Chapter 6: Basic Plasticity
Ch. 6.7

Myoung-Gyu Lee

TA: Gyu Jang Sim (gyujang95@snu.ac.kr)

ENGINEERING
COLLEGE OF ENGINEERING
SEOUL NATIONAL UNIVERSITY
서 울 대 학 교 공 과 대 학

MAMEL
MATERIALS MECHANICS LABORATORY

1. $\dfrac{\partial \mathbf{r}}{\partial \boldsymbol{\sigma}}$

$$\mathbf{r} = \boldsymbol{\sigma}_C - \left(\boldsymbol{\sigma}_B - \Delta\lambda \mathbf{C}\mathbf{a}_C\right) = 0$$ [eq. 6.79]

$\boldsymbol{\sigma}_B$ is fixed, so

$$\Rightarrow \frac{\partial \mathbf{r}}{\partial \boldsymbol{\sigma}_C} = \mathbf{I} + \Delta\lambda \mathbf{C}\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}}\right) = \mathbf{I} + \Delta\lambda \mathbf{C}\left(\frac{\partial^2 f}{\partial \boldsymbol{\sigma}^2}\right)$$

$\mathbf{I}$: *symmetric* 4^{th} *order unit tensor*

In Voigt notation,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $\dfrac{\partial \mathbf{r}}{\partial \Delta\lambda}$

$$\mathbf{r} = \boldsymbol{\sigma}_C - \left(\boldsymbol{\sigma}_B - \Delta\lambda \mathbf{C}\mathbf{a}_C\right) = 0$$

$$\Rightarrow \frac{\partial \mathbf{r}}{\partial \Delta\lambda} = \mathbf{C}\mathbf{a}$$

3. $\dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

$$\Rightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}} = \mathbf{a}$$

4. $\dfrac{\partial f}{\partial \Delta\lambda}$

$$f = \sigma_e(\boldsymbol{\sigma}_C) - \sigma_0(\varepsilon_{ps})$$

$$\Rightarrow \frac{\partial f}{\partial \Delta\lambda} = -\frac{\partial \sigma_0}{\partial \varepsilon_{ps}} = -A'$$

$$0 = \begin{pmatrix} \mathbf{r} \\ f \end{pmatrix} + \begin{bmatrix} \dfrac{\partial \mathbf{r}}{\partial \boldsymbol{\sigma}} & \dfrac{\partial \mathbf{r}}{\partial \Delta\lambda} \\ \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T & \dfrac{\partial f}{\partial \Delta\lambda} \end{bmatrix} \begin{pmatrix} \Delta\boldsymbol{\sigma} \\ \Delta^2\lambda \end{pmatrix}$$

For simplicity, let $\mathbf{Q} \doteq \mathbf{I} + \Delta\lambda \mathbf{C}\left(\dfrac{\partial^2 f}{\partial \boldsymbol{\sigma}^2}\right)$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Ca} \\ \mathbf{a} & -A' \end{bmatrix} \begin{pmatrix} \Delta\boldsymbol{\sigma} \\ \Delta^2\lambda \end{pmatrix} = -\begin{pmatrix} \mathbf{r} \\ f \end{pmatrix} \Rightarrow \begin{cases} \mathbf{0} = \mathbf{r} + \mathbf{Q}\Delta\boldsymbol{\sigma} + \Delta^2\lambda\mathbf{Ca} \\ 0 = f + \mathbf{a}^T\Delta\boldsymbol{\sigma} - A'\Delta^2\lambda \end{cases}$$

[variation of eq.6.80]

[variation of eq.6.82]

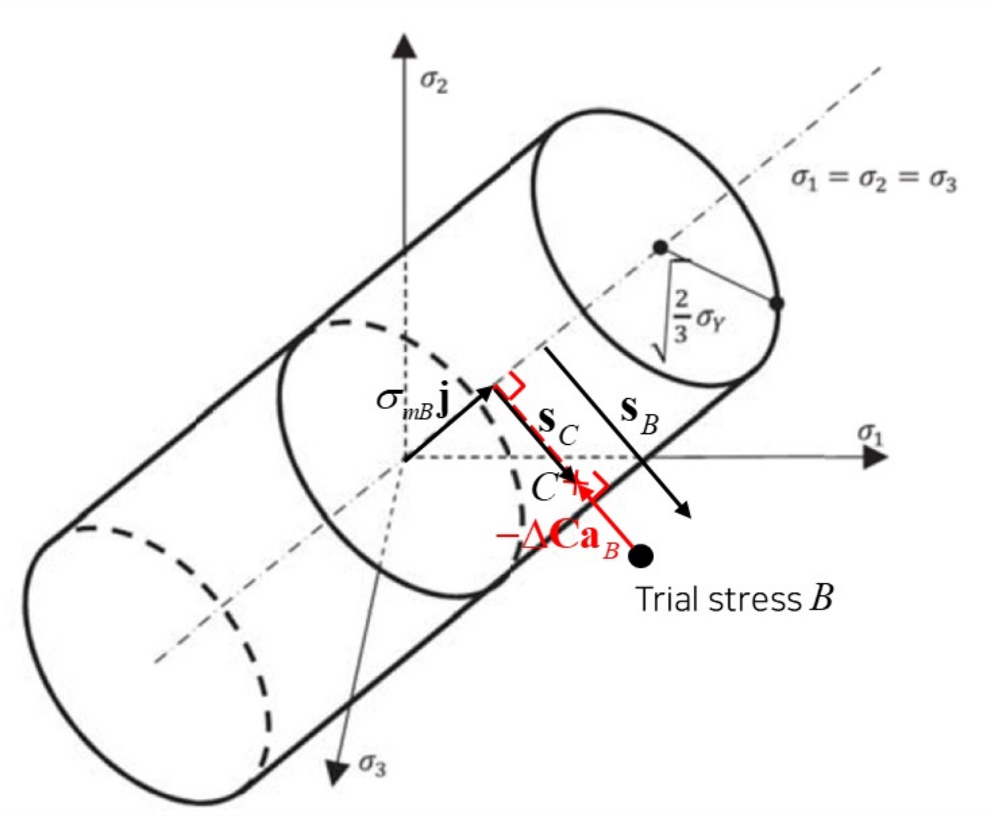
$$\mathbf{0} = \mathbf{r} + \mathbf{Q}\Delta\boldsymbol{\sigma} + \Delta^2\lambda\mathbf{Ca} \Rightarrow \Delta\boldsymbol{\sigma} = -\mathbf{Q}^{-1}\left(\mathbf{r} + \Delta^2\lambda\mathbf{Ca}\right)$$

Substitution to 6.82 $\Rightarrow$ $0 = f - \mathbf{a}^T\mathbf{Q}^{-1}\left(\mathbf{r} + \Delta^2\lambda\mathbf{Ca}\right) - A'\Delta^2\lambda = f - \mathbf{a}^T\mathbf{Q}^{-1}\mathbf{r} - \left(\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{Ca} + A'\right)\Delta^2\lambda$

$$\Rightarrow \Delta^2\lambda = \frac{f - \mathbf{a}^T\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{Ca} + A'}$$ and $$\Delta\boldsymbol{\sigma} = -\mathbf{Q}^{-1}\left(\mathbf{r} + \frac{f - \mathbf{a}^T\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{Ca} + A'}\mathbf{Ca}\right)$$

[variation of eq.6.83]

6.6.7 The radial-return algorithm, a special form of backward-Euler procedure



[Radial return in von Mises criterion]

- From the figure,
  $\mathbf{a}_B$ and $\mathbf{a}_C$ are same.
- Also, their magnitudes are the same.

$$\mathbf{a} = \frac{3}{2\sigma_e}\mathbf{L}\mathbf{s} \quad \text{[eq. 6.33]}$$

$$\left.\begin{aligned} \mathbf{a}_B &= \frac{3}{2\sigma_{eB}}\mathbf{L}\mathbf{s}_B \\ \mathbf{a}_C &= \frac{3}{2\sigma_{eC}}\mathbf{L}\mathbf{s}_C \end{aligned}\right\} \quad \frac{\mathbf{s}_B}{\sigma_{eB}} = \frac{\mathbf{s}_C}{\sigma_{eC}}$$

$$\Rightarrow \quad \mathbf{a}_B = \mathbf{a}_C \quad \text{[eq. 6.91]}$$

※ This is valid only for isotropic hardening. For general case, see *Simo, Hughes, "Computational Inelasticity" ch3.3*

- Backward Euler method is finding roots of these 2(7) equations.

$$\begin{cases} \boldsymbol{\sigma}_C = \boldsymbol{\sigma}_B - \Delta\lambda \mathbf{C}\mathbf{a}_C & \text{[eq. 6.78]} \\ f(\boldsymbol{\sigma}_C) = 0 \end{cases}$$

Set of unknowns (independent variables)

$$\begin{cases} \{\Delta\lambda, \mathbf{a}_C\} \\ \text{or } \{\Delta\lambda, \boldsymbol{\sigma}_C\} & \text{since } \mathbf{a}_C = [\partial f / \partial \boldsymbol{\sigma}]_C \\ \text{or } \{\Delta\lambda, \Delta\boldsymbol{\varepsilon}_p\} & \text{since } \boldsymbol{\sigma}_C = \mathbf{C}(\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}_p) \end{cases}$$

- However, for von Mises criterion, $\mathbf{a}_C$ is already know by $\mathbf{a}_C = \mathbf{a}_B$, so the only unknown is $\Delta\lambda$ and the only equation needed is $f(\boldsymbol{\sigma}_C) = 0$.

$$\boxed{0 = f(\boldsymbol{\sigma}_C) = \sigma_{eC} - \sigma_{oC}}$$

- σ_{eC} can be obtained after some manipulation.

$$\begin{aligned} \boldsymbol{\sigma}_C &= \sigma_{mB}\mathbf{j} + \mathbf{s}_C \\ &= \sigma_{mB}\mathbf{j} + \mathbf{s}_B - \Delta\lambda\mathbf{C}\mathbf{a}_C \\ &= \sigma_{mB}\mathbf{j} + \mathbf{s}_B - \Delta\lambda\mathbf{C}\mathbf{a}_B \\ &= \sigma_{mB}\mathbf{j} + \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\mathbf{s}_B \end{aligned}$$

$$\mathbf{C}\mathbf{a} = \frac{3\mu}{\sigma_e}\mathbf{s} \quad \text{[eq. 6.35]} \quad \longrightarrow$$

$$\Rightarrow \mathbf{s}_C = \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\mathbf{s}_B$$

$$\Rightarrow \boxed{\sigma_{eC} = \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\sigma_{eB}} \qquad \left(\because \sigma_e = \sqrt{\frac{3}{2}}\left(\mathbf{s}^T\mathbf{L}\mathbf{s}\right)^{1/2}\right)$$

- For linear hardening, $\Delta\lambda$ can be obtained even without N-R, since A' is constant.

$$\sigma_{oC} = \sigma_{oB} + A'\Delta\varepsilon_{ps} = \sigma_{oB} + A'\Delta\lambda$$

$$\Rightarrow \quad 0 = f_C$$

$$= \sigma_{eC} - \sigma_{oC}$$

$$= \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\sigma_{eB} - \left(\sigma_{oB} + A'\Delta\lambda\right)$$

$$= f_B - \left(3\mu + A'\right)\Delta\lambda$$

$$\Rightarrow \quad \boxed{\Delta\lambda = \frac{f_B}{3\mu + A'}}$$

- For non-linear hardening, $\Delta\lambda$ can be obtained by N-R.

$$0 = f + \Delta f = f + \frac{\partial f}{\partial \Delta\lambda}\Delta^2\lambda$$

$$\frac{\partial f}{\partial \Delta\lambda} = \frac{\partial}{\partial \Delta\lambda}\left(\sigma_e - \sigma_o\right)$$

$$= \frac{\partial}{\partial \Delta\lambda}\left(\left(1 - \frac{3\mu\Delta\lambda}{\sigma_e^{old}}\right)\sigma_e^{old} - \sigma_o\right)$$

$$= \frac{\partial}{\partial \Delta\lambda}\left(\left(\sigma_e^{old} - 3\mu\Delta\lambda\right) - \sigma_o\right)$$

$$= -3\mu - \frac{\partial}{\partial \Delta\lambda}\left(\sigma_o\right)$$

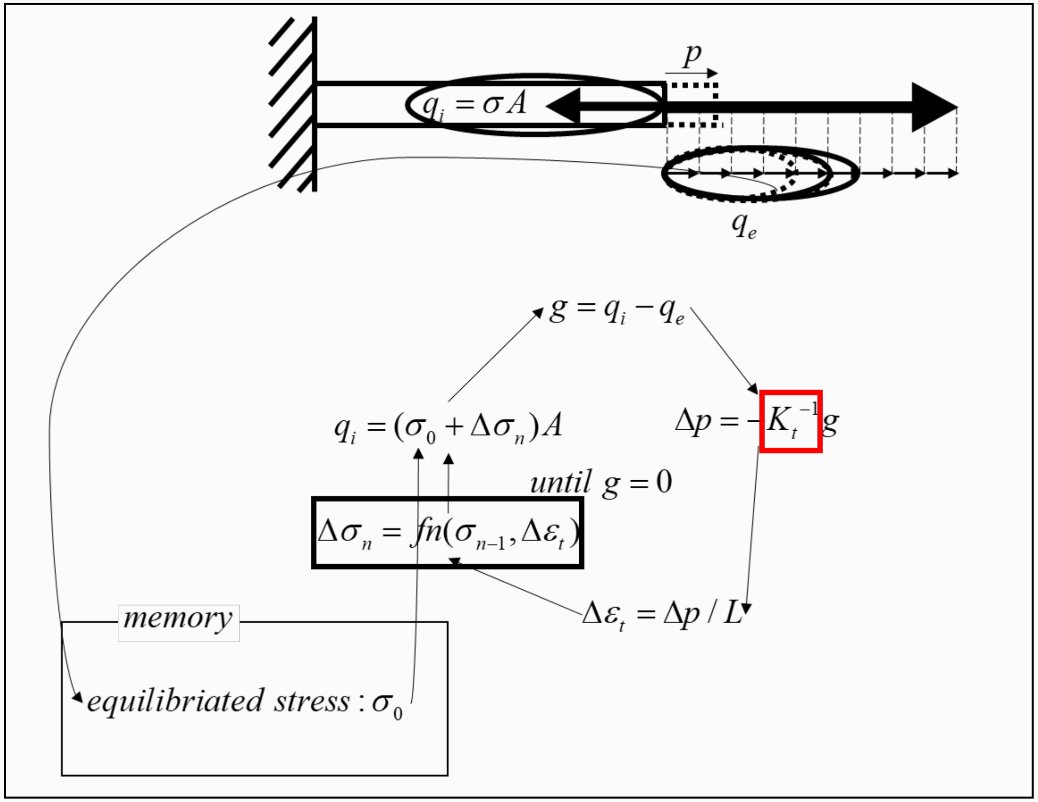
$$= -3\mu - A'$$

※ You may obtain $\Delta\lambda$ analytically depending on the specific form of hardening..

※ For linear hardening, it can be thought as 'N-R converges at 1st try with initial guess $\Delta\lambda = 0$'.

➡ $$\Delta^2\lambda = \frac{f}{3\mu + A'}$$

➡ do $\Delta\lambda \leftarrow \Delta\lambda + \dfrac{f}{3\mu + A'}$ while $|f| > tolerance$



[Equilibrium iteration - Incremental strains]

- To keep 'quadratic convergence' of overall equilibrium iteration, $\mathbf{K}_t$ should also be consistent with **finite strain increment** $\Delta\boldsymbol{\varepsilon}_t$.

$$\mathbf{K}_t = \int \mathbf{B}^T \mathbf{C}^{alg} \mathbf{B} dV + (initial\ stress\ matrix)$$

- In other words, $\mathbf{C}^{alg}$ in the above equation should be consistent with the **finite strain increment**.

6.7.2 A combined formulation

- In section 6.3, modular matrix was derived from

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\dfrac{\partial f}{\partial \boldsymbol{\sigma}} \\ \dot{f} = 0 \end{cases} \Rightarrow \quad \dot{\boldsymbol{\sigma}} = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p\right) = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)$$

$$\Rightarrow \cdots \Rightarrow \quad \dot{\boldsymbol{\sigma}} = \left(\mathbf{C} - \frac{3\mu}{\sigma_e^2\left(1 + \dfrac{A'}{3\mu}\right)}\mathbf{s}\mathbf{s}^T\right)\dot{\boldsymbol{\varepsilon}}$$

modular matrix

- However, since stress update(integration) algorithm is based on **finite strain increment**, formulations should be derived from:

If changing $\Delta\boldsymbol{\varepsilon}_t$ *to* $\Delta\boldsymbol{\varepsilon}_t + \delta\boldsymbol{\varepsilon}$ *results in* $\boldsymbol{\sigma}$ *to* $\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}$ *in the backward Euler scheme, what is the linear relationship between them?*

$$\begin{cases} \boldsymbol{\sigma} = fn(\boldsymbol{\sigma}_o, \Delta\boldsymbol{\varepsilon}_t) \\ \boldsymbol{\sigma} + \delta\boldsymbol{\sigma} = fn(\boldsymbol{\sigma}_o, \Delta\boldsymbol{\varepsilon}_t + \delta\boldsymbol{\varepsilon}) \end{cases} \Rightarrow \quad \delta\boldsymbol{\sigma} = \mathbf{C}^{alg}\delta\boldsymbol{\varepsilon} + O(\delta\boldsymbol{\varepsilon}^2)$$

- Starting from backward Euler scheme:

$$\begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}_B - \Delta\lambda \mathbf{C}\mathbf{a} = \boldsymbol{\sigma}_A + \mathbf{C}\Delta\boldsymbol{\varepsilon}_t - \Delta\lambda\mathbf{C}\mathbf{a} \\ f = 0 \end{cases}$$

$$\Rightarrow \begin{cases} ① \quad \delta\boldsymbol{\sigma} = \delta(\boldsymbol{\sigma}_A + \mathbf{C}\Delta\boldsymbol{\varepsilon}_t - \Delta\lambda\mathbf{C}\mathbf{a}) \\ ② \quad \delta f = 0 \end{cases}$$

① $\Rightarrow$ $\delta\boldsymbol{\sigma} = \mathbf{C}\delta\boldsymbol{\varepsilon} - \delta(\Delta\lambda)\mathbf{C}\mathbf{a} - \Delta\lambda\mathbf{C}\delta\mathbf{a}$ where $\delta\mathbf{a} = \dfrac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}}\delta\boldsymbol{\sigma}$

$\Rightarrow$ $\delta\boldsymbol{\sigma} = \left(\mathbf{I} + \Delta\lambda\mathbf{C}\dfrac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}}\right)^{-1}\mathbf{C}\left(\delta\boldsymbol{\varepsilon} - \delta(\Delta\lambda)\mathbf{a}\right)$ ~ let's simplify $\mathbf{R} \doteq \left(\mathbf{I} + \Delta\lambda\mathbf{C}\dfrac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}}\right)^{-1}\mathbf{C}$

$\Rightarrow$ ③ $\delta\boldsymbol{\sigma} = \mathbf{R}\left(\delta\boldsymbol{\varepsilon} - \delta(\Delta\lambda)\mathbf{a}\right)$

② $\Rightarrow$ $0 = \delta f = \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \delta\boldsymbol{\sigma} + \dfrac{\partial f}{\partial \Delta\lambda}\delta(\Delta\lambda)$

$= \mathbf{a}^T\delta\boldsymbol{\sigma} - A'\delta(\Delta\lambda)$

③ $\Rightarrow$ $0 = \mathbf{a}^T\mathbf{R}\left(\delta\boldsymbol{\varepsilon} - \delta(\Delta\lambda)\mathbf{a}\right) - A'\delta(\Delta\lambda)$

$\Rightarrow$ ④ $\delta(\Delta\lambda) = \dfrac{\mathbf{a}^T\mathbf{R}\delta\boldsymbol{\varepsilon}}{\mathbf{a}^T\mathbf{R}\mathbf{a} + A'}$

③ + ④ ➡ $$\delta\boldsymbol{\sigma} = \mathbf{R}\left(\delta\boldsymbol{\varepsilon} - \frac{\mathbf{a}^T\mathbf{R}\delta\boldsymbol{\varepsilon}}{\mathbf{a}^T\mathbf{R}\mathbf{a} + A'}\mathbf{a}\right)$$

$$= \mathbf{R}\delta\boldsymbol{\varepsilon} - \frac{\left(\mathbf{a}^T\mathbf{R}\delta\boldsymbol{\varepsilon}\right)\mathbf{R}\mathbf{a}}{\mathbf{a}^T\mathbf{R}\mathbf{a} + A'}$$

⬅ $$\left(\mathbf{a}^T\mathbf{R}\delta\boldsymbol{\varepsilon}\right)\mathbf{R}\mathbf{a} = \left(\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}^T\right)\delta\boldsymbol{\varepsilon}$$

$$= \mathbf{R}\delta\boldsymbol{\varepsilon} - \frac{\left(\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}^T\right)\delta\boldsymbol{\varepsilon}}{\mathbf{a}^T\mathbf{R}\mathbf{a} + A'}$$

$$= \left(\mathbf{R} - \frac{\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}^T}{\mathbf{a}^T\mathbf{R}\mathbf{a} + A'}\right)\delta\boldsymbol{\varepsilon}$$

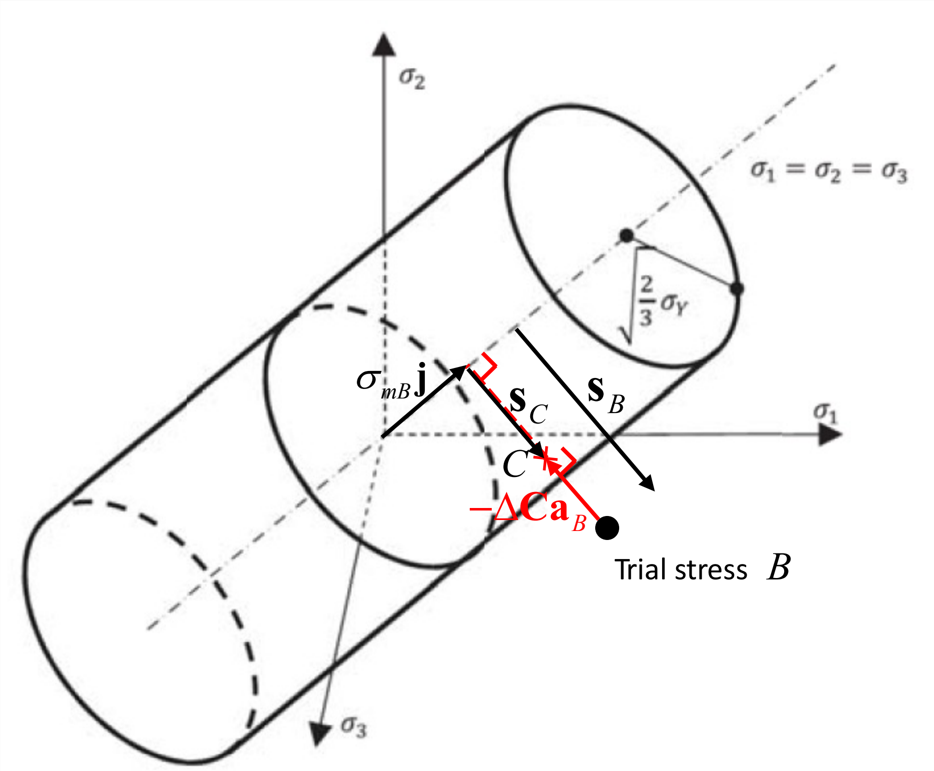
➡ $$\boxed{\mathbf{C}^{alg} = \left(\mathbf{R} - \frac{\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}^T}{\mathbf{a}^T\mathbf{R}\mathbf{a} + A'}\right)}$$

※ Proof of $\left(\mathbf{a}^T\mathbf{R}\delta\boldsymbol{\varepsilon}\right)\mathbf{R}\mathbf{a} = \left(\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}^T\right)\delta\boldsymbol{\varepsilon}$

$$\left(\mathbf{a}^T\mathbf{R}\delta\boldsymbol{\varepsilon}\right)\mathbf{R}\mathbf{a} = \left(a_k R_{kl}\left(\delta\varepsilon\right)_l\right)R_{ij}a_j\mathbf{e}_i = \left(R_{ij}a_j a_k R_{kl}\right)\left(\delta\varepsilon\right)_l\mathbf{e}_i = \left(\mathbf{R}\mathbf{a}\mathbf{a}^T\mathbf{R}^T\right)\delta\boldsymbol{\varepsilon}$$

$\left(\textit{Einstein notation with } 1 \textit{ to } 6\right)$


Thank you!



σ_2
$\sigma_1 = \sigma_2 = \sigma_3$
$\sqrt{\frac{2}{3}}\sigma_Y$
$\sigma_{mB}\mathbf{j}$
$\mathbf{s}_C$
$\mathbf{s}_B$
σ_1
C
$-\Delta C\mathbf{a}_B$
Trial stress B
σ_3

※ There was typo..

1. $\dfrac{\partial \mathbf{r}}{\partial \boldsymbol{\sigma}}$

$$\mathbf{r} = \boldsymbol{\sigma}_C - \left(\boldsymbol{\sigma}_B - \Delta\lambda \mathbf{C}\mathbf{a}_C\right) = 0$$ [eq. 6.79]

$\boldsymbol{\sigma}_B$ is fixed, so

$$\Rightarrow \frac{\partial \mathbf{r}}{\partial \boldsymbol{\sigma}_C} = \mathbf{I} + \Delta\lambda \mathbf{C}\left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}}\right) = \mathbf{I} + \Delta\lambda \mathbf{C}\left(\frac{\partial^2 f}{\partial \boldsymbol{\sigma}^2}\right)$$

$$\mathbf{I} : \textit{symmetric } 4^{th} \textit{ order unit tensor}$$

In Voigt notation,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $\dfrac{\partial \mathbf{r}}{\partial \Delta\lambda}$

$$\mathbf{r} = \boldsymbol{\sigma}_C - \left(\boldsymbol{\sigma}_B - \Delta\lambda \mathbf{C}\mathbf{a}_C\right) = 0$$

$$\Rightarrow \frac{\partial \mathbf{r}}{\partial \Delta\lambda} = \mathbf{C}\mathbf{a}$$

3. $\dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

$$\Rightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}} = \mathbf{a}$$

4. $\dfrac{\partial f}{\partial \Delta\lambda}$

$$f = \sigma_e(\boldsymbol{\sigma}_C) - \sigma_0(\varepsilon_{ps})$$

$$\Rightarrow \frac{\partial f}{\partial \Delta\lambda} = -\frac{\partial \sigma_0}{\partial \varepsilon_{ps}} = -A'$$

$$0=\begin{pmatrix}\mathbf{r}\\ f\end{pmatrix}+\begin{bmatrix}\dfrac{\partial\mathbf{r}}{\partial\boldsymbol{\sigma}} & \dfrac{\partial\mathbf{r}}{\partial\Delta\lambda}\\ \left(\dfrac{\partial f}{\partial\boldsymbol{\sigma}}\right)^T & \dfrac{\partial f}{\partial\Delta\lambda}\end{bmatrix}\begin{pmatrix}\Delta\boldsymbol{\sigma}\\ \Delta^2\lambda\end{pmatrix}$$

For simplicity, let $\mathbf{Q}\doteq\mathbf{I}+\Delta\lambda\mathbf{C}\left(\dfrac{\partial^2 f}{\partial\boldsymbol{\sigma}^2}\right)$

$$\begin{bmatrix}\mathbf{Q} & \mathbf{Ca}\\ \mathbf{a} & -A'\end{bmatrix}\begin{pmatrix}\Delta\boldsymbol{\sigma}\\ \Delta^2\lambda\end{pmatrix}=-\begin{pmatrix}\mathbf{r}\\ f\end{pmatrix} \Rightarrow \begin{cases}\mathbf{0}=\mathbf{r}+\mathbf{Q}\Delta\boldsymbol{\sigma}+\Delta^2\lambda\mathbf{Ca} & \text{[variation of eq.6.80]}\\ 0=f+\mathbf{a}^T\Delta\boldsymbol{\sigma}-A'\Delta^2\lambda & \text{[variation of eq.6.82]}\end{cases}$$

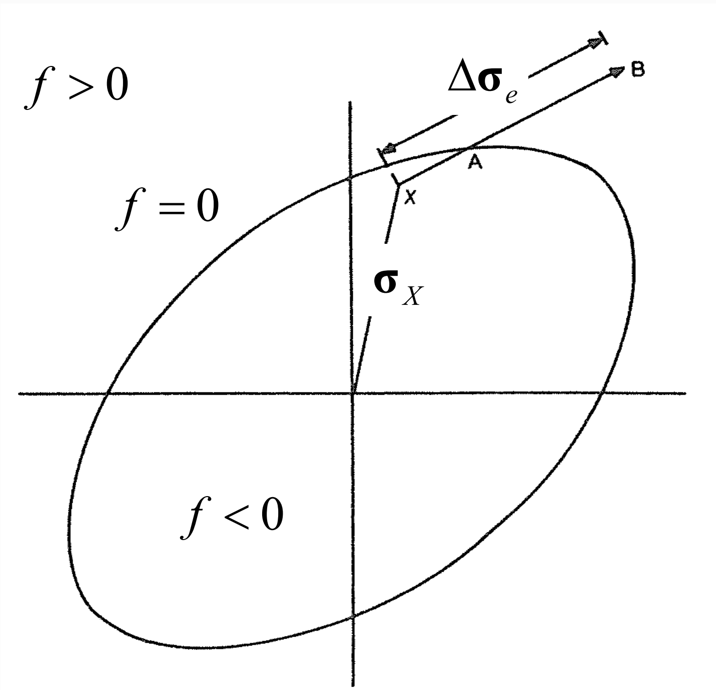
$$\mathbf{0}=\mathbf{r}+\mathbf{Q}\Delta\boldsymbol{\sigma}+\Delta^2\lambda\mathbf{Ca} \Rightarrow \Delta\boldsymbol{\sigma}=-\mathbf{Q}^{-1}\left(\mathbf{r}+\Delta^2\lambda\mathbf{Ca}\right)$$

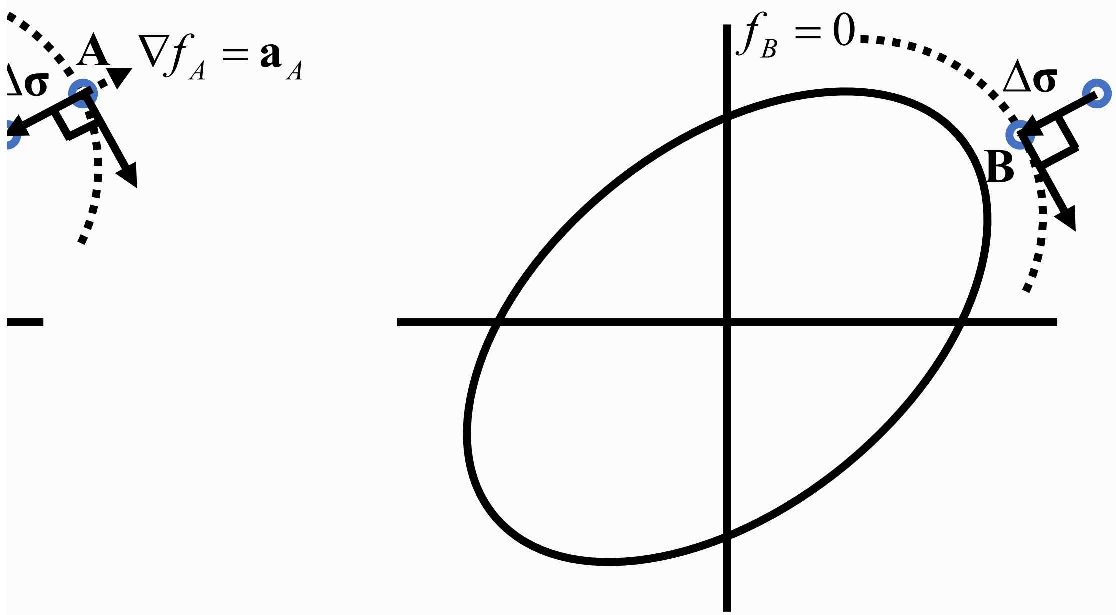
Substitution to 6.82 $\Rightarrow$ $0=f-\mathbf{a}^T\mathbf{Q}^{-1}\left(\mathbf{r}+\Delta^2\lambda\mathbf{Ca}\right)-A'\Delta^2\lambda=f-\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{r}-\left(\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{Ca}+A'\right)\Delta^2\lambda$

$$\Rightarrow \Delta^2\lambda=\frac{f-\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{Ca}+A'} \quad\text{and}\quad \Delta\boldsymbol{\sigma}=-\mathbf{Q}^{-1}\left(\mathbf{r}+\frac{f-\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{r}}{\mathbf{a}^T\mathbf{Q}^{-1}\mathbf{Ca}+A'}\mathbf{Ca}\right)$$

[variation of eq.6.83]

Representation of normality in figure





[Fig 6.9 The forward-Euler procedure]
(a) Locating the intersection print, A

- Since $\nabla f = \mathbf{a}$ is strain quantity, the direction cannot be directly compared with $\Delta\boldsymbol{\sigma}$.
- In these kinds of figures, normality $\left(f_{A\ or\ B} = 0\right) \perp \Delta\boldsymbol{\sigma}$ represents $\Delta\boldsymbol{\sigma} \parallel \left(\mathbf{C}\mathbf{a}_{A\ or\ B}\right)$

$$\boldsymbol{\sigma}_C = \sigma_{mB}\mathbf{j} + \mathbf{s}_B - \Delta\mathbf{C}\mathbf{a}_B = \qquad \text{[eq. 6.94]}$$

$$\mathbf{s}_C = \alpha \mathbf{s}_B = \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\mathbf{s}_B$$ [eq. 6.94]

$$\dot{\mathbf{s}}_C = \alpha \dot{\mathbf{s}}_B + \dot{\alpha}\mathbf{s}_B = 2\mu\alpha\dot{\mathbf{e}}_B + \dot{\alpha}\mathbf{s}_B = 2\mu\alpha\dot{\mathbf{e}}_C + \dot{\alpha}\mathbf{s}_B$$ [eq. 6.95a]

$$\dot{\mathbf{s}}_C = 2\mu\alpha\mathbf{L}^{-1}\dot{\mathbf{e}}_C + \dot{\alpha}\mathbf{s}_B$$ [eq. 6.95b]

$$\dot{\alpha} = -\frac{3\mu\dot{\lambda}}{\sigma_{eB}} + \frac{3\mu\Delta\lambda}{\sigma_{eB}^2}\dot{\sigma}_{eB} = \frac{(1-\alpha)}{\Delta\lambda}\dot{\lambda} + \frac{(1-\alpha)}{\sigma_{eB}}\dot{\sigma}_{eB}$$ [eq. 6.96]

$$\dot{\sigma}_{eB} = \sqrt{\frac{3}{2}}\frac{\mathbf{s}_B : \dot{\mathbf{s}}_B}{\| \mathbf{s}_B \|} = \frac{3}{2\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{s}}_B = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{e}}_B = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{e}}_C = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B^T\dot{\mathbf{e}}_C$$ [eq. 6.97]

$$\dot{f} = \dot{\alpha}\sigma_{eB} + \alpha\dot{\sigma}_{eB} - A_C'\dot{\lambda} = 0$$ [eq. 6.98]

$$\dot{\sigma}_{eB} - \left(3\mu + A_C'\right)\dot{\lambda} = 0$$ [eq. 6.99]

$$\dot{\alpha} = 2\mu\beta\mathbf{s}_B : \dot{\mathbf{e}}_B = 2\mu\beta\mathbf{s}_B^T\dot{\mathbf{e}}_B$$ [eq. 6.100]

$$\beta = \frac{3}{2\sigma_{eB}^2}(1-\alpha)\left(1 - \frac{\sigma_{eB}}{\Delta\lambda\left(3\mu + A_C'\right)}\right) = \frac{3}{2\sigma_{eB}^2}\left(\frac{(1-\alpha)\left(3\mu + A_C'\right) - 3\mu}{\left(3\mu + A_C'\right)}\right)$$ [eq. 6.101]

$$\dot{\mathbf{s}}_C = 2\mu\left(\alpha\mathbf{I} + \beta\mathbf{s}_B \otimes \mathbf{s}_B\right) : \dot{\mathbf{e}}_C = 2\mu\left(\alpha\mathbf{L}^{-1} + \beta\mathbf{s}_B\mathbf{s}_B^T\right)\dot{\mathbf{e}}_C$$ [eq. 6.102]

$$\mathbf{C}_t = \left(k - \frac{2\mu\alpha}{3}\right)(\mathbf{1} \otimes \mathbf{1}) + 2\mu\left(\alpha\mathbf{I} + \beta\mathbf{s}_B \otimes \mathbf{s}_B\right)$$ [eq. 6.103a]

$$\mathbf{C}_t = \left(k - \frac{2\mu\alpha}{3}\right)\left(\mathbf{j}\mathbf{j}^T\right) + 2\mu\left(\alpha\mathbf{I} + \beta\mathbf{s}_B \otimes \mathbf{s}_B\right)$$ **[eq. 6.103b]**

6.7.1 Splitting the deviatoric from the volumetric components

- In radial-return algorithm, basic return is

$$\mathbf{s}_C = \alpha \mathbf{s}_B = \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\mathbf{s}_B \qquad \text{[eq. 6.94]}$$

- To obtain a consistent tangent, eq.6.94 is differentiated.

$$\Delta\mathbf{s}_C = \alpha\Delta\mathbf{s}_B + \Delta\alpha\mathbf{s}_B = 2\mu\alpha\Delta\mathbf{e}_B + \Delta\alpha\mathbf{s}_B = 2\mu\alpha\Delta\mathbf{e}_C + \Delta\alpha\mathbf{s}_B \qquad \text{[eq. 6.95a]}$$

or $$\dot{\mathbf{s}}_C = 2\mu\alpha\mathbf{L}^{-1}\dot{\mathbf{e}}_C + \dot{\alpha}\mathbf{s}_B \qquad \text{[eq. 6.95b]}$$

- $\dot{\alpha}$ can be expressed as

$$\dot{\alpha} = -\frac{3\mu\dot{\lambda}}{\sigma_{eB}} + \frac{3\mu\Delta\lambda}{\sigma_{eB}^2}\dot{\sigma}_{eB} = \frac{(1-\alpha)}{\Delta\lambda}\dot{\lambda} + \frac{(1-\alpha)}{\sigma_{eB}}\dot{\sigma}_{eB} \qquad \text{[eq. 6.96]}$$

- $\dot{\sigma}_{eB}$ can be expressed as

$$\dot{\sigma}_{eB} = \sqrt{\frac{3}{2}}\frac{\mathbf{s}_B : \dot{\mathbf{s}}_B}{\|\mathbf{s}_B\|} = \frac{3}{2\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{s}}_B = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{e}}_B = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{e}}_C = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B^T\dot{\mathbf{e}}_C \qquad \text{[eq. 6.97]}$$

- In radial-return algorithm, basic return is

$$\mathbf{s}_C = \alpha \mathbf{s}_B = \left(1 - \frac{3\mu\Delta\lambda}{\sigma_{eB}}\right)\mathbf{s}_B \qquad \text{[eq. 6.94]}$$

- To obtain a consistent tangent, eq.6.94 is differentiated.

$$\dot{\mathbf{s}}_C = \alpha\dot{\mathbf{s}}_B + \dot{\alpha}\mathbf{s}_B = 2\mu\alpha\dot{\mathbf{e}}_B + \dot{\alpha}\mathbf{s}_B = 2\mu\alpha\dot{\mathbf{e}}_C + \dot{\alpha}\mathbf{s}_B \qquad \text{[eq. 6.95a]}$$

or $$\dot{\mathbf{s}}_C = 2\mu\alpha\mathbf{L}^{-1}\dot{\mathbf{e}}_C + \dot{\alpha}\mathbf{s}_B \qquad \text{[eq. 6.95b]}$$

- $\dot{\alpha}$ can be expressed as

$$\dot{\alpha} = -\frac{3\mu\dot{\lambda}}{\sigma_{eB}} + \frac{3\mu\Delta\lambda}{\sigma_{eB}^2}\dot{\sigma}_{eB} = \frac{(1-\alpha)}{\Delta\lambda}\dot{\lambda} + \frac{(1-\alpha)}{\sigma_{eB}}\dot{\sigma}_{eB} \qquad \text{[eq. 6.96]}$$

- $\dot{\sigma}_{eB}$ can be expressed as

$$\dot{\sigma}_{eB} = \sqrt{\frac{3}{2}}\frac{\mathbf{s}_B : \dot{\mathbf{s}}_B}{\|\mathbf{s}_B\|} = \frac{3}{2\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{s}}_B = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{e}}_B = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B : \dot{\mathbf{e}}_C = \frac{3\mu}{\sigma_{eB}}\mathbf{s}_B^T\dot{\mathbf{e}}_C \qquad \text{[eq. 6.97]}$$