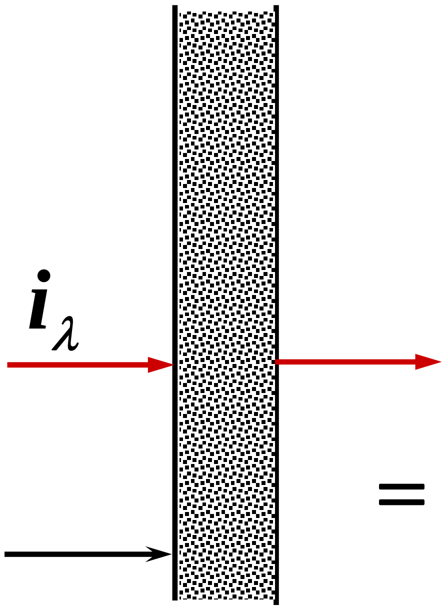
FUNDAMENTALS AND PROPERTIES FOR RADIATION IN ABSORBING, EMITTING, AND SCATTERING MEDIA

- **Attenuation by absorption and scattering**
- **Absorption and scattering coefficients**
- **Augmentation of intensity by emission**
- **Augmentation of intensity by incoming scattering**

Attenuation by Absorption and Scattering



i_λ $\quad$ $i_\lambda + di_\lambda = i_\lambda + \frac{di_\lambda}{ds} ds$

$s \quad s + ds$

$$di_\lambda \propto i_\lambda ds$$

$$di_\lambda = -\kappa_\lambda i_\lambda ds$$

(experimental observation)

or $$\frac{di_\lambda}{ds} = -\kappa_\lambda i_\lambda$$

κ_λ **: extinction coefficient, units of reciprocal length [cm^{-1}]**

$$\kappa_\lambda = \kappa_\lambda(\lambda, T, P, C_i)$$

$$\kappa_\lambda = a_\lambda + \sigma_\lambda$$

a_λ : **absorption coefficient**

σ_λ : **scattering coefficient**

$$\frac{di_\lambda}{ds} = -\kappa_\lambda i_\lambda, \qquad \int_{i_\lambda(0)}^{i_\lambda(s)} \frac{di_\lambda}{i_\lambda} = -\int_0^s \kappa_\lambda (s^*) ds^*$$

$$\ln \frac{i_\lambda(s)}{i_\lambda(0)} = -\int_0^s \kappa_\lambda (s^*) ds^*$$

$$i_\lambda(s) = i_\lambda(0) \exp\left[-\int_0^s \kappa_\lambda (s^*) ds^*\right]$$

Bouguer's law

Radiation mean penetration distance

energy absorbed and scattered at s

$$i_\lambda(s) - i_\lambda(s+ds) = i_\lambda(s) - \left[i_\lambda(s) + \frac{di_\lambda}{ds} ds \right] = -\frac{di_\lambda}{ds} ds$$

fraction of energy absorbed and scattered at s

$$-\frac{1}{i_\lambda(0)} \frac{di_\lambda}{ds} ds = -\frac{1}{i_\lambda(0)} \left(-\kappa_\lambda i_\lambda \right) ds$$

$$= \kappa_\lambda \frac{i_\lambda(s)}{i_\lambda(0)} ds = \kappa_\lambda \exp\left[-\int_0^s \kappa_\lambda(s^*) ds^* \right] ds$$

Mean penetration depth

$$s \cdot \kappa_\lambda(s) \exp\left[-\int_0^s \kappa_\lambda(s^*)ds^*\right]ds$$

$$l_m = \int_0^\infty s \cdot \kappa_\lambda(s) \exp\left[-\int_0^s \kappa_\lambda(s^*)ds^*\right]ds$$

when κ_λ = constant

$$l_m = \kappa_\lambda \int_0^\infty s \exp(-\kappa_\lambda s)ds = \frac{1}{\kappa_\lambda}$$

l_m **: average penetration distance before absorption or scattering**

Optical thickness (Opacity)

$$\tau_\lambda(s) \equiv \int_0^s \kappa_\lambda(s^*)ds^*$$

$$i_\lambda(s) = i_\lambda(0)\exp\left[-\int_0^s \kappa_\lambda(s^*)ds^*\right]$$

$$i_\lambda(s) = i_\lambda(0)\exp\left[-\tau_\lambda(s)\right]$$

for a uniform gas

$$\tau_\lambda = \kappa_\lambda s = \frac{s}{l_m}$$

the number of mean penetration distances

$\tau_\lambda \gg 1$: **optically thick** (photon continuum)

- A volume element within the material is only influenced by surrounding neighbors.
- diffusion approximation (Rosseland approximation)

$\tau_\lambda \ll 1$: **optically thin**

- interact directly with medium boundary
- Radiation emitted within the material is not reabsorbed by the material.
- negligible self-absorption

Limiting cases

$\tau_\lambda = 0$: **transparent medium**

$$\nabla \cdot \vec{q}_r'' = 0$$

$\tau_\lambda \to \infty$: **completely opaque**

$$\vec{q}_r'' = 0$$

The absorption and Scattering Coefficients

Absorption coefficient

if no scattering, $(\sigma_\lambda = 0), \kappa_\lambda = a_\lambda$

$$i_\lambda(s) = i_\lambda(0)\exp\left[-\int_0^s a_\lambda(s^*)ds^*\right]$$

in the case of a uniform medium

$$i_\lambda(s) = i_\lambda(0)\exp(-a_\lambda s)$$

in the electromagnetic theory of radiant energy propagation in conducting media

$$i_\lambda(s) = i_\lambda(0)\exp\left(-\frac{4\pi\kappa s}{\lambda}\right), \quad a_\lambda = \frac{4\pi\kappa}{\lambda}$$

True absorption coefficient

ordinary or spontaneous emission

stimulated or induced emission

(negative absorption): resulting from the presence of the radiation field. At the same frequency and in the same direction as the incident photon

The observed emerging intensity is the result of the actual absorption modified by the addition of induced emission along the path

actual absorbed energy :

true absorption coefficient $a_\lambda^+(\lambda, T, P)$

larger than a_λ **calculated by using observed attenuation media**

for a gas with reflective index $n = 1$

$$a_\lambda = \left[1 - \exp\left(-\frac{hc_0}{k\lambda T}\right)\right] a_\lambda^+$$

$a_\lambda \approx a_\lambda^+$ **except at a large value of** λT

1% deviation $\lambda T < 3120\ \mu\text{m}$

within 5% deviation $\lambda T < 4800\ \mu\text{m}$

Mie Solution

1) Definitions

C_s: scattering cross-section

the ratio of the rate of scattered energy by a spherical particle to the incident energy rate per unit area

C_a: absorption cross-section

C_e: extinction cross-section

$$C_e = C_a + C_s$$

Q : efficiency factor

the ratio of the cross-section to the geometric cross-section

$$Q_a = \frac{C_a}{\pi r^2}$$: absorption efficiency factor

$$Q_s = \frac{C_s}{\pi r^2}$$: scattering efficiency factor

r : radius of the sphere

$$Q_e = Q_a + Q_s$$

2) Mie's solution

$$Q_s = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\left(|a_n|^2 + |b_n|^2\right),$$

$$Q_e = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\,\mathrm{Re}\{a_n + b_n\}$$

Re : real part

x : size parameter, $\dfrac{\pi D}{\lambda}$

a_n, b_n: Mie coefficient
(Riccati-Bessel function)

$$a_n = \frac{\psi_n(x)\left[\psi_n'(y)/\psi_n(y)\right] - m\psi_n'(x)}{\xi_n(x)\left[\psi_n'(y)/\psi_n(y)\right] - m\xi_n'(x)}$$

$$b_n = \frac{m\psi_n(x)\left[\psi_n'(y)/\psi_n(y)\right] - \psi_n'(x)}{m\xi_n(x)\left[\psi_n'(y)/\psi_n(y)\right] - \xi_n'(x)}$$

$$\psi_n(z) = \left(\frac{\pi z}{2}\right)^{1/2} J_{n+\frac{1}{2}}(z),$$

$$\xi_n(z) = \left(\frac{\pi z}{2}\right)^{1/2} J_{n+\frac{1}{2}}(z) + \left(-1\right)^n iJ_{-n-\frac{1}{2}}(z)$$

$z = x$ or y, $y = mx$, $m = n - i\kappa$

Basic parameters

1) index of refraction $m = n - i\kappa$ relative to the surrounding medium ($\kappa = 0$: pure scatter)

2) size parameter: $\frac{\pi D}{\lambda}$

3) scattering angle θ: phase function

For a medium containing N particles per unit volume of the same composition and uniform size

$$a_\lambda = C_a N = \pi r^2 Q_a N$$

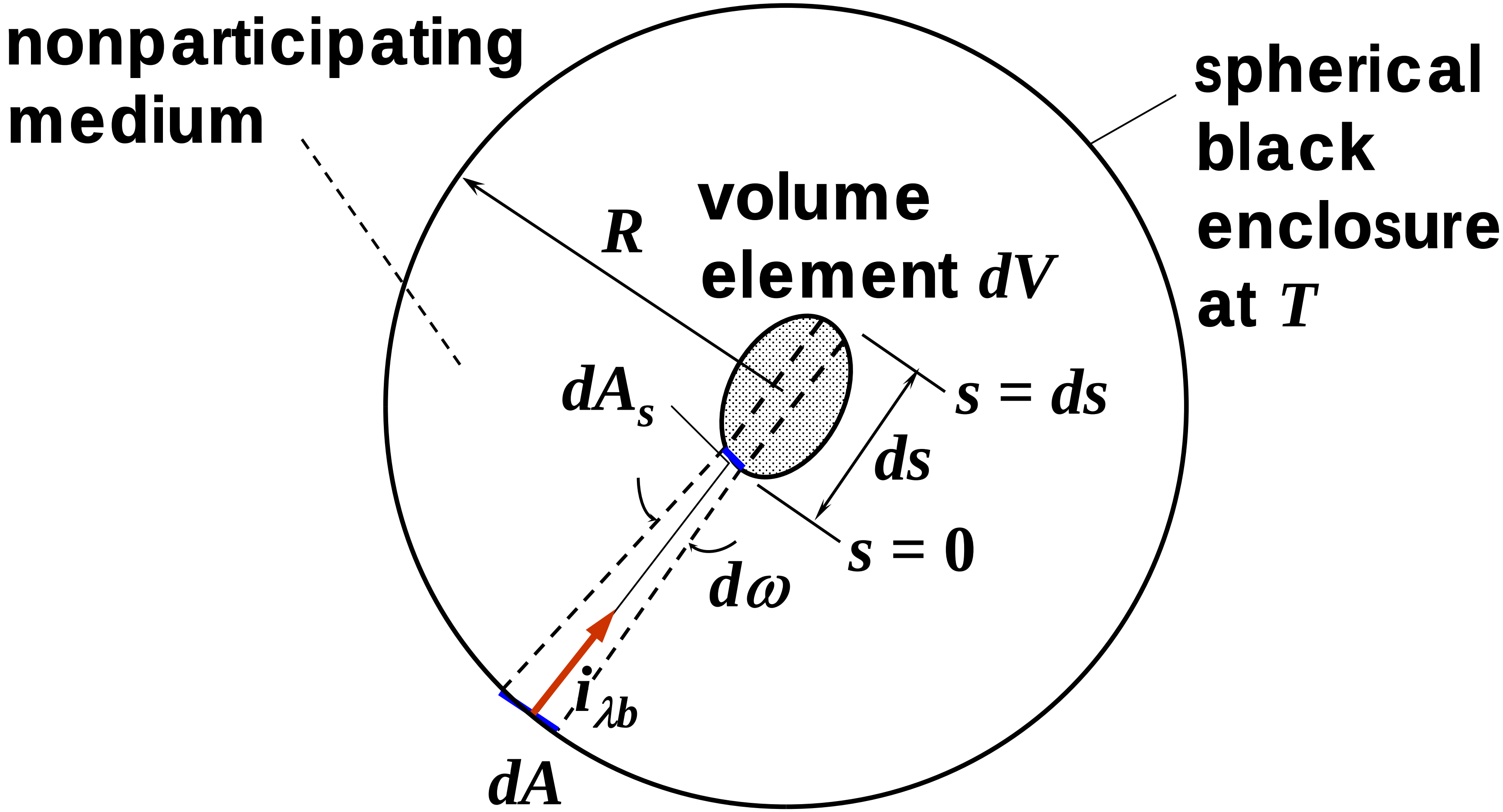
$$\sigma_\lambda = C_s N = \pi r^2 Q_s N$$

For a cloud of particles of different sizes

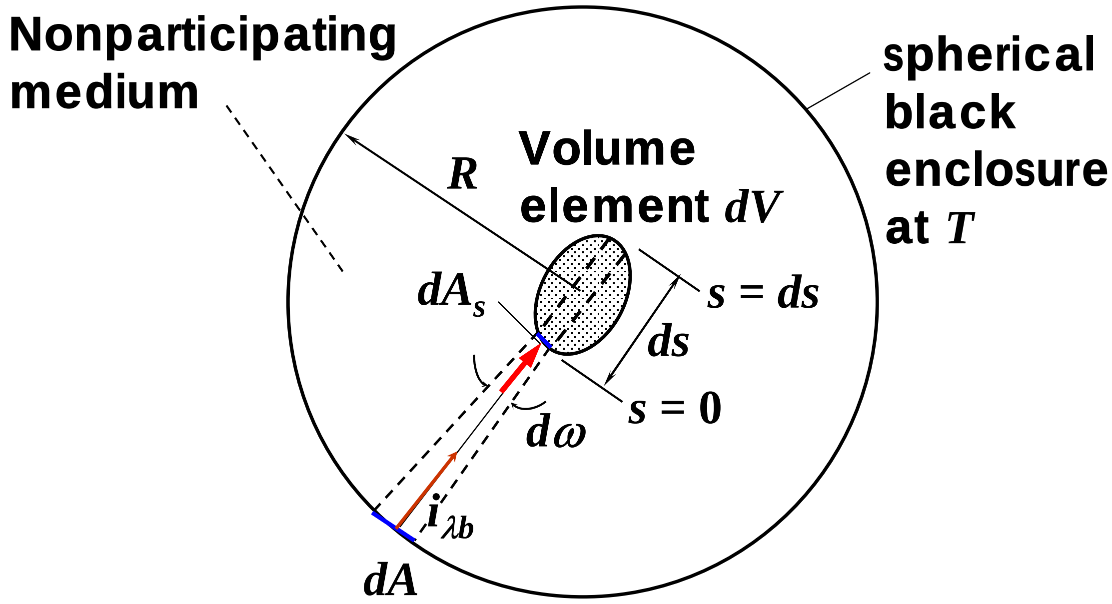
$$a_\lambda = \int_0^\infty C_a dN(r) = \int_0^\infty \pi r^2 Q_a N(r) dr$$

$$\sigma_\lambda = \int_0^\infty C_s dN(r) = \int_0^\infty \pi r^2 Q_s N(r) dr$$

Increase of Intensity by Emission



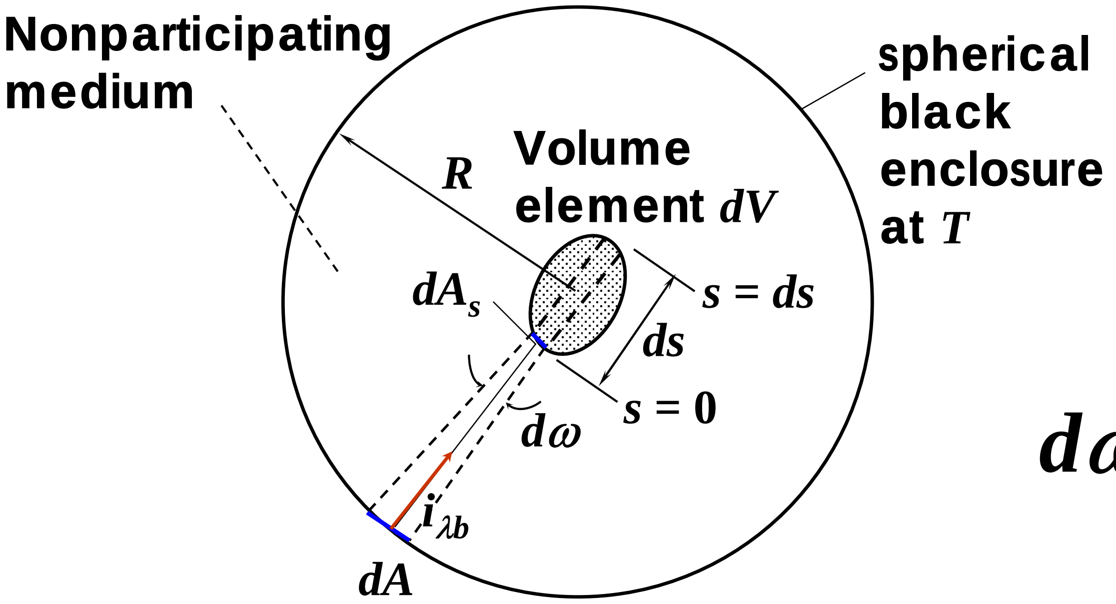
dA_s: a projected area normal to $i_\lambda(0)$



- spectral intensity incident at a location of dA_s on dV, from element dA on the surface of the black enclosure: $i_\lambda(0) = i_{\lambda b}(\lambda, T)$

- the change of this intensity in dV as a result of absorption

$$di_\lambda = -a_\lambda i_\lambda(0) ds = -a_\lambda i_{\lambda b}(\lambda, T) ds$$

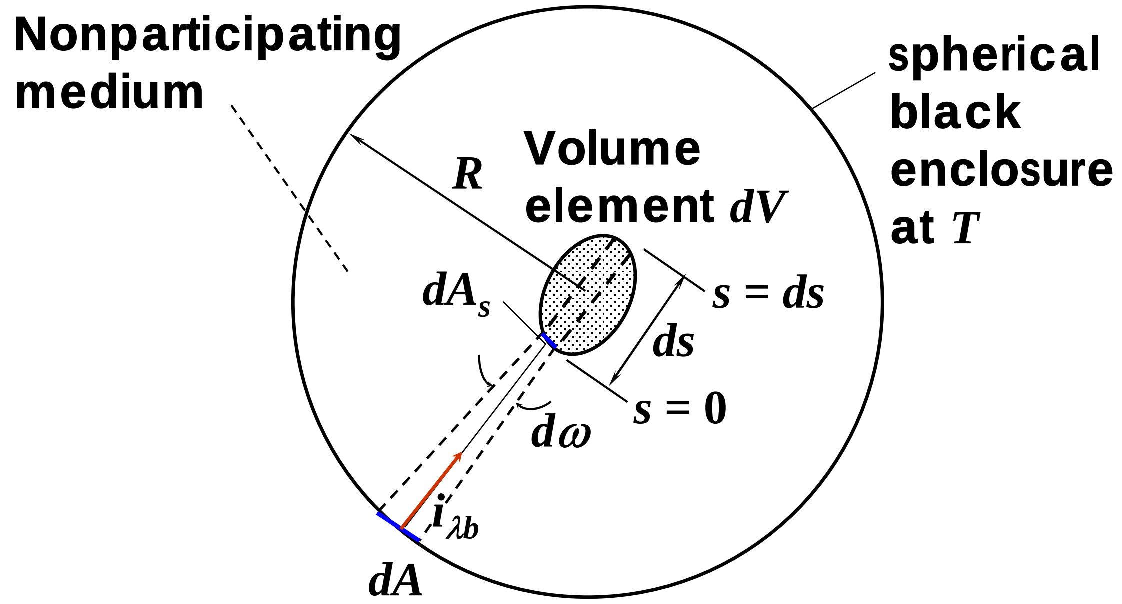


$$d\omega = \frac{dA_s}{R^2}$$

- **the energy absorbed by the differential subvolume** $dsdA_s$ **:** $a_\lambda i_{\lambda b}(\lambda, T)dsdA_s d\omega d\lambda$
- **the energy emitted by** dA **and absorbed by all of** dV, $dV = \int dA_s ds$

$$\int_{dV} a_\lambda i_{\lambda b}(\lambda, T)d\lambda d\omega dA_s ds$$

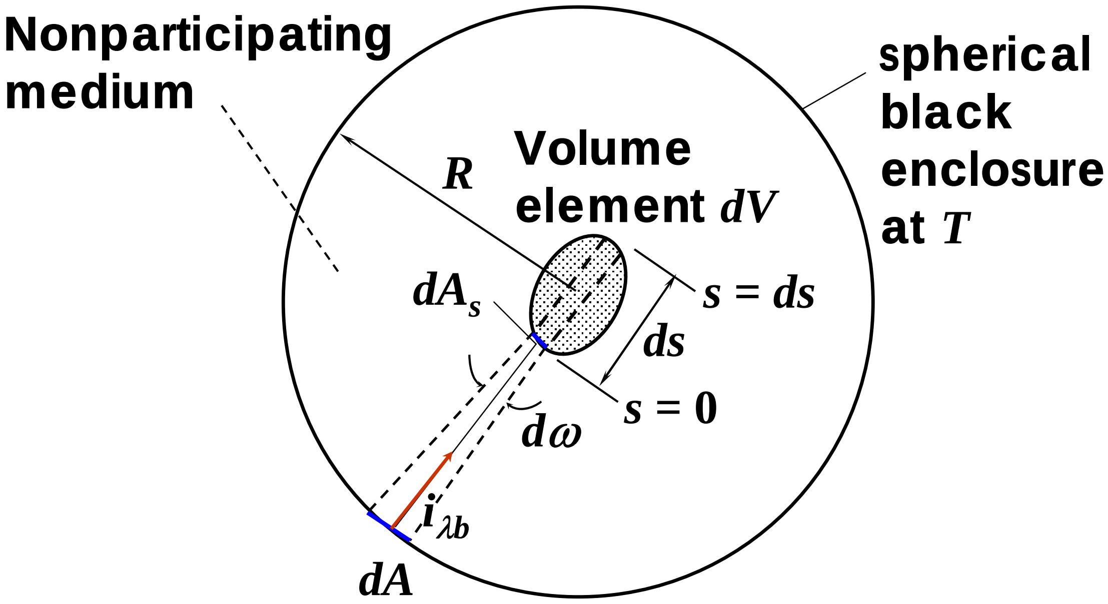
$$= a_\lambda i_{\lambda b}(\lambda, T)d\lambda d\omega \int dA_s ds = a_\lambda i_{\lambda b}(\lambda, T)dVd\lambda d\omega$$



- for all energy incident upon dV from the entire spherical enclosure

$$\int_{\omega=0}^{4\pi} a_\lambda i_{\lambda b}(\lambda,T)\,dV\,d\lambda\,d\omega = a_\lambda i_{\lambda b}(\lambda,T)\,dV\,d\lambda \int_{\omega=0}^{4\pi} d\omega$$

$$= 4\pi a_\lambda i_{\lambda b}(\lambda,T)\,dV\,d\lambda$$



- in equilibrium the energy spontaneously emitted by an isothermal volume element

$$4\pi a_{\lambda} i_{\lambda b}(\lambda, T) dV d\lambda = 4 a_{\lambda} e_{\lambda b}(\lambda, T) dV d\lambda$$

- **the radiation intensity emitted by a volume element into any direction**

$$\int_{4\pi} di_{\lambda,e} dA_p d\lambda d\omega = 4 a_\lambda e_{\lambda b} dV d\lambda$$

dA_p**: the projected area of** dV **normal to the direction of emission**

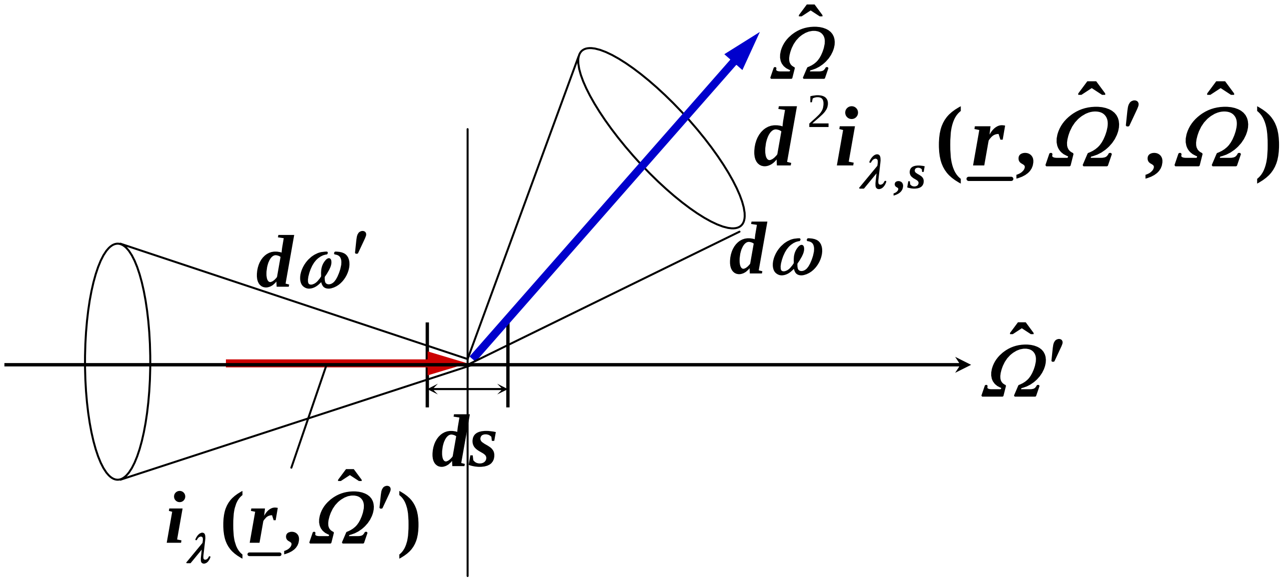
$$ds = \frac{dV}{dA_p}$$ **: the mean thickness of** dV **parallel to the direction of emission**

$$di_{\lambda,e} \cdot 4\pi dA_p d\lambda = 4 a_\lambda e_{\lambda b} dV d\lambda \rightarrow di_{\lambda,e} = a_\lambda i_{\lambda b} ds$$

or $$\frac{di_{\lambda,e}}{ds} = a_\lambda i_{\lambda b}$$

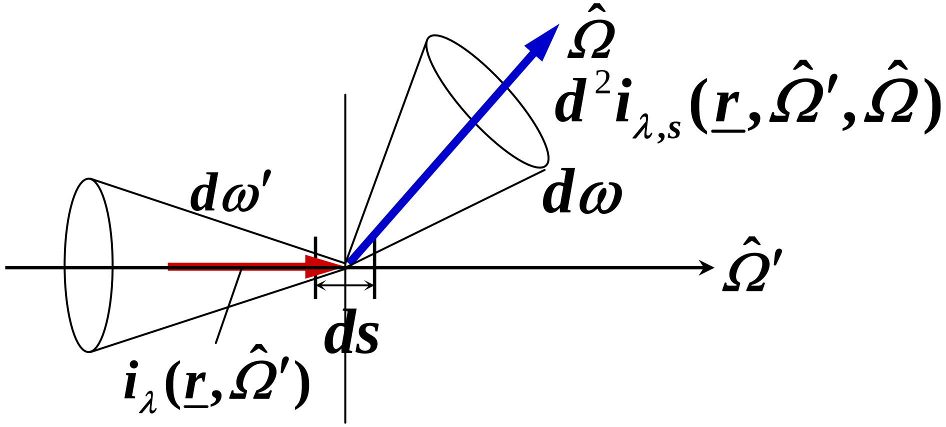
Increase of Intensity by Incoming Scattering

Incoming scattering



scattered intensity in the $\hat{\Omega}$ direction

$$d^2 i_{\lambda,s}(\underline{r},\Omega',\Omega)$$



$$i_\lambda(\underline{r},\hat{\Omega}') + di_\lambda(\underline{r},\hat{\Omega}') = i_\lambda(\underline{r},\hat{\Omega}') - di_{\lambda,s}(\underline{r},\hat{\Omega}')$$

where

$$di_{\lambda,s}(\underline{r},\hat{\Omega}') = \int_{\omega=4\pi} d^2 i_{\lambda,s}(\underline{r},\hat{\Omega}',\hat{\Omega})\,d\omega = \sigma_\lambda i_\lambda(\underline{r},\hat{\Omega}')\,ds$$

phase function: directional distribution of scattered radiation

$$d^2 i_{\lambda,s}(\underline{r},\hat{\Omega}',\hat{\Omega}) \equiv di_{\lambda,s}(\underline{r},\hat{\Omega}')\frac{1}{4\pi}P_\lambda(\hat{\Omega}',\hat{\Omega})$$

$$\int_{\omega=4\pi} d^2 i_{\lambda,s}(\underline{r},\hat{\Omega}',\hat{\Omega})d\omega$$

$$= \frac{1}{4\pi} di_{\lambda,s}(\underline{r},\hat{\Omega}')\int_{\omega=4\pi} P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega$$

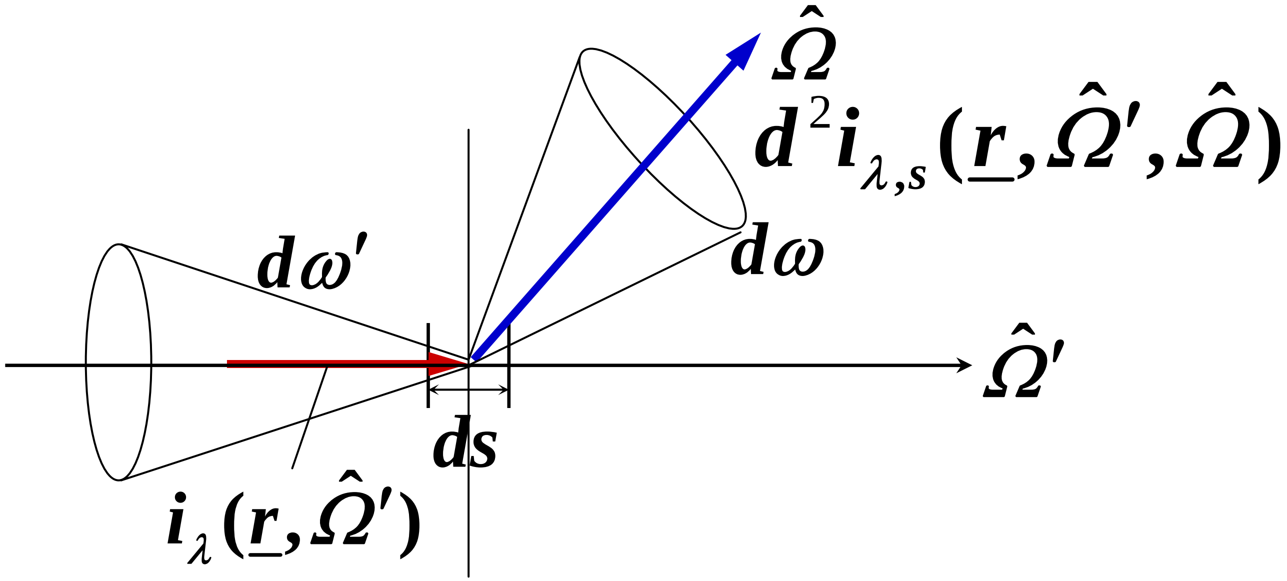
$$\frac{1}{4\pi}\int_{\omega=4\pi} P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega = 1$$

isotropic scattering $\frac{1}{4\pi} P_\lambda \int_{\omega=4\pi} d\omega = P_\lambda = 1$

physical interpretation $\frac{1}{4\pi}\int_{\omega=4\pi} P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega$

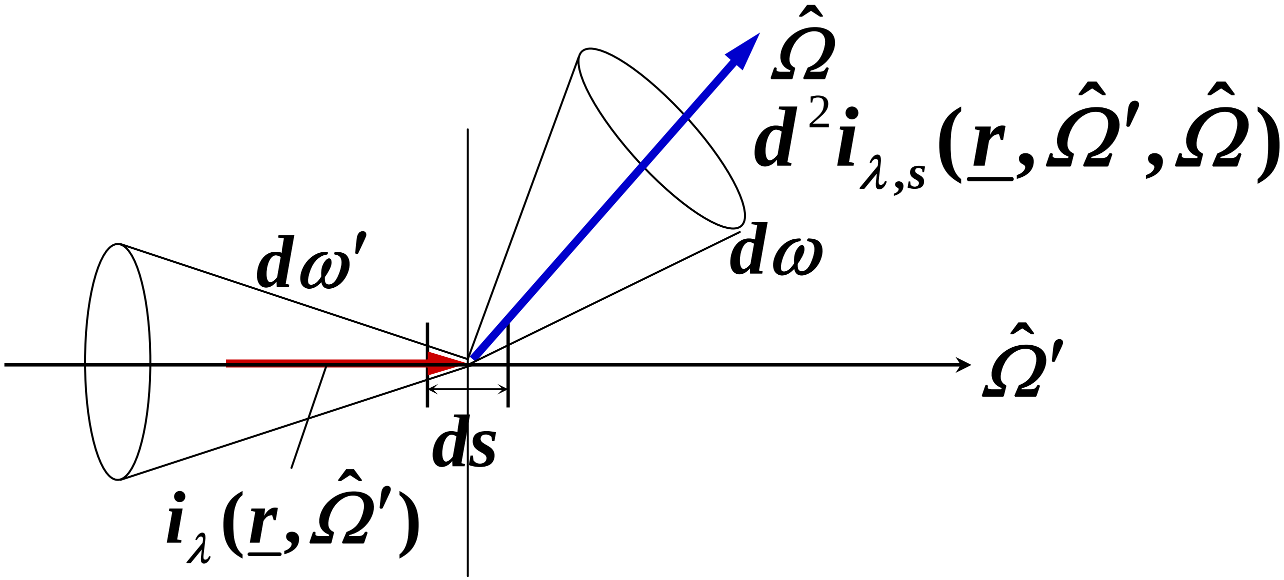
: probability that the incident radiation at $\hat{\Omega}'$ **will be scattered into the element of solid angle** $d\omega$ **about the direction** $\hat{\Omega}$

Augmentation by incoming scattering



scattering of the incident radiation per unit time, per unit wavelength, per unit volume, into an element of solid angle $d\omega$ about $\hat{\Omega}$

$$d^2 i_{\lambda,s}(\underline{r},\hat{\Omega}',\hat{\Omega}) = \left[\sigma_\lambda i_\lambda(\underline{r},\hat{\Omega}')ds\right]\frac{1}{4\pi}P_\lambda(\hat{\Omega}',\hat{\Omega})$$



$$d^2 i_{\lambda,s}(\underline{r},\hat{\Omega}',\hat{\Omega}) = \left[\sigma_\lambda i_\lambda(\underline{r},\hat{\Omega}')ds\right]\frac{1}{4\pi}P_\lambda(\hat{\Omega}',\hat{\Omega})$$

All scattered intensity in the direction $\hat{\Omega}$

$$di_{\lambda,s}(\underline{r},\hat{\Omega}) = \frac{ds}{4\pi}\sigma_\lambda\int_{\omega'=4\pi} i_\lambda(\underline{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

or $$\frac{di_{\lambda,s}}{ds} = \frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\underline{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

Radiative Transfer Equation (RTE)

$$\frac{di_{\lambda,a+s}}{ds} = -\kappa_\lambda i_\lambda, \quad \frac{di_{\lambda,e}}{ds} = a_\lambda i_{\lambda b}$$

$$\frac{di_{\lambda,is}}{ds} = \frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\underline{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

$$\frac{di_\lambda(\underline{r},\hat{\Omega})}{ds} = -\kappa_\lambda i_\lambda(\underline{r},\hat{\Omega}) + a_\lambda i_{\lambda b}(\underline{r})$$

$$+\frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\underline{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$