THE EQUATIONS OF ENERGY TRANSFER FOR ABSORBING, EMITTING, AND SCATTERING MEDIA

- **Radiative Transfer Equation**
- **Equations of Transfer and Flux for Planar Media**
- **Equations for Gray Medium**
- **Solution for Gray Medium in Radiative Equilibrium between Black or Diffuse-Gray Walls at Specified Temperatures**

Equations of Energy Transfer for Participating Media

Radiative Transfer Equation

$$\frac{di_\lambda(\vec{r},\hat{\Omega})}{ds} = -\left(a_\lambda + \sigma_\lambda\right) i_\lambda(\vec{r},\hat{\Omega}) + a_\lambda i_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}') P_\lambda(\hat{\Omega}',\hat{\Omega}) d\omega'$$

in terms of optical thickness

$$\tau_\lambda(s) = \int_0^s \kappa_\lambda(s')ds' = \int_0^s \left(a_\lambda + \sigma_\lambda\right) ds'$$

$$\frac{d}{ds} = \frac{d}{d\tau_\lambda}\frac{d\tau_\lambda}{ds} = \kappa_\lambda \frac{d}{d\tau_\lambda}$$

$$\kappa_\lambda \frac{di_\lambda(\vec{r},\hat{\Omega})}{d\tau_\lambda} = -\kappa_\lambda i_\lambda(\vec{r},\hat{\Omega}) + a_\lambda i_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

or $$\frac{di_\lambda(\vec{r},\hat{\Omega})}{d\tau_\lambda} = -i_\lambda(\vec{r},\hat{\Omega}) + \frac{a_\lambda}{\kappa_\lambda} i_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi\kappa_\lambda}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

scattering albedo

$$\omega_{0\lambda} = \frac{\sigma_\lambda}{\kappa_\lambda} = \frac{\sigma_\lambda}{a_\lambda + \sigma_\lambda} \rightarrow \frac{a_\lambda}{\kappa_\lambda} = \frac{\kappa_\lambda - \sigma_\lambda}{\kappa_\lambda} = \left(1 - \omega_{0_\lambda}\right)$$

$$\frac{di_\lambda(\vec{r},\hat{\Omega})}{d\tau_\lambda} + i_\lambda(\vec{r},\hat{\Omega}) = \left(1-\omega_{0\lambda}\right) i_{\lambda b}(\vec{r}) + \frac{\omega_{0\lambda}}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

source function

$$S_\lambda(\vec{r},\hat{\Omega}) = \left(1-\omega_{0\lambda}\right) i_{\lambda b}(\vec{r}) + \frac{\omega_{0\lambda}}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

$$\frac{di_\lambda(\vec{r},\hat{\Omega})}{d\tau_\lambda} + i_\lambda(\vec{r},\hat{\Omega}) = S_\lambda(\vec{r},\hat{\Omega})$$

Formal solution of RTE in a given direction

integrating factor $\exp(\tau_\lambda)$

$$i_\lambda(\tau_\lambda) = i_\lambda(0)\exp(-\tau_\lambda) + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)\exp\left[-(\tau_\lambda - \tau'_\lambda)\right] d\tau'_\lambda$$

Radiative heat flux vector

$$\vec{q}''_\lambda = \int_{\omega=4\pi} i_\lambda(\vec{r},\hat{\Omega})\hat{\Omega}\, d\omega$$

$$\vec{q}'' = \int_0^\infty \int_{\omega=4\pi} i_\lambda(\vec{r},\hat{\Omega})\hat{\Omega}\, d\omega\, d\lambda$$

$$q''_\lambda = \vec{q}''_\lambda \cdot \hat{n} = \int_{\omega=4\pi} i_\lambda(\vec{r},\hat{\Omega})\cos\theta\, d\omega$$

$$q'' = \int_0^\infty \int_{\omega=4\pi} i_\lambda(\vec{r},\hat{\Omega})\cos\theta\, d\omega\, d\lambda$$

Divergence of radiative heat flux vector

$$\nabla \cdot \vec{q}'' = \nabla \cdot \int_0^{\infty} \vec{q}''_{\lambda} d\lambda = \int_0^{\infty} \nabla \cdot \vec{q}''_{\lambda} d\lambda$$

$$\nabla \cdot \vec{q}''_{\lambda} = \nabla \cdot \int_{\omega=4\pi} i_{\lambda}(\vec{r}, \hat{\Omega}) \hat{\Omega} d\omega$$

$$= \int_{\omega=4\pi} \nabla \cdot \left(i_{\lambda}(\vec{r}, \hat{\Omega}) \hat{\Omega} \right) d\omega$$

$$\nabla \cdot \left(i_{\lambda}(\vec{r}, \hat{\Omega}) \hat{\Omega} \right) = \left(\nabla i_{\lambda}(\vec{r}, \hat{\Omega}) \right) \cdot \hat{\Omega} + i_{\lambda}(\vec{r}, \hat{\Omega}) \nabla \cdot \hat{\Omega}$$

$$= \left(\nabla i_{\lambda}(\vec{r}, \hat{\Omega}) \right) \cdot \hat{\Omega} = \frac{d i_{\lambda}(\vec{r}, \hat{\Omega})}{ds}$$

$$\nabla \cdot \vec{q}''_{\lambda} = \int_{\omega=4\pi} \frac{d i_{\lambda}(\vec{r}, \hat{\Omega})}{ds} d\omega$$

$$\frac{di_\lambda(\vec{r},\hat{\Omega})}{ds} = -\left(a_\lambda + \sigma_\lambda\right) i_\lambda(\vec{r},\hat{\Omega}) + a_\lambda i_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

$$\nabla \cdot \vec{q}''_\lambda = \int_{\omega=4\pi} \frac{di_\lambda(\vec{r},\hat{\Omega})}{ds} d\omega$$

$$\nabla \cdot \vec{q}''_\lambda = \int_{\omega=4\pi} \left(-\kappa_\lambda i_\lambda(\vec{r},\hat{\Omega})\right) d\omega + \int_{\omega=4\pi} a_\lambda i_{\lambda b}(\vec{r}) d\omega + \int_{\omega=4\pi} \left[\frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'\right] d\omega$$

define

$$\overline{P}_\lambda(\hat{\Omega}') = \frac{1}{4\pi}\int_{\omega=4\pi} P_\lambda(\hat{\Omega}', \hat{\Omega})\,d\omega$$

: measure of how much scattering occurs for radiation incident from $\hat{\Omega}'$

Incident radiation

$$G_\lambda(\vec{r}) = \int_{\omega=4\pi} i_\lambda(\vec{r}, \hat{\Omega})\,d\omega$$

$$\nabla\cdot\vec{q}''_\lambda=\int_{\omega=4\pi}\left(-\kappa_\lambda i_\lambda(\vec{r},\hat{\Omega})\right)d\omega+\int_{\omega=4\pi}a_\lambda i_{\lambda b}(\vec{r})d\omega$$

$$+\int_{\omega=4\pi}\left[\frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi}i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'\right]d\omega$$

$$=-\kappa_\lambda\int_{\omega=4\pi}i_\lambda(\vec{r},\hat{\Omega})d\omega+a_\lambda i_{\lambda b}(\vec{r})\int_{\omega=4\pi}d\omega$$

$$+\sigma_\lambda\int_{\omega'=4\pi}i_\lambda(\vec{r},\hat{\Omega}')\left[\frac{1}{4\pi}\int_{\omega=4\pi}P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega\right]d\omega'$$

$$\nabla\cdot\vec{q}''_\lambda=-\kappa_\lambda G_\lambda(\vec{r})+4a_\lambda e_{\lambda b}(\vec{r})$$

$$+\sigma_\lambda\int_{\omega'=4\pi}i_\lambda(\vec{r},\hat{\Omega}')\overline{P}_\lambda(\hat{\Omega}')d\omega'$$

for isotropic scattering

$$\bar{P}_\lambda(\hat{\Omega}') = \frac{1}{4\pi}\int_{\omega=4\pi} P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega = 1$$

$$\nabla\cdot\vec{q}''_\lambda = -\kappa_\lambda G_\lambda(\vec{r}) + 4a_\lambda e_{\lambda b}(\vec{r})$$

$$+\sigma_\lambda\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')\bar{P}_\lambda(\hat{\Omega}')d\omega'$$

$$\nabla\cdot\vec{q}''_\lambda = -\kappa_\lambda G_\lambda + 4a_\lambda e_{\lambda b} + \sigma_\lambda G_\lambda$$

$$= 4a_\lambda\left(e_{\lambda b} - \frac{G_\lambda}{4}\right) = a_\lambda\left(4\pi i_{\lambda b} - G_\lambda\right)$$

Energy equation : summary

$$\rho c_p\left(\frac{\partial T}{\partial t}+\vec{u}\cdot\nabla T\right)=\beta T\left(\frac{\partial P}{\partial t}+\vec{u}\cdot\nabla P\right)$$

$$-\nabla\cdot\vec{q}_c''-\nabla\cdot\vec{q}_r''+\dot{q}+\Phi$$

$$\nabla\cdot\vec{q}_c''=-\nabla\cdot(k\nabla T)$$

$$\nabla\cdot\vec{q}_r''=$$

$$\int_0^\infty\left\{-\kappa_\lambda G_\lambda+4a_\lambda e_{\lambda b}+\sigma_\lambda\int_{\omega'=4\pi}i_\lambda(\vec{r},\hat{\Omega}')\bar{P}_\lambda(\hat{\Omega}')d\omega'\right\}d\lambda$$

where $\bar{P}_\lambda(\hat{\Omega}')=\frac{1}{4\pi}\int_{\omega=4\pi}P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega$

$$G_\lambda(\vec{r})=\int_{\omega=4\pi}i_\lambda(\vec{r},\hat{\Omega})d\omega$$

and

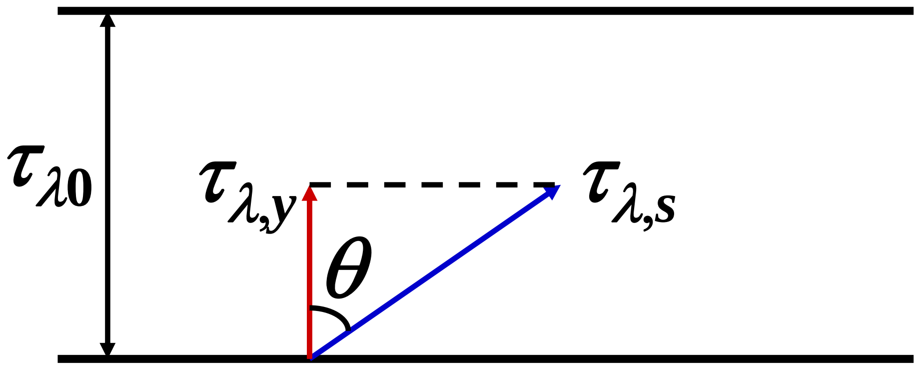
$$\nabla i_\lambda(\vec{r},\hat{\Omega})\cdot\hat{\Omega} = -\left(a_\lambda + \sigma_\lambda\right) i_\lambda(\vec{r},\hat{\Omega}) + a_\lambda i_{\lambda b}(\vec{r})$$

$$+\frac{\sigma_\lambda}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\Omega')P_\lambda(\Omega',\Omega)\,d\omega'$$

$$e_{\lambda b}(T) = \pi i_{\lambda b}(T) = \frac{2\pi C_1}{\lambda^5\left(e^{C_2/\lambda T}-1\right)}$$

Equations of Transfer and Flux for Planar Media

Radiative transfer equation



$$\tau_{\lambda,y} = \tau_{\lambda,s}\cos\theta,$$

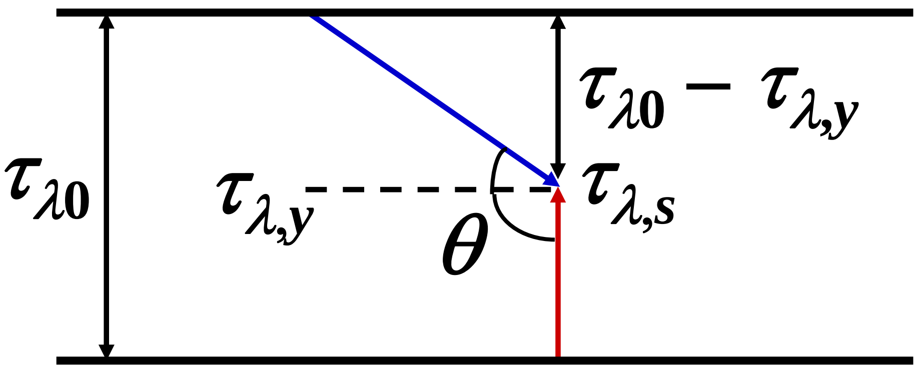
$$\frac{d\tau_{\lambda,y}}{d\tau_{\lambda,s}} = \cos\theta$$

$$\frac{di_\lambda^+}{d\tau_{\lambda,s}} + i_\lambda^+ = S_\lambda$$

$$\frac{di_\lambda^+}{d\tau_{\lambda,s}} = \frac{di_\lambda^+}{d\tau_{\lambda,y}}\frac{d\tau_{\lambda,y}}{d\tau_{\lambda,s}} = \cos\theta\frac{di_\lambda^+}{d\tau_{\lambda,y}}$$

Let $\cos\theta = \mu$ and omit subscript y from now on

$$\mu \frac{di_\lambda^+}{d\tau_\lambda} + i_\lambda^+ = S_\lambda$$



$$\tau_{\lambda 0} - \tau_{\lambda,y} = \tau_{\lambda,s}\cos(\pi - \theta)$$
$$= -\tau_{\lambda,s}\cos\theta$$

$$\frac{d\tau_{\lambda,y}}{d\tau_{\lambda,s}} = \cos\theta$$

$$\mu \frac{di_\lambda^-}{d\tau_\lambda} + i_\lambda^- = S_\lambda$$

isotropic scattering

$$S_\lambda(\vec{r},\hat{\Omega}) = \frac{a_\lambda}{\kappa_\lambda} i_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi\kappa_\lambda}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')P_\lambda(\hat{\Omega}',\hat{\Omega})d\omega'$$

$$S_\lambda(\vec{r}) = \frac{a_\lambda}{\kappa_\lambda} i_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi\kappa_\lambda}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')d\omega'$$

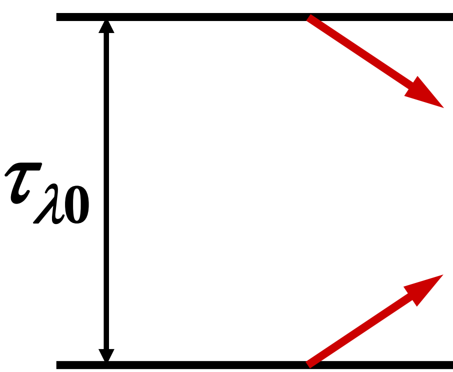
$$= \frac{a_\lambda}{\pi\kappa_\lambda} e_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4\pi\kappa_\lambda} G_\lambda(\vec{r})$$

$$= \frac{1}{\pi\kappa_\lambda}\left[a_\lambda e_{\lambda b}(\vec{r}) + \frac{\sigma_\lambda}{4} G_\lambda(\vec{r})\right]$$

$$\mu \frac{di_\lambda^+}{d\tau_\lambda} + i_\lambda^+ = \frac{1}{\pi\kappa_\lambda}\left[a_\lambda e_{\lambda b} + \frac{\sigma_\lambda}{4} G_\lambda\right]$$

$$\mu \frac{di_\lambda^-}{d\tau_\lambda} + i_\lambda^- = \frac{1}{\pi\kappa_\lambda}\left[a_\lambda e_{\lambda b} + \frac{\sigma_\lambda}{4} G_\lambda\right]$$

boundary conditions

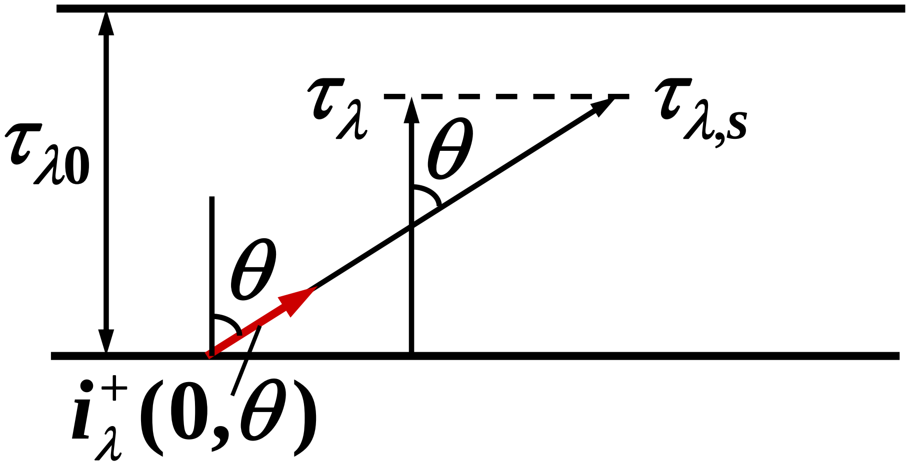


$i_\lambda^-(\tau_\lambda, \mu) = i_\lambda^-(\tau_{\lambda 0}, \mu)$ **at** $\tau_\lambda = \tau_{\lambda 0}$

$i_\lambda^+(\tau_\lambda, \mu) = i_\lambda^+(0, \mu)$ **at** $\tau_\lambda = 0$

Formal solution

for $0 \le \theta \le \pi/2$



$$\tau_{\lambda,s}\cos\theta = \tau_\lambda$$

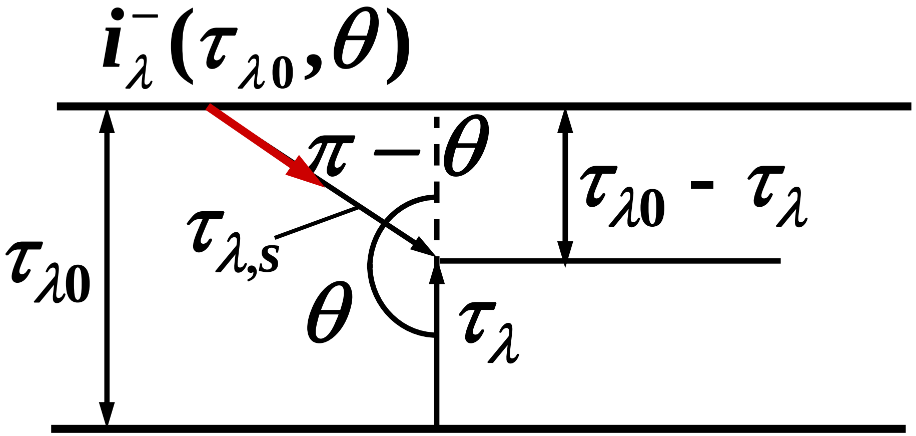
$$\tau'_{\lambda,s}\cos\theta = \tau'_\lambda$$

$$\tau_{\lambda,s} = \frac{\tau_\lambda}{\mu},\quad d\tau'_{\lambda,s} = \frac{1}{\mu}d\tau'_\lambda$$

$$i_\lambda(\tau_{\lambda,s}) = i_\lambda(0)\exp(-\tau_{\lambda,s}) + \int_0^{\tau_{\lambda,s}} S_\lambda(\tau'_{\lambda,s})\exp\left[-\left(\tau_{\lambda,s} - \tau'_{\lambda,s}\right)\right]d\tau'_{\lambda,s}$$

$$i_\lambda^+(\tau_\lambda,\mu) = i_\lambda^+(0,\mu)\exp\left(-\frac{\tau_\lambda}{\mu}\right) + \int_0^{\tau_\lambda}\frac{1}{\mu}S(\tau'_\lambda,\mu)\exp\left[-\frac{1}{\mu}\left(\tau_\lambda - \tau'_\lambda\right)\right]d\tau'_\lambda,\quad \mu > 0$$

for $\pi / 2 \le \theta \le \pi$



$\tau_{\lambda,s} \cos(\pi - \theta) = \tau_{\lambda 0} - \tau_\lambda$

or $-\tau_{\lambda,s} \cos\theta = \tau_{\lambda 0} - \tau_\lambda$

$\tau'_{\lambda,s} \cos(\pi - \theta) = \tau_{\lambda 0} - \tau'_\lambda$

or $-\tau'_{\lambda,s} \cos\theta = \tau_{\lambda 0} - \tau'_\lambda$

$$\tau_{\lambda,s} = -\frac{\tau_{\lambda 0} - \tau_\lambda}{\mu}, \;\; \tau'_{\lambda,s} = -\frac{\tau_{\lambda 0} - \tau'_\lambda}{\mu}$$

$$i_\lambda^-(\tau_\lambda, \mu) = i_\lambda^-(\tau_{\lambda 0}, \mu) \exp\left[\frac{1}{\mu}\left(\tau_{\lambda 0} - \tau_\lambda\right)\right]$$

$$+\int_{\tau_{\lambda 0}}^{\tau_\lambda} \frac{1}{\mu} S(\tau'_\lambda, \mu) \exp\left[-\frac{1}{\mu}\left(\tau_\lambda - \tau'_\lambda\right)\right] d\tau'_\lambda, \;\; \mu < 0$$

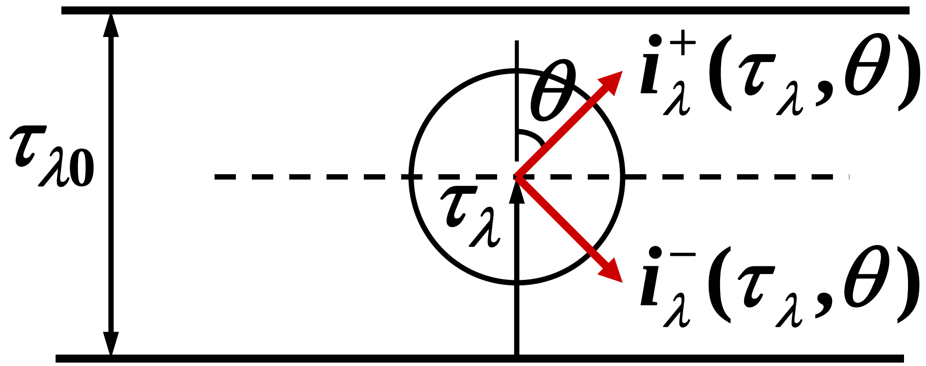
or

$$i_\lambda^-(\tau_\lambda,-\mu)=i_\lambda^-(\tau_{\lambda 0},-\mu)\exp\left[-\frac{1}{\mu}\left(\tau_{\lambda 0}-\tau_\lambda\right)\right]$$

$$+\int_{\tau_\lambda}^{\tau_{\lambda 0}}\frac{1}{\mu}S(\tau_\lambda',-\mu)\exp\left[-\frac{1}{\mu}\left(\tau_\lambda'-\tau_\lambda\right)\right]d\tau_\lambda',\ \ \mu>0$$

Heat flux

$$q''_\lambda = \vec{q}''_\lambda \cdot \hat{n} = \int_{\omega=4\pi} i_\lambda(\vec{r},\hat{\Omega})\cos\theta d\omega$$



$$q''_\lambda = \int_0^{2\pi}\int_0^{\pi} i_\lambda(\tau_\lambda,\mu)\mu\sin\theta d\theta d\phi = 2\pi\int_{-1}^{1} i_\lambda(\tau_\lambda,\mu)\mu d\mu$$

$$= 2\pi\left[\int_0^1 i^+_\lambda(\tau_\lambda,\mu)\mu d\mu + \int_{-1}^0 i^-_\lambda(\tau_\lambda,\mu)\mu d\mu\right]$$

or $$= 2\pi\left[\int_0^1 i^+_\lambda(\tau_\lambda,\mu)\mu d\mu - \int_0^1 i^-_\lambda(\tau_\lambda,-\mu)\mu d\mu\right]$$

$$\int_0^1 i_\lambda^+(\tau_\lambda,\mu)\mu d\mu$$

$$=\int_0^1 \left[i_\lambda^+(0,\mu)\exp\left(-\frac{\tau_\lambda}{\mu}\right)\right.$$

$$\left.+\int_0^{\tau_\lambda} \frac{1}{\mu} S_\lambda(\tau'_\lambda,\mu)\exp\left(-\frac{1}{\mu}(\tau_\lambda-\tau'_\lambda)\right)d\tau'_\lambda\right]\mu d\mu$$

$$=\int_0^1 i_\lambda^+(0,\mu)\exp\left(-\frac{\tau_\lambda}{\mu}\right)\mu d\mu$$

$$+\int_0^1\left[\int_0^{\tau_\lambda} \frac{1}{\mu} S_\lambda(\tau'_\lambda,\mu)\exp\left(-\frac{1}{\mu}(\tau_\lambda-\tau'_\lambda)\right)d\tau'_\lambda\right]\mu d\mu$$

isotropic scattering $S_\lambda(\tau_\lambda,\mu) = S_\lambda(\tau_\lambda)$

$$\int_0^1 \left[\int_0^{\tau_\lambda} \frac{1}{\mu} S_\lambda(\tau'_\lambda,\mu) \exp\left(-\frac{1}{\mu}\left(\tau_\lambda - \tau'_\lambda\right) \right) d\tau'_\lambda \right] \mu d\mu$$

$$= \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda) \left[\int_0^1 \frac{1}{\mu} \exp\left(-\frac{1}{\mu}\left(\tau_\lambda - \tau'_\lambda\right) \right) \mu d\mu \right] d\tau'_\lambda$$

$$= \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda) E_2(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda$$

exponential integral function

$$E_n(\tau) = \int_0^1 \mu^{n-2} \exp\left(-\frac{\tau}{\mu} \right) d\mu$$

$$
\begin{aligned}
q''_\lambda = 2\pi &\left[\int_0^1 i_\lambda^+(0,\mu)\exp\left(-\frac{\tau_\lambda}{\mu}\right)\mu d\mu\right. \\
&\left. + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda) E_2(\tau_\lambda - \tau'_\lambda) d\tau'_\lambda\right] \\
-2\pi &\left[\int_0^1 i_\lambda^-(\tau_{\lambda 0},\mu)\exp\left(-\frac{\tau_{\lambda 0}-\tau_\lambda}{\mu}\right)\mu d\mu\right. \\
&\left. + \int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda) E_2(\tau'_\lambda - \tau_\lambda) d\tau'_\lambda\right]
\end{aligned}
$$

isotropic scattering, boundary intensity independent of direction

$$q''_\lambda = 2\pi\left[i^+_\lambda(0)E_3(\tau_\lambda) + \int_0^{\tau_\lambda} S_\lambda(\tau'_\lambda)E_2(\tau_\lambda - \tau'_\lambda)d\tau'_\lambda \right]$$

$$-2\pi\left[i^-_\lambda(\tau_{\lambda 0})E_3(\tau_{\lambda 0} - \tau_\lambda) + \int_{\tau_\lambda}^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_2(\tau'_\lambda - \tau_\lambda)d\tau'_\lambda \right]$$

non-scattering, boundary intensity independent of direction

$$S_\lambda(\tau_\lambda) = i_{\lambda b}(\tau_\lambda)$$

$$q''_\lambda = 2\pi\left[i^+_\lambda(0)E_3(\tau_\lambda) + \int_0^{\tau_\lambda} i_{\lambda b}(\tau'_\lambda)E_2(\tau_\lambda - \tau'_\lambda)d\tau'_\lambda \right]$$

$$-2\pi\left[i^-_\lambda(\tau_{\lambda 0})E_3(\tau_{\lambda 0} - \tau_\lambda) + \int_{\tau_\lambda}^{\tau_{\lambda 0}} i_{\lambda b}(\tau'_\lambda)E_2(\tau'_\lambda - \tau_\lambda)d\tau' \right]$$

Divergence of heat flux

$$\nabla\cdot\vec{q}''_\lambda = -\kappa_\lambda G_\lambda(\vec{r}) + 4a_\lambda e_{\lambda b}(\vec{r})$$

$$+\sigma_\lambda \int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')\bar{P}_\lambda(\hat{\Omega}')\,d\omega'$$

isotropic scattering

$$\nabla\cdot\vec{q}''_\lambda = -\kappa_\lambda G_\lambda + 4a_\lambda e_{\lambda b} + \sigma_\lambda G_\lambda = a_\lambda\left(4\pi i_{\lambda b} - G_\lambda\right)$$

$$\nabla\cdot\vec{q}''_\lambda = \frac{dq''_\lambda}{dy} = a_\lambda\left(4\pi i_{\lambda b} - G_\lambda\right) = \kappa_\lambda \frac{dq''_\lambda}{d\tau_\lambda}$$

$$\frac{dq''_\lambda}{d\tau_\lambda} = \left(1-\omega_{0\lambda}\right)\left(4\pi i_{\lambda b} - G_\lambda\right)$$

$$G_\lambda(\vec{r}) = \int_{\omega=4\pi} i_\lambda(\vec{r},\Omega)d\omega = 2\pi\int_{-1}^{1} i_\lambda(\tau_\lambda,\mu)d\mu$$

$$= 2\pi\left[\int_0^1 i_\lambda^+(\tau_\lambda,\mu)d\mu + \int_{-1}^0 i_\lambda^-(\tau_\lambda,\mu)d\mu\right]$$

$$= 2\pi\left[\int_0^1 i_\lambda^+(\tau_\lambda,\mu)d\mu + \int_0^1 i_\lambda^-(\tau_\lambda,-\mu)d\mu\right]$$

$$i_\lambda^+(\tau_\lambda,\mu) = i_\lambda^+(0,\mu)\exp\left(-\frac{\tau_\lambda}{\mu}\right)$$

$$+\int_0^{\tau_\lambda} \frac{1}{\mu} S(\tau'_\lambda)\exp\left[-\frac{1}{\mu}\left(\tau_\lambda - \tau'_\lambda\right)\right]d\tau'_\lambda$$

$$i_\lambda^-(\tau_\lambda,-\mu) = i_\lambda^-(\tau_{\lambda 0},-\mu)\exp\left[-\frac{1}{\mu}\left(\tau_{\lambda 0} - \tau_\lambda\right)\right]$$

$$+\int_{\tau_\lambda}^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda)\exp\left[-\frac{1}{\mu}\left(\tau'_\lambda - \tau_\lambda\right)\right]d\tau'_\lambda$$

$$\int_0^{\tau_\lambda} \frac{1}{\mu} S(\tau'_\lambda) \exp\left[-\frac{1}{\mu}\left(\tau_\lambda - \tau'_\lambda\right)\right] d\tau'_\lambda + \int_{\tau_\lambda}^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda) \exp\left[-\frac{1}{\mu}\left(\tau'_\lambda - \tau_\lambda\right)\right] d\tau'_\lambda$$

$$= \int_0^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda) \exp\left[-\frac{1}{\mu}\left|\tau_\lambda - \tau'_\lambda\right|\right] d\tau'_\lambda$$

$$\int_0^1 \left\{ \int_0^{\tau_{\lambda 0}} \frac{1}{\mu} S(\tau'_\lambda) \exp\left[-\frac{1}{\mu}\left|\tau_\lambda - \tau'_\lambda\right|\right] d\tau'_\lambda \right\} d\mu$$

$$= \int_0^{\tau_{\lambda 0}} \left\{ \int_0^1 \frac{1}{\mu} S(\tau'_\lambda) \exp\left[-\frac{1}{\mu}\left|\tau_\lambda - \tau'_\lambda\right|\right] d\mu \right\} d\tau'_\lambda$$

$$= \int_0^{\tau_{\lambda 0}} S(\tau'_\lambda) \left\{ \int_0^1 \frac{1}{\mu} \exp\left[-\frac{1}{\mu}\left|\tau_\lambda - \tau'_\lambda\right|\right] d\mu \right\} d\tau'_\lambda$$

$$= \int_0^{\tau_{\lambda 0}} S(\tau'_\lambda) E_1(\left|\tau_\lambda - \tau'_\lambda\right|) d\tau'_\lambda$$

$$
\begin{aligned}
\frac{dq''_\lambda}{d\tau_\lambda} = {} & \left(1-\omega_{0\lambda}\right)4\pi i_{\lambda b}(\tau_\lambda) \\
& -\left(1-\omega_{0\lambda}\right)2\pi\Bigg[\int_0^1 i_\lambda^+(0,\mu)\exp\left(-\frac{\tau_\lambda}{\mu}\right)d\mu \\
& +\int_0^1 i_\lambda^-(\tau_{\lambda 0},-\mu)\exp\left(-\frac{\tau_{\lambda 0}-\tau_\lambda}{\mu}\right)d\mu \\
& +\int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_1(\left|\tau_\lambda-\tau'_\lambda\right|)d\tau'_\lambda\Bigg]
\end{aligned}
$$

isotropic scattering, boundary intensity independent of direction

$$\frac{dq''_\lambda}{d\tau_\lambda} = \left(1-\omega_{0\lambda}\right)4\pi i_{\lambda b}(\tau_\lambda)$$
$$-\left(1-\omega_{0\lambda}\right)2\pi\Big[i_\lambda^+(0)E_2(\tau_\lambda)+i_\lambda^-(\tau_{\lambda 0})E_2(\tau_{\lambda 0}-\tau_\lambda)$$
$$+\int_0^{\tau_{\lambda 0}} S_\lambda(\tau'_\lambda)E_1(|\tau_\lambda-\tau'_\lambda|)d\tau'_\lambda\Big]$$

non-scattering, boundary intensity independent of direction

$$\frac{dq''_\lambda}{d\tau_\lambda} = 4\pi i_{\lambda b} - 2\pi\Big[i_\lambda^+(0)E_2(\tau_\lambda)$$
$$+i_\lambda^-(\tau_{\lambda 0})E_2(\tau_{\lambda 0}-\tau_\lambda)+\int_0^{\tau_{\lambda 0}} i_{\lambda b}(\tau'_\lambda)E_1(|\tau_\lambda-\tau'_\lambda|)d\tau'_\lambda\Big]$$

Equations for Gray Medium

Gray medium with isotropic scattering

$$S_\lambda(\vec{r}) = \left(1-\omega_{0\lambda}\right) i_{\lambda b}(\vec{r}) + \frac{\omega_{0\lambda}}{4\pi}\int_{\omega'=4\pi} i_\lambda(\vec{r},\hat{\Omega}')d\omega'$$

$$S(\tau) = \left(1-\omega_0\right) i_b(\tau) + \frac{\omega_0}{2}\int_{-1}^{1} i(\tau,\mu)\mu d\mu$$

Heat flux

$$q''(\tau) = 2\pi\left[\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)\mu d\mu + \int_0^\tau S(\tau')E_2(\tau-\tau')d\tau'\right]$$

$$-2\pi\left[\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)\mu d\mu + \int_\tau^{\tau_0} S(\tau')E_2(\tau'-\tau)d\tau'\right]$$

for diffuse boundaries

$$q''(\tau) = 2\pi\left[i^{+}(0)E_3(\tau) + \int_0^{\tau} S(\tau')E_2(\tau-\tau')d\tau' \right]$$

$$-2\pi\left[i^{-}(\tau_0)E_3(\tau_0-\tau) + \int_{\tau}^{\tau_0} S(\tau')E_2(\tau'-\tau)d\tau' \right]$$

Divergence of heat flux

$$\frac{dq''}{d\tau} = \left(1-\omega_0\right)\left[4\pi i_b - G\right]$$

$$= \left(1-\omega_0\right)4\pi i_b - \left(1-\omega_0\right)2\pi\left[\int_0^1 i^{+}(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu\right.$$

$$\left.+\int_0^1 i^{-}(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu + \int_0^{\tau_0} S(\tau')E_1(\left|\tau-\tau'\right|)d\tau'\right]$$

$$= 4\pi\left[\left(1-\omega_0\right)i_b(\tau)+\frac{\omega_0}{2}\left\{\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu\right.\right.$$

$$\left.\left.+\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu+\int_0^{\tau_0} S(\tau')E_1(|\tau-\tau'|)d\tau'\right\}\right]$$

$$-2\pi\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu$$

$$-2\pi\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu$$

$$-2\pi\int_0^{\tau_0} S(\tau')E_1(|\tau-\tau'|)d\tau'$$

since

$$S(\tau)=\left(1-\omega_0\right)i_b(\tau)+\frac{\omega_0}{2}\left[\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu\right.$$

$$\left.+\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu+\int_0^{\tau_0} S(\tau')E_1(|\tau-\tau'|)d\tau'\right]$$

$$\frac{dq''}{d\tau}=4\pi S(\tau)-2\pi\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu$$

$$-2\pi\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu$$

$$-2\pi\int_0^{\tau_0} S(\tau')E_1(|\tau-\tau'|)d\tau'$$

for diffuse boundaries

$$\frac{dq''}{d\tau} = 4\pi S(\tau) - 2\pi \Big[i^{+}(0)E_2(\tau)$$

$$+ i^{-}(\tau_0)E_2(\tau_0 - \tau) + \int_0^{\tau_0} S(\tau')E_1(|\tau - \tau'|)d\tau' \Big]$$

$$\frac{dq''}{d\tau} = (1 - \omega_0)[4\pi i_b - G]$$

$$= 4\pi(1 - \omega_0)i_b + \omega_0 G - G$$

$$= 4\pi\left[(1 - \omega_0)i_b + \frac{\omega_0}{4\pi}G\right] - G$$

$$= 4\pi S - G$$

$$S(\vec{r}) = (1 - \omega_0)i_b(\vec{r}) + \frac{\omega_0}{4\pi}\int_{\omega'=4\pi} i(\vec{r}, \hat{\Omega}')d\omega'$$

Gray Medium in Radiative Equilibrium

$$\nabla \cdot \vec{q}_r'' = 0 \rightarrow \frac{dq''}{d\tau} = 0$$

isotropic scattering

$$\frac{dq''}{d\tau} = (1-\omega_0)[4\pi i_b - G] = 4\pi S - G = 0$$

$$G = 4\pi i_b = 4\pi S$$

gray medium in radiative equilibrium between diffuse gray boundaries

$$i^+(0,\mu) = \frac{J_1}{\pi},\ i^-(\tau_0,-\mu) = \frac{J_2}{\pi}$$

$$S(\tau) = \frac{G(\tau)}{4\pi},\ G(\tau) = 4\pi i_b = 4\sigma T^4,\ S(\tau) = i_b$$

$$q''(\tau) = 2\pi\left[i^{+}(0)E_3(\tau) + \int_0^{\tau} S(\tau')E_2(\tau - \tau')d\tau' \right]$$

$$-2\pi\left[i^{-}(\tau_0)E_3(\tau_0 - \tau) + \int_{\tau}^{\tau_0} S(\tau')E_2(\tau' - \tau)d\tau' \right]$$

$$i^{+}(0,\mu) = \frac{J_1}{\pi},\;\; i^{-}(\tau_0,-\mu) = \frac{J_2}{\pi}$$

$$S(\tau) = \frac{G(\tau)}{4\pi},\;\; G(\tau) = 4\pi i_b = 4\sigma T^4,\;\; S(\tau) = i_b = \frac{\sigma T^4}{\pi}$$

$$q''(\tau) = 2\left[J_1E_3(\tau) + \int_0^{\tau} \sigma T^4(\tau')E_2(\tau - \tau')d\tau' \right]$$

$$-2\left[J_2E_3(\tau_0 - \tau) + \int_{\tau}^{\tau_0} \sigma T^4(\tau')E_2(\tau' - \tau)d\tau' \right]$$

$$\frac{dq''}{d\tau} = \left(1-\omega_0\right)4\pi i_b - \left(1-\omega_0\right)2\pi\left[\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu\right.$$

$$\left.+\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu + \int_0^{\tau_0} S(\tau')E_1(\left|\tau-\tau'\right|)d\tau'\right]$$

$$\frac{dq''}{d\tau} = \left(1-\omega_0\right)4\pi i_b - \left(1-\omega_0\right)2\pi\left[i^+(0)E_2(\tau)\right.$$

$$\left.+i^-(\tau_0)E_2(\tau_0-\tau) + \int_0^{\tau_0} S(\tau')E_1(\left|\tau-\tau'\right|)d\tau'\right] = 0$$

$$4\pi i_b = G(\tau) = 2\pi\left[\int_0^1 i^+(0,\mu)\exp\left(-\frac{\tau}{\mu}\right)d\mu\right.$$

$$\left.+\int_0^1 i^-(\tau_0,-\mu)\exp\left(-\frac{\tau_0-\tau}{\mu}\right)d\mu + \int_0^{\tau_0} S(\tau')E_1(\left|\tau-\tau'\right|)d\tau'\right]$$

$$4\pi i_b = G(\tau) = 2\pi\Big[i^+(0)E_2(\tau) + i^-(\tau_0)E_2(\tau_0 - \tau) + \int_0^{\tau_0} S(\tau')E_1(|\tau - \tau'|)d\tau' \Big]$$

$$G(\tau) = 2\Big[J_1E_2(\tau) + J_2E_2(\tau_0 - \tau) + \int_0^{\tau_0} \sigma T^4(\tau')E_1(|\tau - \tau'|)d\tau' \Big]$$

$$\sigma T^4(\tau) = \frac{1}{2}\Big[J_1E_2(\tau) + J_2E_2(\tau_0 - \tau) + \int_0^{\tau_0} \sigma T^4(\tau')E_1(|\tau - \tau'|)d\tau' \Big]$$

Solution for Gray Medium in Radiative Equilibrium between Black or Diffuse-Gray Walls at Specified Temperatures

Between black walls

$$q''(\tau) = 2\left[J_1 E_3(\tau) + \int_0^{\tau} \sigma T^4(\tau') E_2(\tau - \tau') d\tau'\right]$$

$$-2\left[J_2 E_3(\tau_0 - \tau) + \int_{\tau}^{\tau_0} \sigma T^4(\tau') E_2(\tau' - \tau) d\tau'\right]$$

$$J_1 = \sigma T_1^4,\ J_2 = \sigma T_2^4$$

$$q''(\tau) = q''(0)$$

$$= \sigma T_1^4 - 2\sigma T_2^4 E_3(\tau_0) - 2\int_0^{\tau_0} \sigma T^4(\tau') E_2(\tau') d\tau' \quad \left(E_3(0) = \frac{1}{2}\right)$$

$$\sigma T^4(\tau) = \frac{1}{2}\left[J_1 E_2(\tau) + J_2 E_2(\tau_0 - \tau)\right.$$

$$\left. + \int_0^{\tau_0} \sigma T^4(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

$$T^4(\tau) = \frac{1}{2}\left[T_1^4 E_2(\tau) + T_2^4 E_2(\tau_0 - \tau) + \int_0^{\tau_0} T^4(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

$$\phi_b(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4}, \quad \psi_b = \frac{q''}{\sigma\left(T_1^4 - T_2^4\right)}$$

$$T^4(\tau) - T_2^4 = \frac{1}{2}\left[T_1^4 E_2(\tau) + T_2^4 E_2(\tau_0 - \tau)\right.$$

$$\left. + \int_0^{\tau_0} \left(T^4(\tau') - T_2^4\right) E_1(|\tau - \tau'|) d\tau' + T_2^4 \int_0^{\tau_0} E_1(|\tau - \tau'|) d\tau' \right] - T_2^4$$

$$\int_0^{\tau_0} E_1(|\tau-\tau'|)d\tau' = \int_0^{\tau} E_1(\tau-\tau')d\tau' + \int_{\tau}^{\tau_0} E_1(\tau'-\tau)d\tau'$$

$$= \left[E_2(\tau-\tau')\right]_0^{\tau} + \left[-E_2(\tau'-\tau)\right]_{\tau}^{\tau_0} \quad \left(\int E_n(x)dx = -E_{n+1}(x)\right)$$

$$= E_2(0) - E_2(\tau) - E_2(\tau_0-\tau) + E_2(0)$$

$$= 2 - E_2(\tau) - E_2(\tau_0-\tau) \quad \left(E_2(0)=1\right)$$

$$T^4(\tau) - T_2^4 = \frac{1}{2}\Big[T_1^4 E_2(\tau) + \cancel{T_2^4 E_2(\tau_0-\tau)}$$

$$+\int_0^{\tau_0}\left(T^4(\tau')-T_2^4\right)E_1(|\tau-\tau'|)d\tau'$$

$$+T_2^4\left\{\cancel{2} - E_2(\tau) - \cancel{E_2(\tau_0-\tau)}\right\}\Big] - \cancel{T_2^4}$$

$$= \frac{1}{2}\left[\left(T_1^4 - T_2^4\right)E_2(\tau) + \int_0^{\tau_0}\left(T^4(\tau')-T_2^4\right)E_1(|\tau-\tau'|)d\tau'\right]$$

$$\phi_b(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4} = \frac{1}{2}\left[E_2(\tau) + \int_0^{\tau_0} \phi_b(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

$$q''(\tau) = \sigma T_1^4 - 2\sigma T_2^4 E_3(\tau_0) - 2\int_0^{\tau_0} \sigma T^4(\tau') E_2(\tau') d\tau'$$

$$q''(\tau) = \sigma\left(T_1^4 - T_2^4\right) - \cancel{2\sigma T_2^4 E_3(\tau_0)}$$

$$-2\int_0^{\tau_0} \sigma\left(T^4(\tau') - T_2^4\right) E_2(\tau') d\tau' + \cancel{\sigma T_2^4} - 2\int_0^{\tau_0} \sigma T_2^4 E_2(\tau') d\tau'$$

$$\int_0^{\tau_0} \sigma T_2^4 E_2(\tau') d\tau' = \sigma T_2^4 \int_0^{\tau_0} E_2(\tau') d\tau' = \sigma T_2^4 \left[-E_3(\tau')\right]_0^{\tau_0}$$

$$= \sigma T_2^4 \left[E_3(0) - E_3(\tau_0)\right] = \sigma T_2^4 \left[\cancel{\frac{1}{2}} - \cancel{E_3(\tau_0)}\right]$$

$$q''(\tau) = \sigma\left(T_1^4 - T_2^4\right) - 2\int_0^{\tau_0} \sigma\left(T^4(\tau') - T_2^4\right)E_2(\tau')d\tau'$$

$$\psi_b(\tau) = \frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = 1 - 2\int_0^{\tau_0} \phi_b(\tau')E_2(\tau')d\tau'$$

$$\phi_b(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4} = \frac{1}{2}\left[E_2(\tau) + \int_0^{\tau_0} \phi_b(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

$$\psi_b(\tau) = \frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = 1 - 2\int_0^{\tau_0} \phi_b(\tau') E_2(\tau') d\tau'$$

numerical solution available

limiting behavior

$$q''(\tau_0 \to 0) = \sigma\left(T_1^4 - T_2^4\right)$$

$$\frac{T^4(\tau_0 \to 0) - T_2^4}{T_1^4 - T_2^4} = \frac{1}{2}\left[E_2(0)\right] = \frac{1}{2} \quad \left(E_2(0) = 1\right)$$

$$T^4(\tau_0 \to 0) = \frac{T_1^4 - T_2^4}{2} + T_2^4, \;\; T^4(\tau_0 \to 0) = \frac{T_1^4 + T_2^4}{2}$$

Temperature jump at the wall

$$\phi_b(\tau) = \frac{T^4(\tau) - T_2^4}{T_1^4 - T_2^4} = \frac{1}{2}\left[E_2(\tau) + \int_0^{\tau_0} \phi_b(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

$$T^4(\tau) - T_2^4$$

$$= \frac{1}{2}\left[\left(T_1^4 - T_2^4\right) E_2(\tau) + \int_0^{\tau_0} \left(T^4(\tau') - T_2^4\right) E_1(|\tau - \tau'|) d\tau' \right]$$

$$T^4(\tau = 0) - T_2^4$$

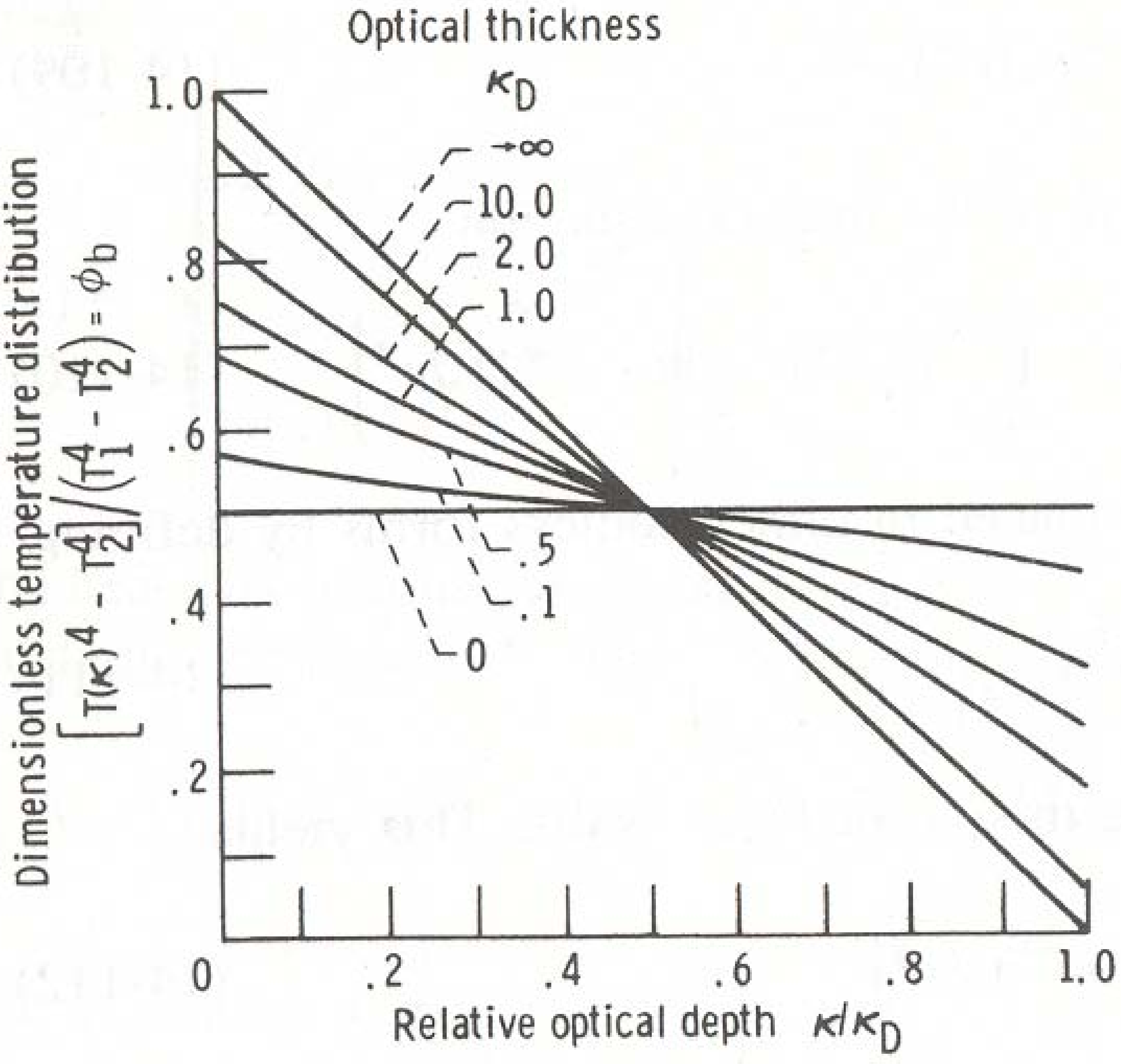
$$= \frac{1}{2}\left[\left(T_1^4 - T_2^4\right) E_2(0) + \int_0^{\tau_0} \left(T^4(\tau') - T_2^4\right) E_1(\tau') d\tau' \right]$$

$$= \frac{1}{2}\left[T_1^4 - T_2^4 - \int_0^{\tau_0} T_2^4 E_1(\tau') d\tau' + \int_0^{\tau_0} T^4(\tau') E_1(\tau') d\tau' \right]$$

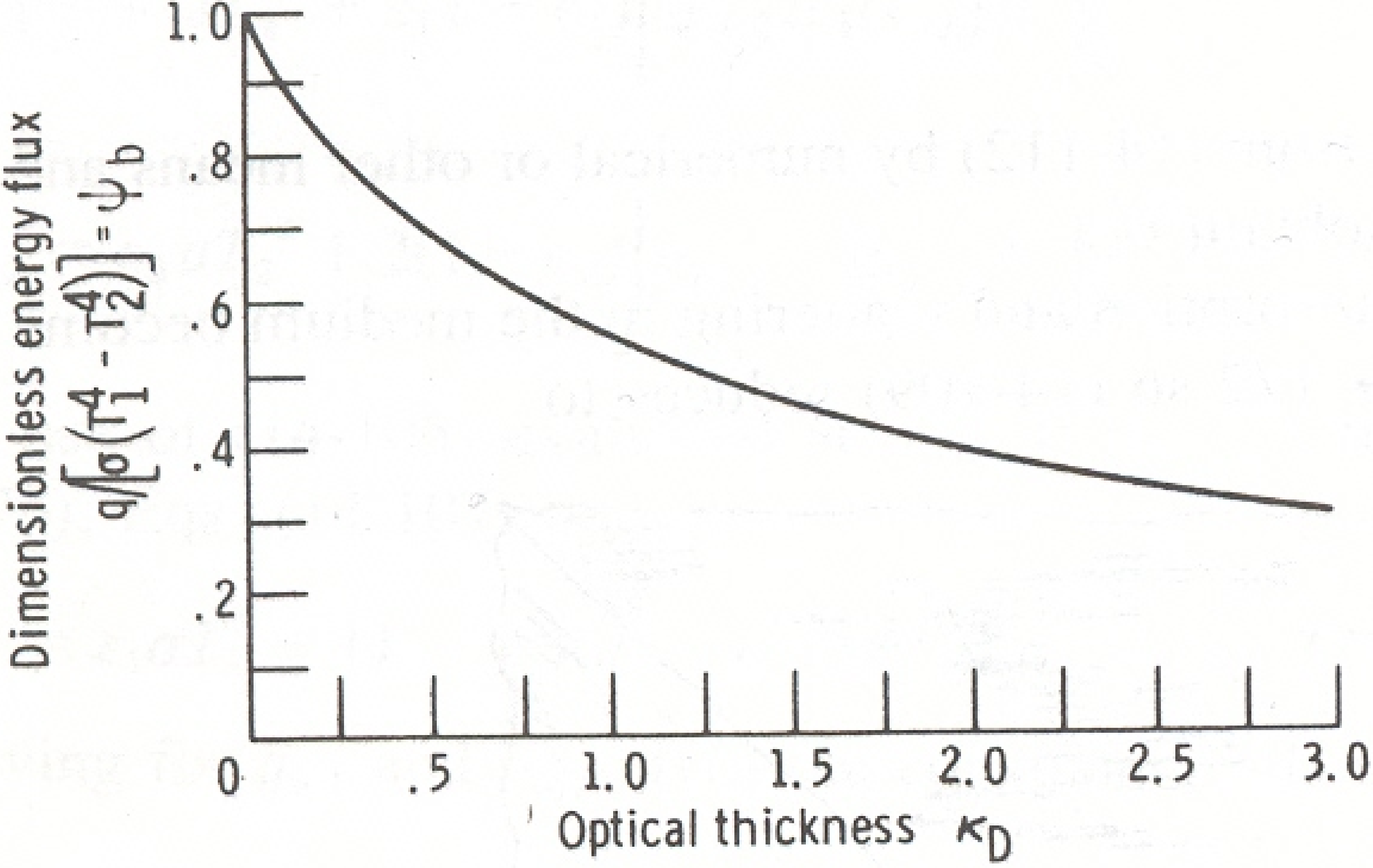
$$\int_0^{\tau_0} E_1(\tau') d\tau' = \left[-E_2(\tau') \right]_0^{\tau_0} = \left[E_2(0) - E_2(\tau_0) \right]_0^{\tau_0} = 1 - E_2(\tau_0)$$

$$T^4(\tau = 0) = \frac{1}{2}\left[T_1^4 + T_2^4 E_2(\tau_0) + \int_0^{\tau_0} T^4(\tau') E_1(\tau') d\tau' \right]$$

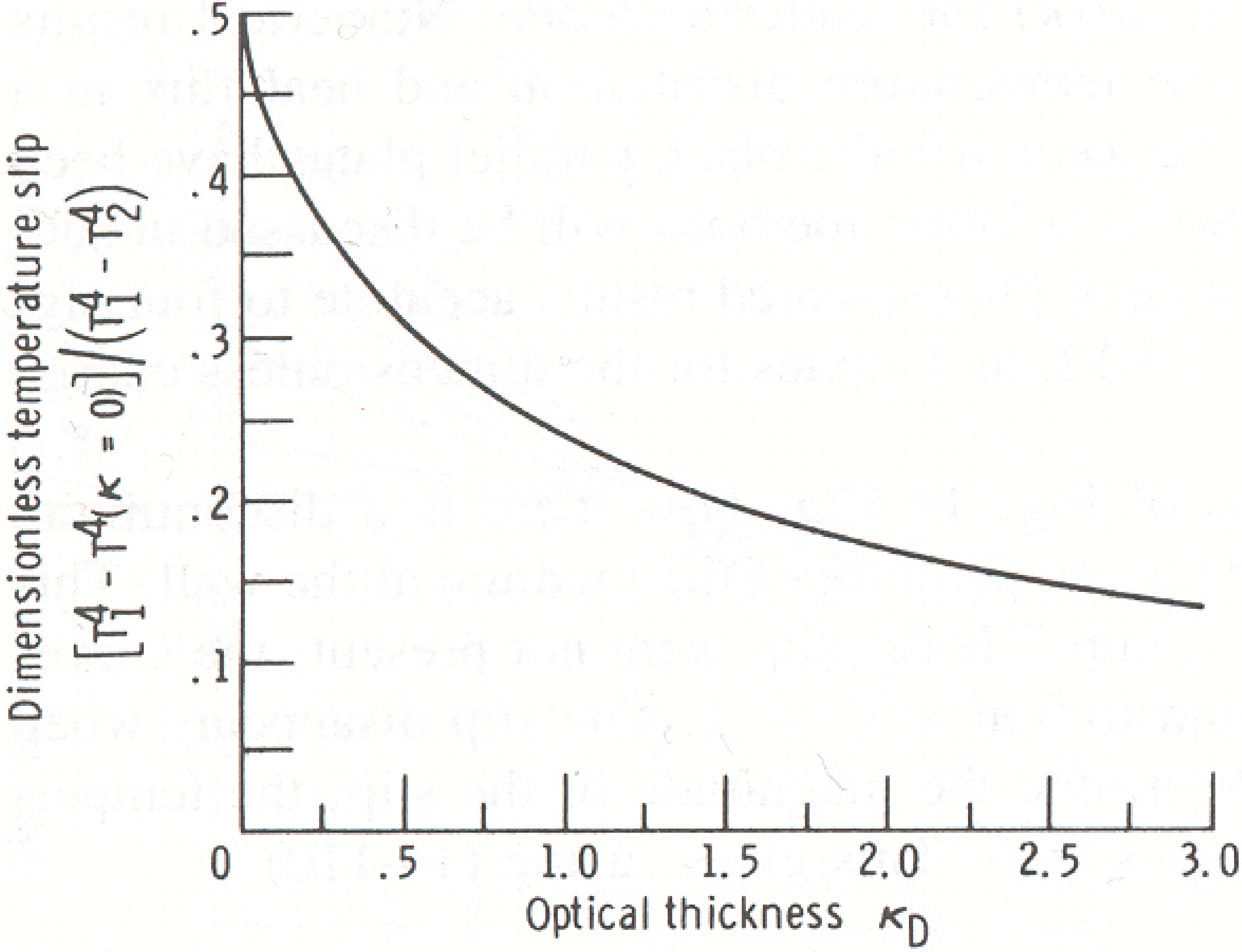
$$T^4(\tau = 0) - T_1^4 = \frac{1}{2}\left[-T_1^4 + T_2^4 E_2(\tau_0) + \int_0^{\tau_0} T^4(\tau') E_1(\tau') d\tau' \right]$$



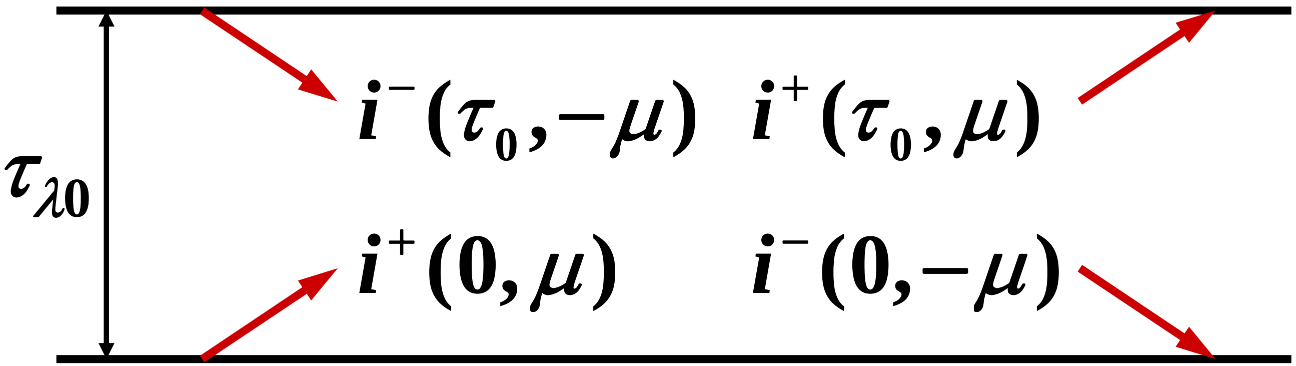
Optical thickness
κ_D
→∞
10.0
2.0
1.0
.5
.1
0
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
Dimensionless temperature distribution
$\left[T(\kappa)^4 - T_2^4\right] / \left(T_1^4 - T_2^4\right) = \phi_b$
Relative optical depth κ/κ_D



Dimensionless energy flux
$q/[\sigma(T_1^4 - T_2^4)] = \psi_b$
1.0
.8
.6
.4
.2
0
.5
1.0
1.5
2.0
2.5
3.0
Optical thickness κ_D


Dimensionless temperature slip
$\left[T_1^4 - T^4(\kappa = 0)\right] / \left(T_1^4 - T_2^4\right)$
.5
.4
.3
.2
.1
0
.5
1.0
1.5
2.0
2.5
3.0
Optical thickness κ_D

Between diffuse-gray walls



$$J_1 = \int_{\omega=2\pi} i^+(0,\mu)\cos\theta d\omega = \pi i^+(0) \quad \text{or} \quad i^+(0) = \frac{J_1}{\pi}$$

$$J_2 = \int_{\omega=2\pi} i^-(\tau_0,-\mu)\cos\theta d\omega = \pi i^-(\tau_0) \quad \text{or} \quad i^-(\tau_0) = \frac{J_2}{\pi}$$

$$J_1 = \varepsilon_1 \int_{\omega=2\pi} i_{b1}(T_1)\cos\theta d\omega + \left(1-\varepsilon_1\right)\int_{\omega'=2\pi} i^-(0,-\mu')\cos\theta' d\omega'$$

$$= \pi\varepsilon_1 i_{b1}(T_1) + \left(1-\varepsilon_1\right)G_1 = \varepsilon_1\sigma T_1^4 + \left(1-\varepsilon_1\right)G_1$$

$$G_1 = \int_{\omega'=2\pi} i^-(0,-\mu')\cos\theta' d\omega'$$

$$i^-(\tau,-\mu) = i^-(\tau_0,-\mu)\exp\left[-\frac{1}{\mu}\left(\tau_0-\tau\right)\right]$$

$$+\int_{\tau}^{\tau_0}\frac{1}{\mu}S(\tau',-\mu)\exp\left[-\frac{1}{\mu}\left(\tau'-\tau\right)\right]d\tau'$$

$$G_1 = \int_{\omega'=2\pi}\left\{i^-(\tau_0,-\mu')\exp\left(-\frac{\tau_0}{\mu'}\right)\right.$$

$$\left.+\int_0^{\tau_0}\frac{1}{\mu'}S(\tau',-\mu')\exp\left(-\frac{\tau'}{\mu'}\right)d\tau'\right\}\cos\theta' d\omega'$$

$$\int_{\omega'=2\pi} i^-(\tau_0,-\mu')\exp\left(-\frac{\tau_0}{\mu'}\right)\cos\theta' d\omega'$$

$$= 2\pi i^-(\tau_0)\int_0^1 \mu'\exp\left(-\frac{\tau_0}{\mu'}\right)d\mu' = 2J_2E_3(\tau_0)$$

for isotropic scattering

$$\int_{\omega=2\pi}\int_0^{\tau_0}\frac{1}{\mu'}S(\tau',-\mu')\exp\left(-\frac{\tau'}{\mu'}\right)d\tau'\cos\theta' d\omega'$$

$$=\int_0^{\tau_0}S(\tau')\left\{\int_{\omega=2\pi}\frac{1}{\mu'}\exp\left(-\frac{\tau'}{\mu'}\right)\cos\theta' d\omega'\right\}d\tau'$$

$$=\int_0^{\tau_0}S(\tau')\left\{2\pi\int_0^{1}\exp\left(-\frac{\tau'}{\mu'}\right)d\mu'\right\}d\tau'$$

$$=2\int_0^{\tau_0}\pi S(\tau')E_2(\tau')d\tau' \quad \leftarrow S=\frac{1}{\pi\kappa}\left[ae_b+\frac{\sigma_s}{4}G\right]$$

$$=2\int_0^{\tau_0}\frac{1}{\kappa}\left[ae_b+\frac{\sigma_s}{4}G\right]E_2(\tau')d\tau'$$

$$G_1 = 2J_2E_3(\tau_0) + 2\int_0^{\tau_0} \frac{1}{\kappa}\left[ae_b + \frac{\sigma_s}{4}G\right]E_2(\tau')d\tau'$$

$$J_1 = \varepsilon_1\sigma T_1^4$$
$$+2\left(1-\varepsilon_1\right)\left[J_2E_3(\tau_0) + \int_0^{\tau_0}\frac{1}{\kappa}\left\{ae_b + \frac{\sigma_s}{4}G\right\}E_2(\tau')d\tau'\right]$$

$$J_2 = \varepsilon_2\sigma T_2^4$$
$$+2\left(1-\varepsilon_2\right)\left[J_1E_3(\tau_0) + \int_0^{\tau_0}\frac{1}{\kappa}\left\{ae_b + \frac{\sigma_s}{4}G\right\}E_2(\tau_0-\tau')d\tau'\right]$$

in radiative equilibrium, $G = 4\pi i_b = 4e_b$

$$J_1 = \varepsilon_1\sigma T_1^4 + 2\left(1-\varepsilon_1\right)\left[J_2E_3(\tau_0) + \int_0^{\tau_0}\sigma T^4(\tau')E_2(\tau')d\tau'\right]$$

$$J_2 = \varepsilon_2\sigma T_2^4 + 2\left(1-\varepsilon_2\right)\left[J_1E_3(\tau_0) + \int_0^{\tau_0}\sigma T^4(\tau')E_2(\tau_0-\tau')d\tau'\right]$$

$$\sigma T^4(\tau) = \frac{1}{2}\left[J_1E_2(\tau) + J_2E_2(\tau_0-\tau) + \int_0^{\tau_0}\sigma T^4(\tau')E_1(|\tau-\tau'|)d\tau'\right]$$

$$q'' = q''(0) = q''(\tau_0)$$

$$q''(\tau) = 2\left[J_1 E_3(\tau) + \int_0^{\tau} \sigma T^4(\tau') E_2(\tau - \tau') d\tau' \right]$$
$$-2\left[J_2 E_3(\tau_0 - \tau) + \int_{\tau}^{\tau_0} \sigma T^4(\tau') E_2(\tau' - \tau) d\tau' \right]$$

$$q''(0) = J_1 - 2J_2 E_3(\tau_0) - 2\int_0^{\tau_0} \sigma T^4(\tau') E_2(\tau') d\tau'$$

$$q''(\tau_0) = 2J_1 E_3(\tau_0) + 2\int_0^{\tau_0} \sigma T^4(\tau') E_2(\tau_0 - \tau') d\tau' - J_2$$

Between diffuse-gray walls

$$q''(\tau) = 2\left[J_1 E_3(\tau) + \int_0^{\tau} \sigma T^4(\tau') E_2(\tau - \tau') d\tau' \right]$$

$$-2\left[J_2 E_3(\tau_0 - \tau) + \int_{\tau}^{\tau_0} \sigma T^4(\tau') E_2(\tau' - \tau) d\tau' \right]$$

$$\sigma T^4(\tau) = \frac{1}{2}\left[J_1 E_2(\tau) + J_2 E_2(\tau_0 - \tau) + \int_0^{\tau_0} \sigma T^4(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

Between black walls

$$q''(\tau) = 2\left[\sigma T_1^4 E_3(\tau) + \int_0^{\tau} \sigma T^4(\tau') E_2(\tau - \tau') d\tau' \right]$$

$$-2\left[\sigma T_2^4 E_3(\tau_0 - \tau) + \int_{\tau}^{\tau_0} \sigma T^4(\tau') E_2(\tau' - \tau) d\tau' \right]$$

$$T^4(\tau) = \frac{1}{2}\left[T_1^4 E_2(\tau) + T_2^4 E_2(\tau_0 - \tau) + \int_0^{\tau_0} T^4(\tau') E_1(|\tau - \tau'|) d\tau' \right]$$

$$\phi_b(\tau)=\frac{T^4(\tau)-T_2^4}{T_1^4-T_2^4}=\frac{1}{2}\left[E_2(\tau)+\int_0^{\tau_0}\phi_b(\tau')E_1(|\tau-\tau'|)d\tau'\right]$$

$$\psi_b(\tau)=\frac{q''}{\sigma\left(T_1^4-T_2^4\right)}=1-2\int_0^{\tau_0}\phi_b(\tau')E_2(\tau')d\tau'$$

Let $\phi=\dfrac{\sigma T^4-J_2}{J_1-J_2}$, $\psi=\dfrac{q''}{J_1-J_2}$, **then** ϕ **and** ψ **have the same expression as** ϕ_b **and** ψ_b **, respectively.**

$$\phi(\tau)=\frac{\sigma T^4(\tau)-J_2}{J_1-J_2}=\frac{1}{2}\left[E_2(\tau)+\int_0^{\tau_0}\phi(\tau')E_1(|\tau-\tau'|)d\tau'\right]$$

$$\psi(\tau)=\frac{q''}{J_1-J_2}=1-2\int_0^{\tau_0}\phi(\tau')E_2(\tau')d\tau'$$

$$\phi_b = \frac{\sigma T^4 - J_2}{J_1 - J_2}, \;\; \psi_b = \frac{q''}{J_1 - J_2}$$

$$J_1 = \varepsilon_1 \sigma T_1^4 + \left(1 - \varepsilon_1\right) G_1, \;\; q_1'' = J_1 - G_1 \equiv q''$$

$$J_1 = \varepsilon_1 \sigma T_1^4 - \left(1 - \varepsilon_1\right)\left(q'' - J_1\right)$$

$$= \varepsilon_1 \sigma T_1^4 - \left(1 - \varepsilon_1\right) q'' + \left(1 - \varepsilon_1\right) J_1$$

$$\varepsilon_1 J_1 = \varepsilon_1 \sigma T_1^4 - \left(1 - \varepsilon_1\right) q'', \quad J_1 = \sigma T_1^4 - \frac{1 - \varepsilon_1}{\varepsilon_1} q''$$

$$J_2 = \varepsilon_2 \sigma T_2^4 + \left(1 - \varepsilon_2\right) G_2, \;\; q_2'' = J_2 - G_2 = -q''$$

$$J_2 = \sigma T_2^4 + \frac{1 - \varepsilon_2}{\varepsilon_2} q''$$

$$\psi_b = \frac{q''}{J_1 - J_2}$$

$$= \frac{q''}{\left[\sigma T_1^4 - q''\left(1-\varepsilon_1\right)/\varepsilon_1\right] - \left[\sigma T_2^4 + q''\left(1-\varepsilon_2\right)/\varepsilon_2\right]}$$

$$= \frac{q''}{\sigma\left(T_1^4 - T_2^4\right) - q''\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}$$

$$= \frac{1}{\sigma\left(T_1^4 - T_2^4\right)/q'' - \left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}$$

$$\frac{1}{\psi_b} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{q''} - \left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)$$

$$\frac{1}{\psi_b} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{q''} - \left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)$$

$$\frac{\sigma\left(T_1^4 - T_2^4\right)}{q''} = \frac{1}{\psi_b} + \left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)$$

$$= \frac{1 + \psi_b\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}{\psi_b}$$

final solution

$$\frac{q''}{\sigma\left(T_1^4 - T_2^4\right)} = \frac{\psi_b}{1 + \psi_b\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}$$

$$\frac{T^4 - T_2^4}{T_1^4 - T_2^4} = \frac{\phi_b + \left[\left(1 - \varepsilon_2\right)/\varepsilon_2\right]\psi_b}{1 + \psi_b\left(1/\varepsilon_1 + 1/\varepsilon_2 - 2\right)}$$