 726.8 \text{ K}$$

$$T_5 = 740.8 \text{ K}$$

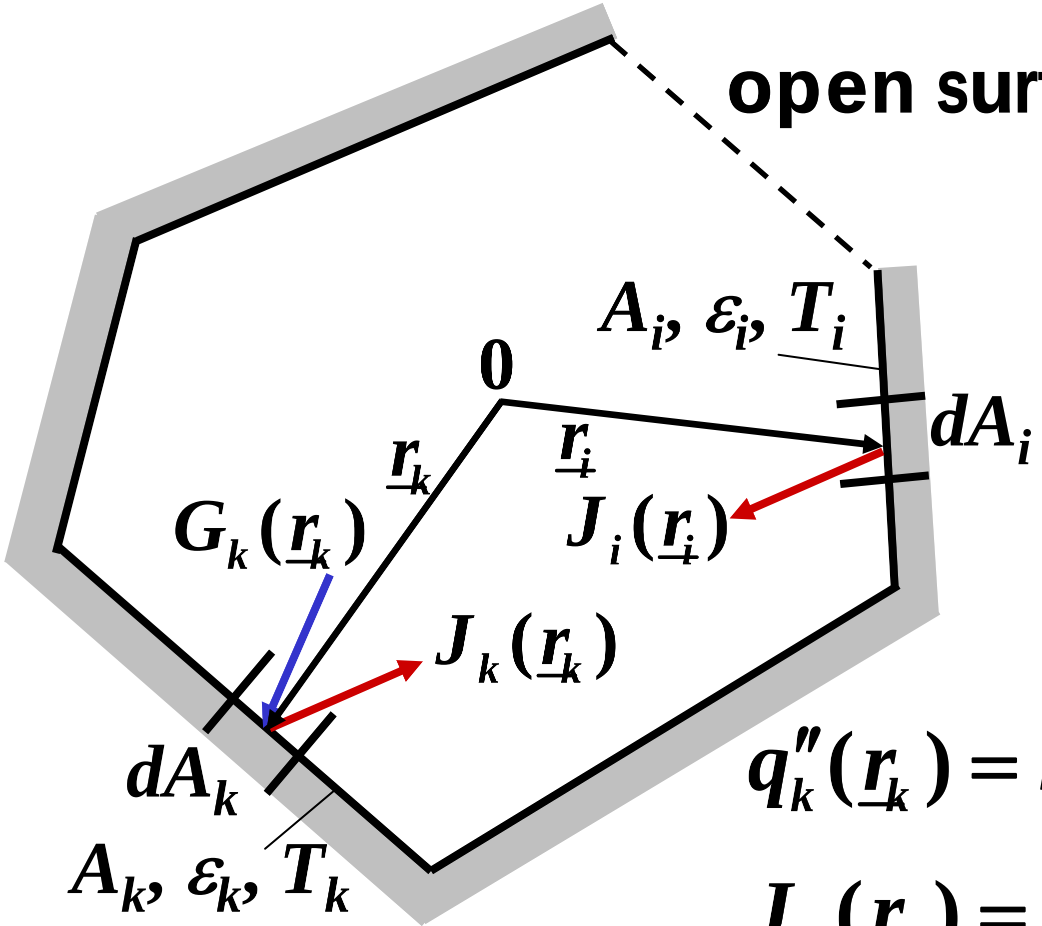


$$q''_1 = -4101 \text{ W/m}^2$$

$$q''_2 = -425 \text{ W/m}^2$$

$$T_3 = 733.9 \text{ K}$$

Generalized Zone Analysis

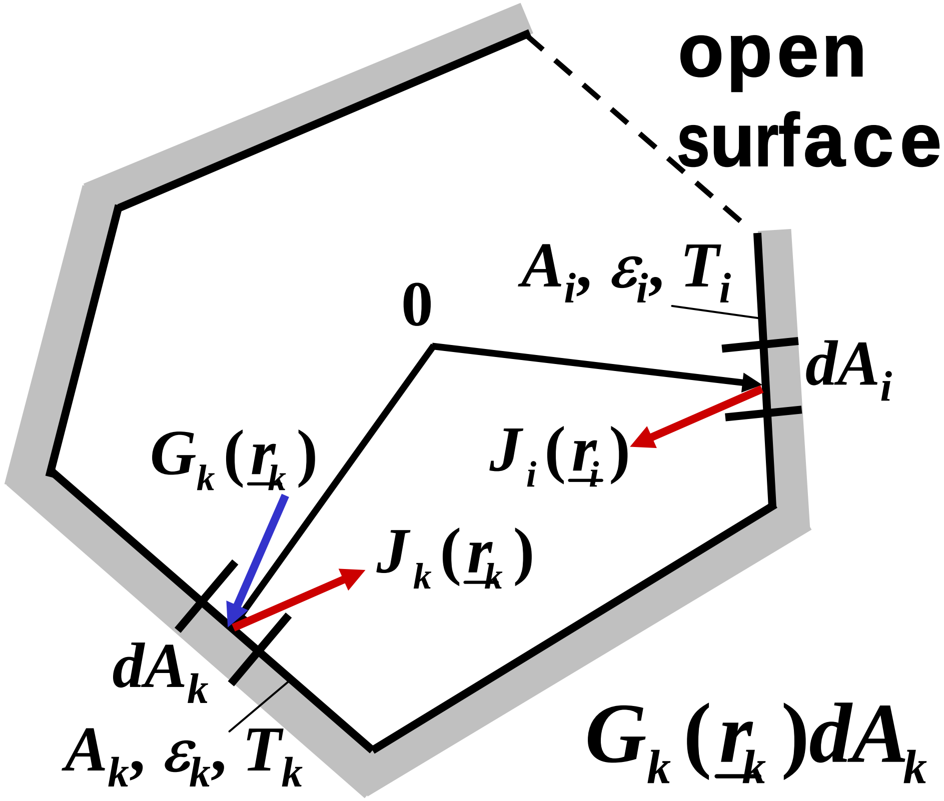


temperature, properties: uniform over each surface

$$q_k''(\underline{r}_k) = J_k(\underline{r}_k) - G_k(\underline{r}_k)$$

$$J_k(\underline{r}_k) = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k) G_k(\underline{r}_k)$$

$$q_k''(\underline{r}_k) = \frac{\varepsilon_k}{1-\varepsilon_k}\left[\sigma T_k^4 - J_k(\underline{r}_k)\right]$$



$$G_k(\underline{r}_k)dA_k = \sum_{i=1}^{n}\int_{A_i} J_i(\underline{r}_i)dA_i dF_{di-dk}$$

$$= \sum_{i=1}^{n}\int_{A_i} J_i(\underline{r}_i)dA_k dF_{dk-di}$$

$$G_k(\underline{r}_k) = \sum_{i=1}^{n}\int_{A_i} J_i(\underline{r}_i)dF_{dk-di}$$

$$J_k(\underline{r}_k) = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k)\sum_{i=1}^{n}\int_{A_i} J_i(\underline{r}_i)\, dF_{dk-di}$$

$$K(\underline{r}_i, \underline{r}_k) \equiv \frac{dF_{dk-di}(\underline{r}_i, \underline{r}_k)}{dA_i}$$

$$J_k(\underline{r}_k) = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k)\sum_{i=1}^{n}\int_{A_i} J_i(\underline{r}_i) K(\underline{r}_i, \underline{r}_k)\, dA_i$$

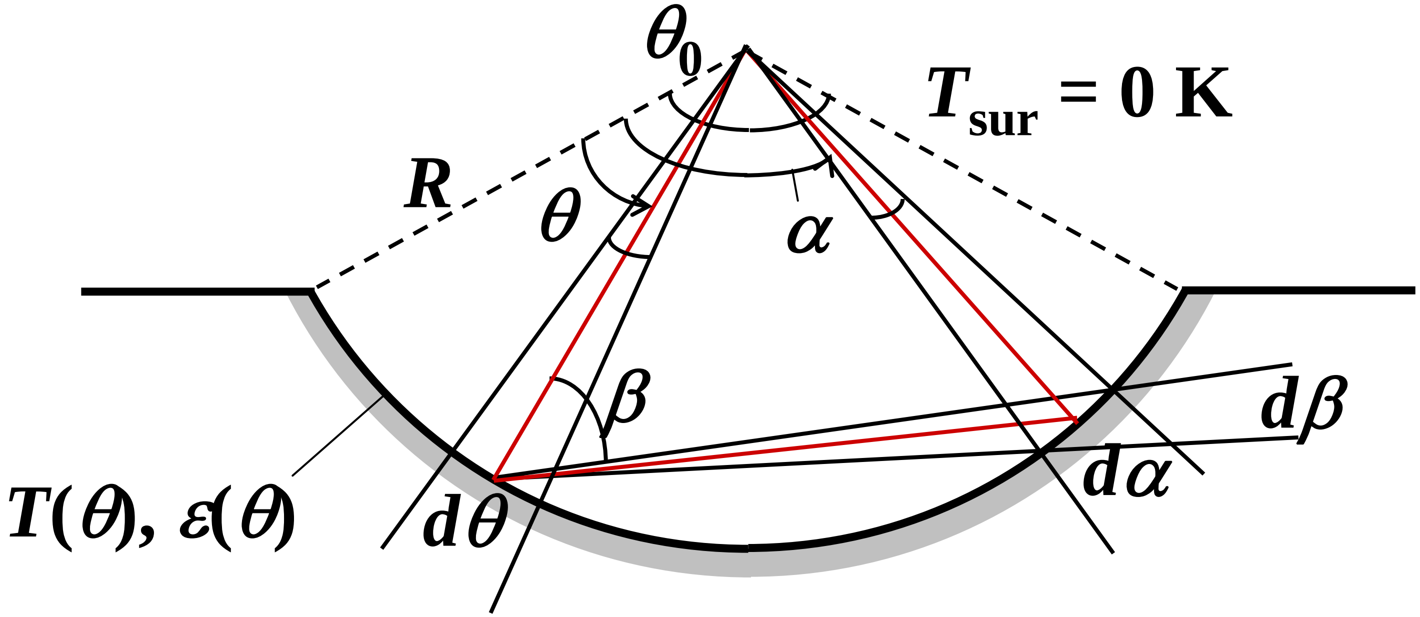
Summary

$$q_k''(\underline{r}_k) = \frac{\varepsilon_k}{1-\varepsilon_k}\left[\sigma T_k^4 - J_k(\underline{r}_k)\right]$$

$$J_k(\underline{r}_k) = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k)\sum_{i=1}^{n}\int_{A_i} J_i(\underline{r}_i) K(\underline{r}_i, \underline{r}_k)\, dA_i$$

$$k = 1, 2, 3, \cdots, n$$

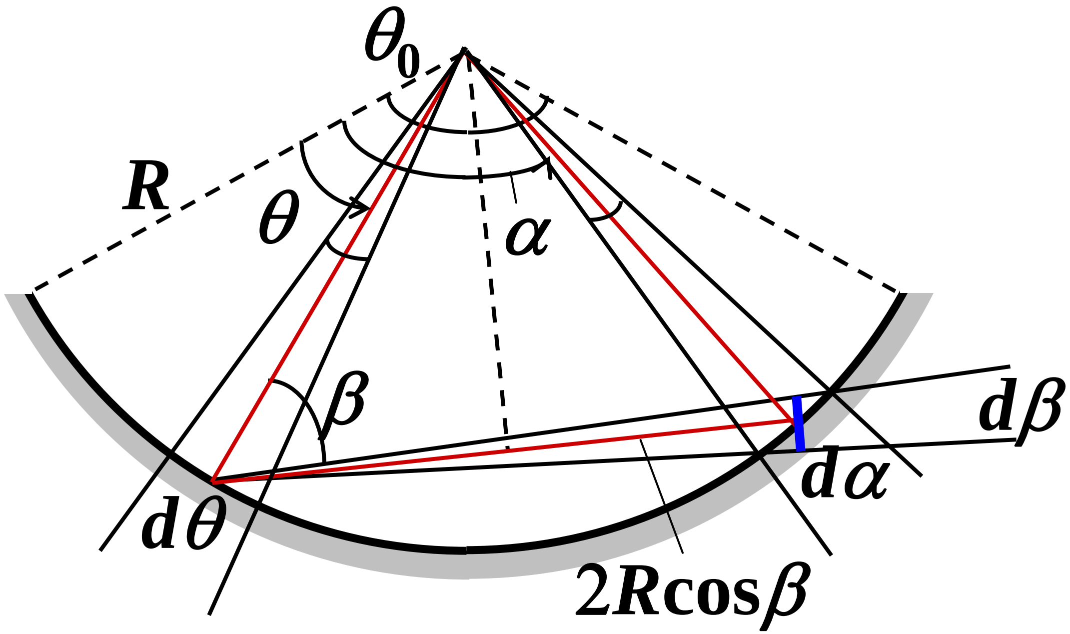
Exact solution: cylindrical circular cavity



$$q''(\theta) = \frac{\varepsilon(\theta)}{1-\varepsilon(\theta)}\left[\sigma T^4(\theta) - J(\theta)\right]$$

$$J(\theta) = \varepsilon(\theta)\sigma T^4(\theta) + \left(1-\varepsilon(\theta)\right)\int_0^{\theta_0} J(\alpha)\,dF_{\theta-\alpha}$$

$$dF_{\theta-\alpha} = \frac{1}{2}d(\sin\beta) = \frac{1}{2}\cos\beta\, d\beta$$



$$dF_{\theta-\alpha} = \frac{1}{2}\cos\beta d\beta$$

$$\beta = \frac{\pi}{2} - \frac{1}{2}|\alpha-\theta|,$$

$$\cos\beta = \sin\frac{1}{2}|\alpha-\theta|$$

$$2R\cos\beta d\beta = Rd\alpha\cos\beta \rightarrow d\beta = \frac{d\alpha}{2}$$

$$dF_{\theta-\alpha} = \frac{1}{4}\sin\frac{1}{2}|\alpha-\theta|d\alpha$$

$$J(\theta) = \varepsilon(\theta)\sigma T^4(\theta) + \left(1-\varepsilon(\theta)\right)\int_0^{\theta_0}\frac{1}{4}J(\alpha)\sin\frac{1}{2}|\alpha-\theta|d\alpha$$

Let $I(\theta) = \int_0^{\theta_0} J(\alpha) \sin\frac{1}{2}|\alpha - \theta| d\alpha$

$$= \int_0^{\theta} J(\alpha) \sin\frac{1}{2}(\theta - \alpha) d\alpha + \int_{\theta}^{\theta_0} J(\alpha) \sin\frac{1}{2}(\alpha - \theta) d\alpha$$

Then, $\dfrac{d^2 I(\theta)}{d\theta^2} + \dfrac{1}{4}\varepsilon(\theta) I(\theta) = \varepsilon(\theta)\sigma T^4(\theta)$

When ε and T are constants,

$$\frac{d^2 I(\theta)}{d\theta^2} + \frac{1}{4}\varepsilon I(\theta) = \varepsilon\sigma T^4, \text{ b.c. } I(0) = I(\theta_0), \ \frac{dI}{d\theta}\left(\frac{\theta_0}{2}\right) = 0$$

$$J(\theta) = \varepsilon\sigma T^4 + \frac{1}{4}(1 - \varepsilon) I(\theta)$$

$$q''(\theta) = \frac{\varepsilon}{1 - \varepsilon}\left[\sigma T^4 - J(\theta)\right]$$

Methods for Solving Integral Equations

(Hildebrand "Methods of Applied Mathematics")

$$J(\theta) = \varepsilon\sigma T^4 + (1-\varepsilon)\int_0^{\theta_0} J(\alpha)\sin\frac{1}{2}|\alpha-\theta|\,d\alpha$$

$$\phi(x) = f(x) + \lambda\int_a K(x,\eta)\phi(\eta)\,d\eta$$

1) Reduction to sets of algebraic equations

integral $\rightarrow$ summation (finite difference)
$\rightarrow$ algebraic linear equations

- **trapezoidal rule**
- **Simpson's rule**
- **Gaussian quadrature**

2) Successive approximation

(iterative method)

initial guess $\phi_0(x)$ and get first approximation $\phi_1(x)$

$$\phi_1(x) = f(x) + \lambda \int_a^b K(x,\eta)\phi_0(\eta)d\eta$$

$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,\eta)\phi_1(\eta)d\eta$$

$$\vdots$$

$$\phi_n(x) = f(x) + \lambda \int_a^b K(x,\eta)\phi_{n-1}(\eta)d\eta$$

$$\phi_n(x) = f(x) + \lambda L f(x) + \lambda^2 L^2 f(x) + \cdots + \lambda^{n-1} L^{n-1} f(x) + \lambda^n L^n \phi_0(x)$$

$$L f(x) \equiv \int_a^b K(x,\eta) f(\eta) d\eta,$$

$$L^2 f(x) \equiv \int_a^b K(x,\eta) \int_a^b K(\eta,\eta_1) f(\eta_1) d\eta_1 d\eta, \cdots$$

as $n \to \infty$, $\lim_{n\to\infty} \phi_n(x) = \phi(x)$

when $|\lambda| < \dfrac{1}{(b-a)M}$

M: maximum value of the kernel $K(x,\eta)$

if $\lambda << 1$ rapidly converge

(proof: Hildebrand pp.421-424)

3) Variational method

(Courant & Hilbert "Methods of Mathematical Physics"p.205-)

$$I \equiv \lambda \int_a^b \int_a^b K(x,\eta)\phi(x)\phi(\eta)dxd\eta$$

$$+2\int_{-1/2}^{1/2} \phi(x)dx - \int_{-1/2}^{1/2} \phi^2(x)dx$$

extremum of I:

$\phi(x)$ satisfies the integral equation.

approximate solution: Ritz method

(Hildebrand p.187)

$$\phi(x) = \sum_{k=1}^{n} C_k \psi_k(x)$$

with a proper choice of $\psi_k(x)$

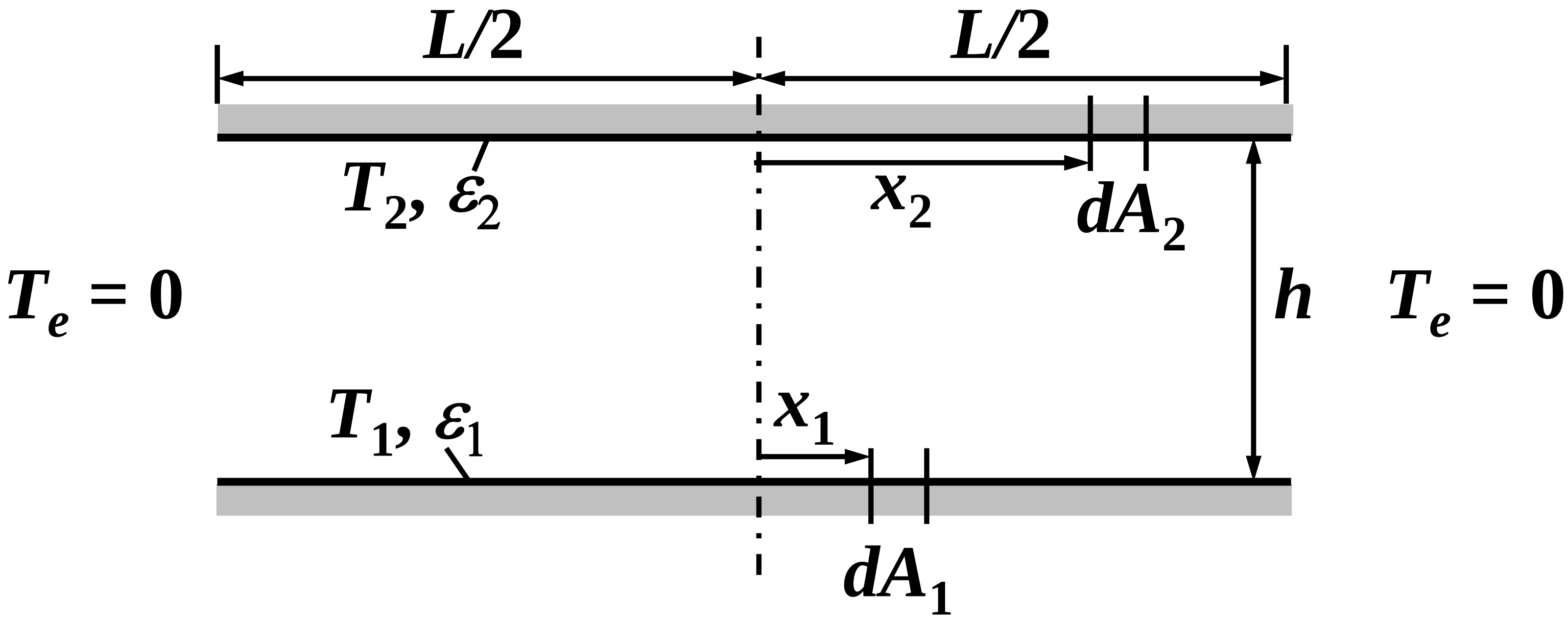
$$I = I(C_1, C_2, \cdots, C_n)$$

$$\frac{\partial I}{\partial C_k} = 0, \quad k = 1, 2, \cdots, n$$

n simultaneous algebraic equations

Ex

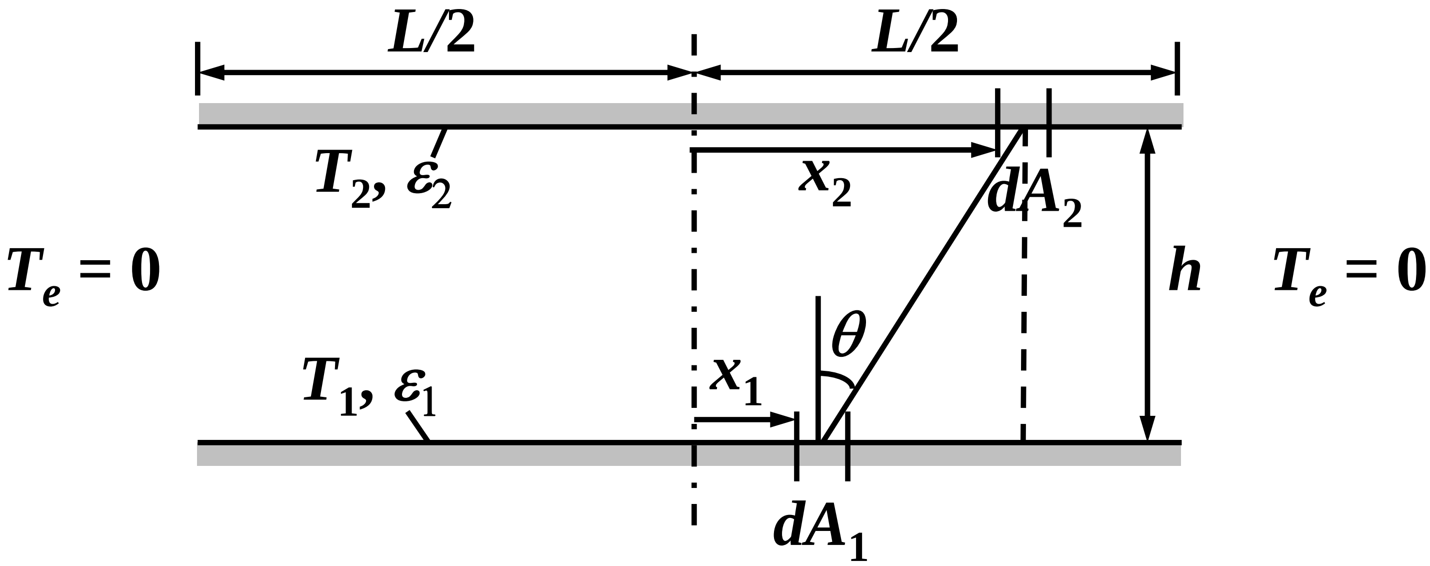
Radiative exchange between parallel plates: variational method



$$q_1''(x_1) = \frac{\varepsilon_1}{1-\varepsilon_1}\left[\sigma T_1^4 - J_1(x_1)\right]$$

$$J_1(x_1) = \varepsilon_1 \sigma T_1^4 + (1-\varepsilon_1)\int_{-L/2}^{L/2} J_2(x_2)\, dF_{d1-d2}$$

$$J_2(x_2) = \varepsilon_2 \sigma T_2^4 + (1-\varepsilon_2)\int_{-L/2}^{L/2} J_1(x_1)\, dF_{d2-d1}$$



$$dF_{d1-d2} = \frac{1}{2} d(\sin\theta), \quad \sin\theta = \frac{x_2 - x_1}{\left[\left(x_2 - x_1\right)^2 + h^2\right]^{1/2}}$$

$$\frac{1}{2} d(\sin\theta) = \frac{1}{2} \frac{h^2}{\left[\left(x_2 - x_1\right)^2 + h^2\right]^{3/2}} dx_2$$

$$J_1(x_1) = \varepsilon_1 \sigma T_1^4 + \frac{1}{2}\left(1 - \varepsilon_1\right) \int_{-L/2}^{L/2} \frac{h^2}{\left[\left(x_2 - x_1\right)^2 + h^2\right]^{3/2}} J_2(x_2) dx_2$$

Let $x = \frac{x_1}{L}, \eta = \frac{x_2}{L}, \gamma = \frac{h}{L}, \phi = \frac{J_1(x_1)}{\varepsilon_1 \sigma T_1^4}$

when $T_1 = T_2,\ \varepsilon_1 = \varepsilon_2$

$$\phi(x) = 1 + \lambda \int_{-1/2}^{1/2} K(x,\eta)\phi(\eta)d\eta$$

$$K(x,\eta) = \frac{\gamma^2}{\left[\left(x-\eta\right)^2 + \gamma^2\right]^{3/2}},\ \lambda = \frac{1-\varepsilon}{2}$$

$$\frac{q''(x)}{\varepsilon \sigma T^4} = \frac{1}{1-\varepsilon}\left[1 - \varepsilon \phi(x)\right]$$

Solution by variational method

$$I \equiv \lambda \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} K(x,\eta)\phi(x)\phi(\eta)dx d\eta$$
$$+2\int_{-1/2}^{1/2}\phi(x)dx - \int_{-1/2}^{1/2}\phi^2(x)dx$$

symmetry condition

$$\phi(x) = c_1 + c_2 x^2$$

$$I = (1-\varepsilon)\left(c_1^2 a_1 + c_1 c_2 a_2 + c_2^2 a_3\right)$$
$$-c_1^2 - \frac{1}{6}c_1 c_2 - \frac{1}{80}c_2^2 + 2c_1 + \frac{c_2}{6}$$

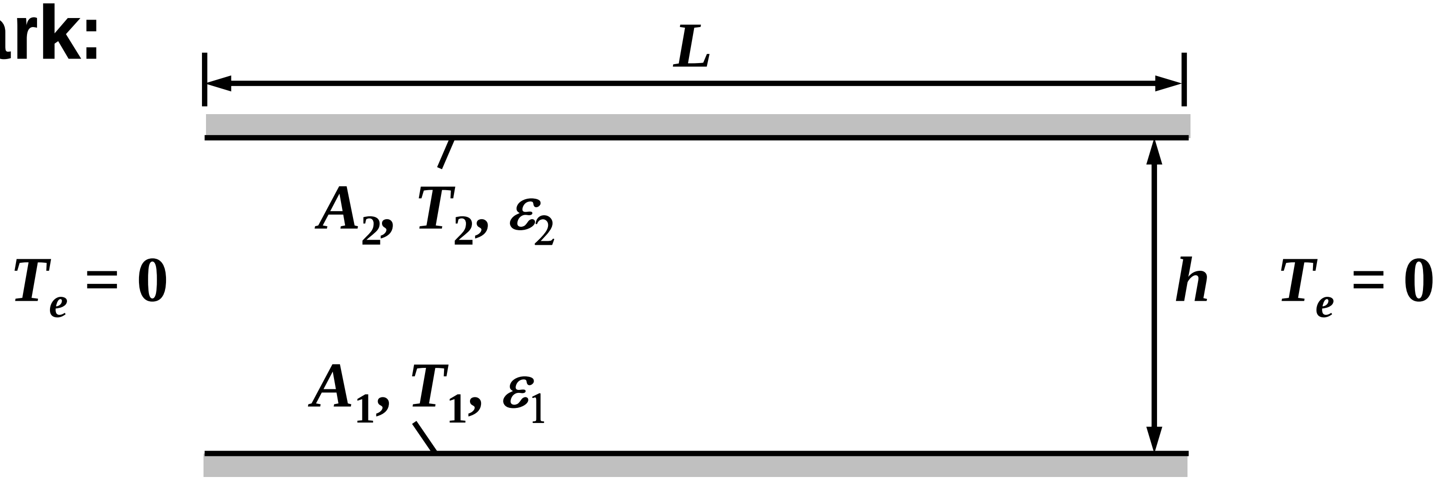
a_1, a_2, a_3: constant functions of γ

$$\frac{\partial I}{\partial c_1} = 0 \rightarrow 2c_1\left[a_1\left(1-\varepsilon\right)-1\right] + c_2\left[a_2\left(1-\varepsilon\right)-\frac{1}{6}\right] = -2$$

$$\frac{\partial I}{\partial c_2} = 0 \rightarrow c_1\left[a_2\left(1-\varepsilon\right)-\frac{1}{6}\right] + 2c_2\left[a_3\left(1-\varepsilon\right)-\frac{1}{80}\right] = -\frac{1}{6}$$

$$\frac{q''(x)}{\varepsilon\sigma T^4} = \frac{1}{1-\varepsilon}\left[1-\varepsilon\left(c_1 + c_2 x^2\right)\right]$$

Remark:



$$q_1'' = \frac{\varepsilon_1}{1-\varepsilon_1}\left(\sigma T_1^4 - J_1\right), \qquad J_1 = \varepsilon_1 \sigma T_1^4 + \left(1-\varepsilon\right) J_2 F_{12}$$

when $T_1 = T_2 = T, \varepsilon_1 = \varepsilon_2 = \varepsilon \rightarrow J_1 = J_2 = J$

$$F_{12} = \frac{\sqrt{L^2 + h^2} - h}{L} = \sqrt{1 + \left(\frac{h}{L}\right)^2} - \frac{h}{L}$$

$$J = \varepsilon \sigma T^4 + \left(1-\varepsilon\right) J F_{12}$$

$$\left[1 - \left(1-\varepsilon\right) F_{12}\right] J = \varepsilon \sigma T^4 \rightarrow J = \frac{\varepsilon \sigma T^4}{1 - \left(1-\varepsilon\right) F_{12}}$$

$$q'' = \frac{\varepsilon}{1-\varepsilon}\left[\sigma T^4 - \frac{\varepsilon\sigma T^4}{1-(1-\varepsilon)F_{12}}\right]$$

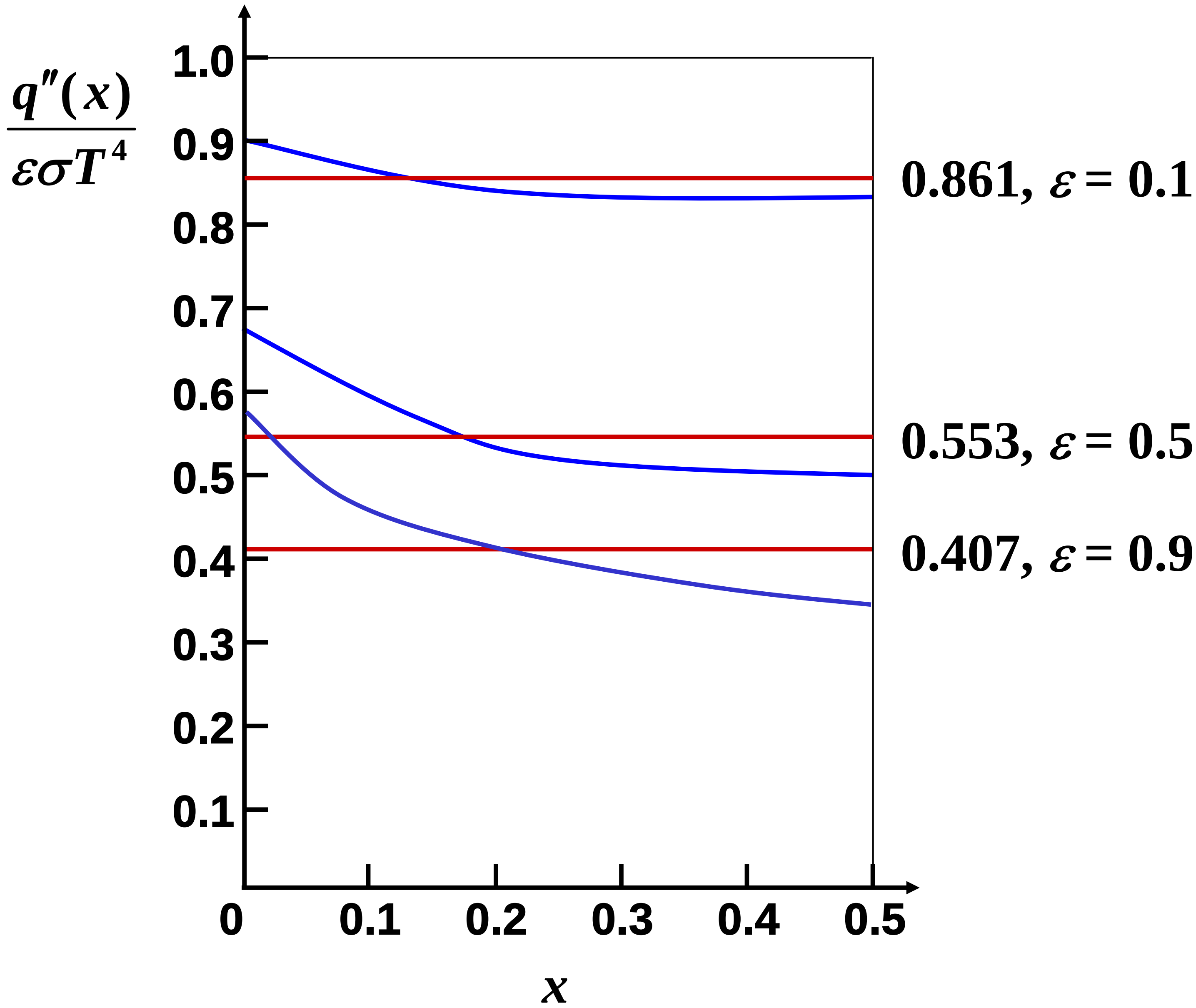
or $$\frac{q''}{\varepsilon\sigma T^4} = \frac{1}{1-\varepsilon}\left[1-\frac{\varepsilon}{1-(1-\varepsilon)F_{12}}\right]$$

when $$\frac{h}{L} = 0.5 \rightarrow F_{12} = 0.618$$

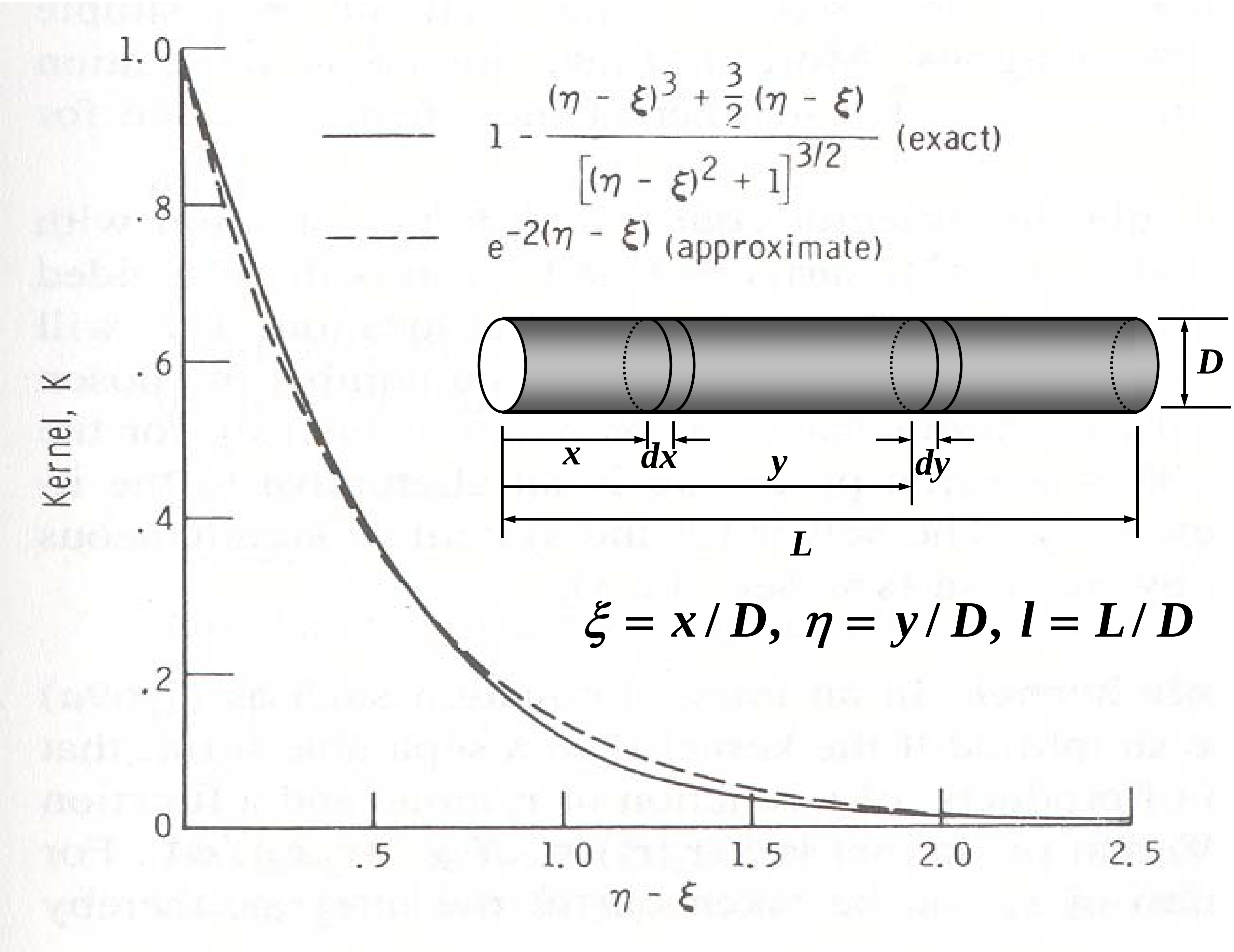
$$\varepsilon = 0.1: \quad \frac{q''}{\varepsilon\sigma T^4} = 0.861$$

$$\varepsilon = 0.5: \quad \frac{q''}{\varepsilon\sigma T^4} = 0.553$$

$$\varepsilon = 0.9: \quad \frac{q''}{\varepsilon\sigma T^4} = 0.407$$

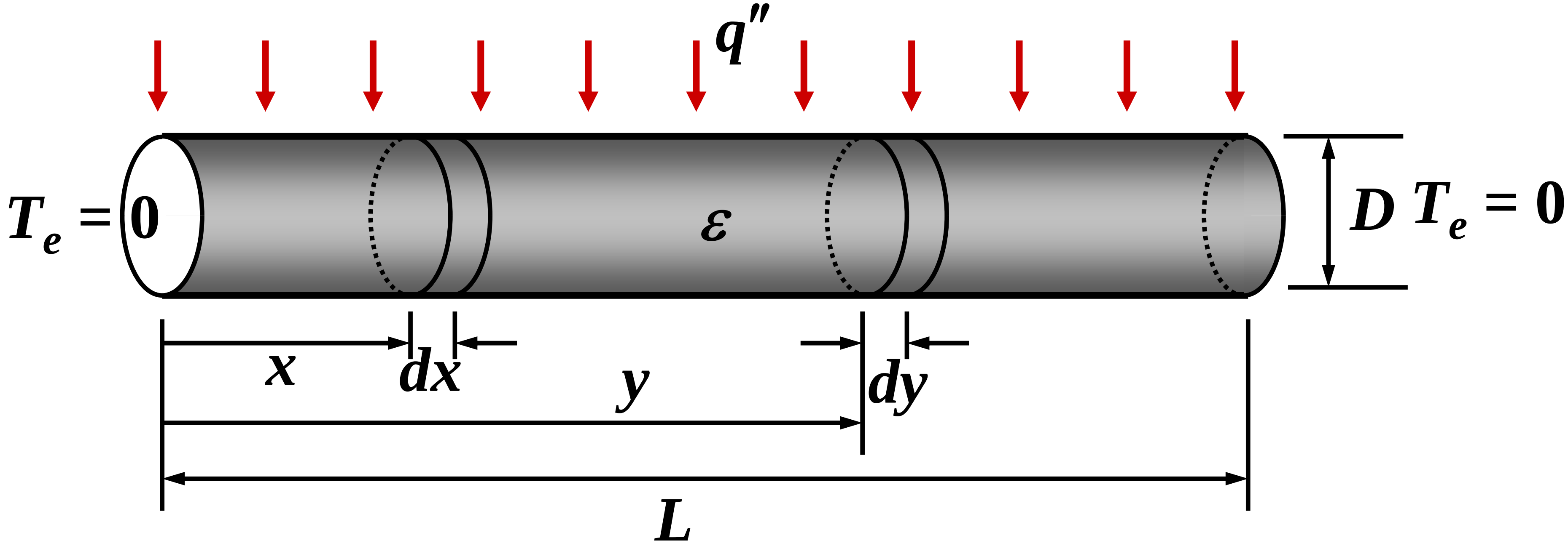


q″(x) / εσT⁴
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
x
0.861, ε = 0.1
0.553, ε = 0.5
0.407, ε = 0.9



$$\xi = x / D,\ \ \eta = y / D,\ l = L / D$$

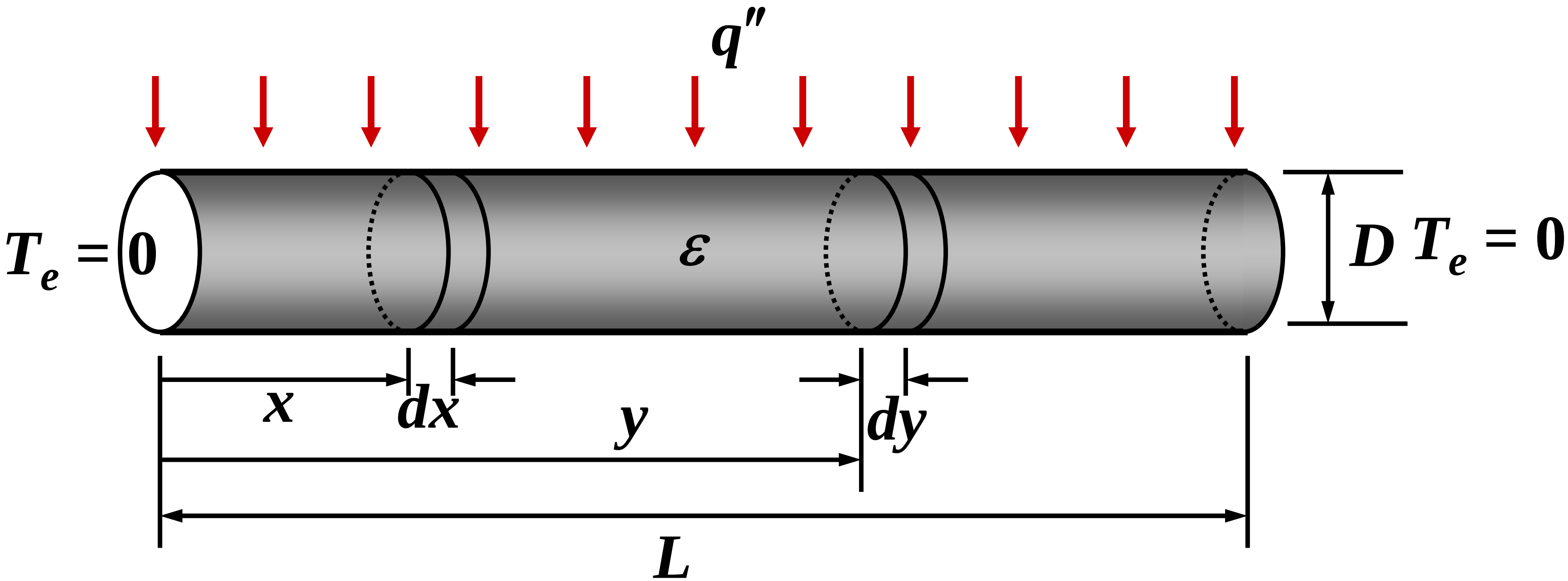
Ex 7-23 **Circular tube with uniform heat flux: approximation of kernel**



Tube surface temperature $T = ?$

$$\xi = x/D,\ \eta = y/D,\ l = L/D$$

$$q'' = \frac{\varepsilon}{1-\varepsilon}\left[\sigma T^4(\xi) - J(\xi)\right] \rightarrow \sigma T^4(\xi) = \frac{1-\varepsilon}{\varepsilon} q'' + J(\xi)$$



$$J(\xi) = \varepsilon\sigma T^4(\xi) + (1-\varepsilon)\int_{A_\eta} J(\eta)\, dF_{d\xi - d\eta}$$

$$\varepsilon\sigma T^4(\xi) = (1-\varepsilon)q'' + \varepsilon J(\xi)$$

$$dF_{d\xi - d\eta} = \left\{ 1 - \frac{|\eta - \xi|^3 + \frac{3}{2}|\eta - \xi|}{\left[(\eta - \xi)^2 + 1\right]^{3/2}} \right\} d\eta$$

$$J(\xi) = (1-\varepsilon)q'' + \varepsilon J(\xi) + (1-\varepsilon)\int_0^l K(\eta,\xi) J(\eta) d\eta$$

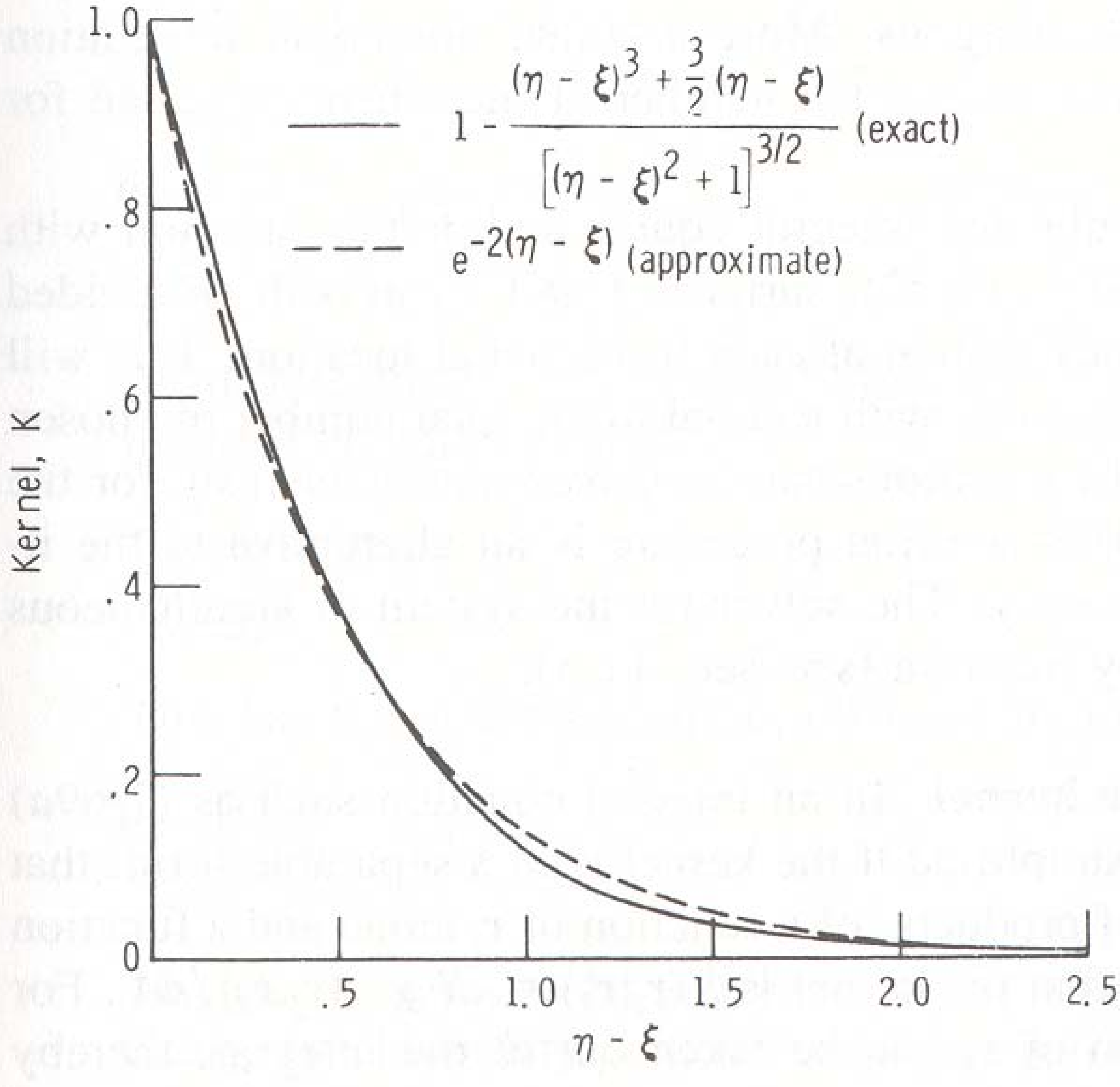
$$K(\eta,\xi) = 1 - \frac{|\eta-\xi|^3 + \frac{3}{2}|\eta-\xi|}{\left[(\eta-\xi)^2+1\right]^{3/2}}$$

$$(1-\varepsilon)J(\xi) = (1-\varepsilon)q'' + (1-\varepsilon)\int_0^l K(\eta,\xi) J(\eta) d\eta$$

$$J(\xi) = q'' + \int_o^l K(\eta,\xi) J(\eta) d\eta$$

$$T(\xi) = \left\{\frac{1}{\sigma}\left[\frac{1-\varepsilon}{\varepsilon}q'' + J(\xi)\right]\right\}^{1/4}$$

A



1.0
.8
.6
.4
.2
0
.5
1.0
1.5
2.0
2.5
Kernel, K
η - ξ
$1 - \frac{(\eta - \xi)^3 + \frac{3}{2}(\eta - \xi)}{\left[(\eta - \xi)^2 + 1\right]^{3/2}}$ (exact)
$e^{-2(\eta - \xi)}$ (approximate)

$$J(\xi) = q''\left[l + 1 + \varepsilon\left(\xi l - \xi^2\right)\right]$$

$$T(\xi) = \left\{\frac{1}{\sigma}\left[\frac{1-\varepsilon}{\varepsilon}q'' + J(\xi)\right]\right\}^{1/4}$$

$$= \left\{\frac{1}{\sigma}\left[\frac{1-\varepsilon}{\varepsilon}q'' + q''\left[l + 1 + \varepsilon\left(\xi l - \xi^2\right)\right]\right]\right\}^{1/4}$$

$$= \left\{\frac{q''}{\sigma}\left[\frac{1}{\varepsilon} + l + \varepsilon\left(\xi l - \xi^2\right)\right]\right\}^{1/4}$$

5) Taylor series expansion

$$K(x,\eta) = K(|x-\eta|):$$

decreases very rapidly as $|x-\eta|$ increases

$$\phi(\eta) = \phi(x) + \frac{d\phi}{dx}\bigg|(\eta - x) + \frac{1}{2}\frac{d^2\phi}{dx^2}\bigg|_x (\eta - x)^2 + \cdots$$

substitute into integral equation

$$\phi(x) = f(x) + \lambda\bigg[\phi(x)\int_a^b K(x,\eta)d\eta + \frac{d\phi(x)}{dx}\int_a^b (\eta - x) K(x,\eta)d\eta + \frac{d^2\phi(x)}{dx^2}\frac{1}{2}\int_a^b (\eta - x)^2 K(x,\eta)d\eta + \cdots\bigg]$$

three-term expansion

$$\left[\frac{1}{2}\int_a^b (\eta - x)^2 K(x,\eta)d\eta\right]\frac{d^2\phi(x)}{dx^2}$$

$$+\left[\int_a^b (\eta - x) K(x,\eta)d\eta\right]\frac{d\phi(x)}{dx}$$

$$+\left[\int_a^b K(x,\eta)d\eta - \frac{1}{\lambda}\right]\phi(x) = -\frac{f(x)}{\lambda}$$