Paraxial Beam Transport with Space Charge

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Transverse force on sheet beams by applied electric field

- Transverse component of static electric field:

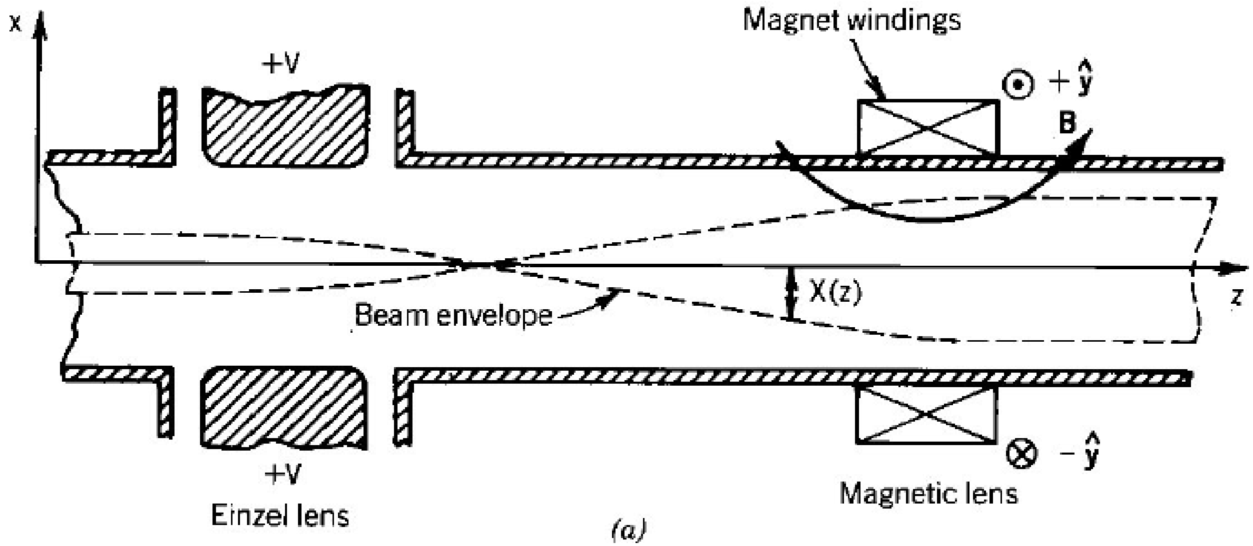
$$E_x(x,z) \approx -x\frac{\partial E_z(0,z)}{\partial z}$$

- Change in the kinetic energy by axial electric field:

$$\frac{\partial(\gamma m_0 c^2)}{\partial z} = qE_z(0,z)$$

- The applied transverse electric force on the envelope:

$$\boxed{F_x \cong -X(m_0 c^2)\gamma''}$$



Transverse force on sheet beams by applied magnetic field

- Because there are no y-directed forces, the canonical momentum of particles in y is a conserved quantity:

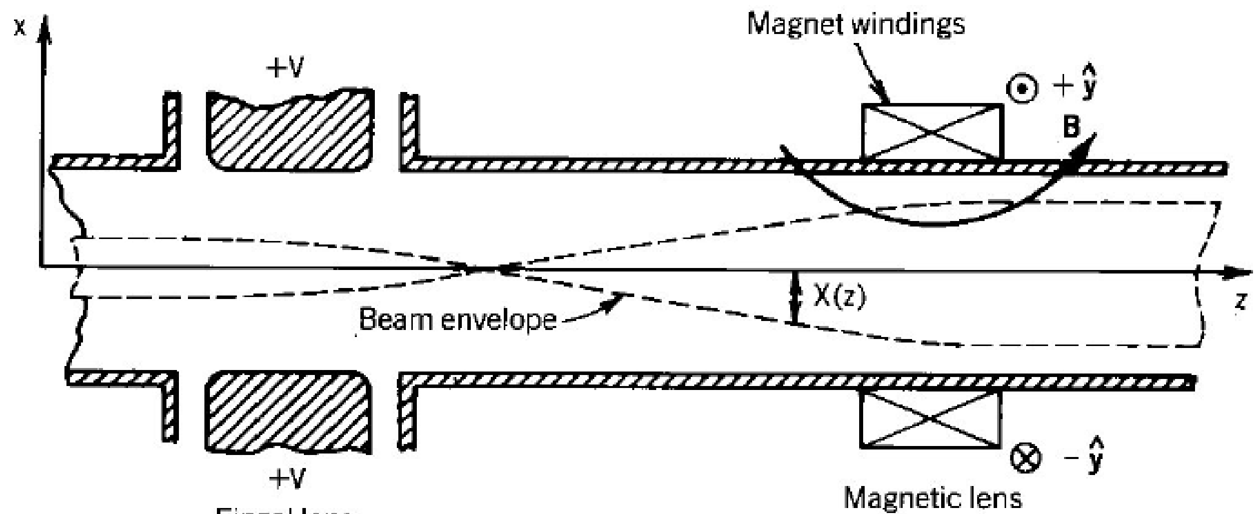
$$P_y = \gamma m_0 v_y + qA_y \approx \gamma m_0 v_y + qB_z x = P_0$$

- If there is no magnetic field at the source and particles leave perpendicular to the surface ($v_y = 0$), then all particles have zero canonical momentum, $P_0 = 0$, then:

$$v_y(z) \approx -\frac{qB_z(0,z)}{\gamma m_0} X$$

- The applied transverse magnetic force on the envelope:

$$\boxed{F_x \cong -\frac{q^2 B_z^2(0,z)}{\gamma m_0} X}$$



(a)

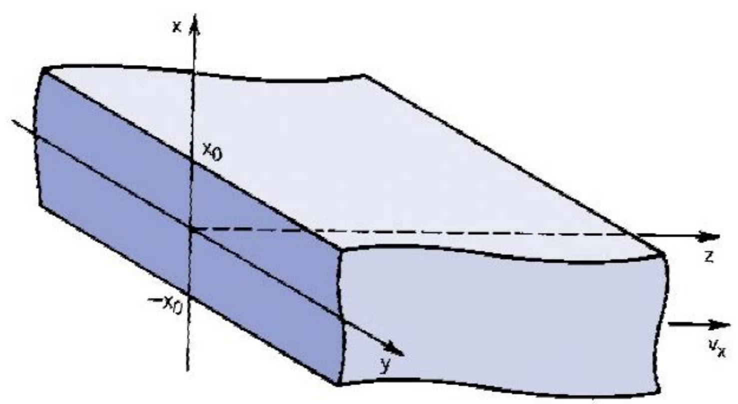
Transverse force on sheet beams by self-generating forces

- The electric and magnetic forces acting on the envelope of a sheet beam carrying a current per unit length (along y) of J (A/m) is:

$$F_{x0}(electric) = qE_{x0} = \frac{qJ}{2\epsilon_0 \beta c} \qquad F_{x0}(magnetic) = -qv_z B_{y0} = -\frac{q\beta c\mu_0 J}{2}$$

- The total beam-generated force on the envelope:

$$\boxed{F_x = \gamma m_0 (\beta c)^2 K_x} \qquad K_x \equiv \frac{qJ}{2\epsilon_0 m_0 \beta\gamma c}\ (generalized\ perveance)$$



Envelope equation for sheet beams

- The beam envelope follows an equation of motion of the form:

$$\frac{d}{dt}\left[\gamma m_0\left(\frac{dX}{dt}\right)\right]=\frac{d}{dt}[\gamma m_0\beta c X']=m_0\beta c^2[\gamma\beta X''+\gamma\beta' X'+\gamma'\beta X']=\sum F_x$$

$$\gamma\beta'+\beta\gamma'=\gamma'/\beta$$

- We obtain the following equation:

$$\gamma m_0(\beta c)^2\left[X''+\frac{\gamma'}{\gamma\beta^2}X'\right]=-X(m_0c^2)\gamma''-\frac{q^2B_z^2(0,z)}{\gamma m_0}X+\gamma m_0(\beta c)^2K_x+\epsilon_x^2\frac{\gamma m_0(\beta c)^2}{X^3}$$

- Finally, we obtain the envelop equation for sheet beams:

$$X''=-\frac{\gamma'}{\gamma\beta^2}X'-\frac{\gamma''}{\gamma\beta^2}X-\left(\frac{qB_z}{\gamma m_0\beta c}\right)^2X+K_x+\frac{\epsilon_x^2}{X^3}$$

Emittance force

Focusing by electrostatic lens

Defocusing by beam-generated forces

Decrease in the envelope angle by beam acceleration

Focusing by magnetic lens

Paraxial ray equation

- In a cylindrical system, symmetry permits only certain components of electric and magnetic field:

  1. axial and radial components of the applied electric field,
  2. radial electric field resulting from space-charge,
  3. axial and radial magnetic field components generated by axi-centered circular coils, and
  4. beam-generated toroidal magnetic field.

- In the paraxial limit, we can relate the radial components of applied fields to the axial field by:

$$E_r(r,z) \approx -\frac{r}{2}\left(\frac{\partial E_z}{\partial z}\right)_{r=0} \qquad B_r(r,z) \approx -\frac{r}{2}\left(\frac{\partial B_z}{\partial z}\right)_{r=0}$$

- Particles gain azimuthal velocity when they move through the radial magnetic fields of a solenoidal lens. For forces with cylindrical symmetry, the canonical angular momentum is a constant of particle motion:

$$\gamma m_0 r v_\theta + qrA_\theta = P_\theta = \text{constant}$$

Paraxial ray equation

- We can derive the following equation for axial variation of the envelope of a cylindrical beam:

$$\psi_0 = \int_0^{R_s} 2\pi R B_z(R, Z_s)\,dR$$

$$R'' = -\frac{\gamma'}{\gamma\beta^2}R' - \frac{\gamma''}{2\gamma\beta^2}R - \left(\frac{qB_z}{2\gamma m_0 \beta c}\right)^2 R + \frac{\epsilon^2}{R^3} + \left(\frac{q\psi_0}{2\pi\gamma m_0\beta c}\right)^2 \frac{1}{R^3} + \frac{K}{R}$$

Decrease in the envelope angle by beam acceleration

Electrostatic focusing from radial components of applied electric fields

Magnetic focusing from applied solenoidal fields

Emittance force

Non-zero angular momentum

Defocusing by beam-generated forces

Addition of approximate term for periodic focusing systems

- The displacement of a particle orbit at the boundary of the nth lens in an array obeys the equation:

$$r_n = r_0 \cos(n\mu_0 + \phi) \qquad \mu_0\text{: vacuum phase advance per lens}$$

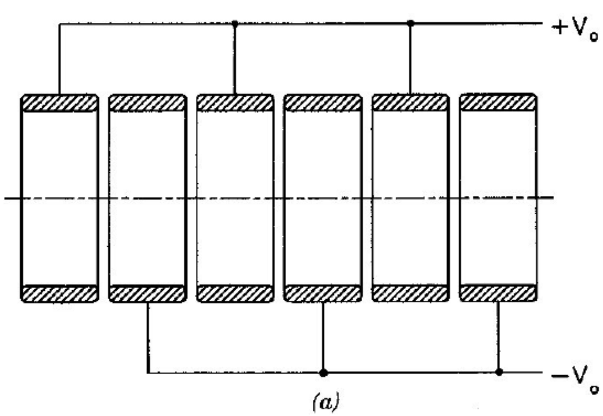
- If the length of a focusing cell is L, the long-term harmonic motion follows the equation:

$$r(z) \cong r_0 \cos((\mu_0/L)z + \phi) \qquad r'' = -(\mu_0/L)^2 r$$

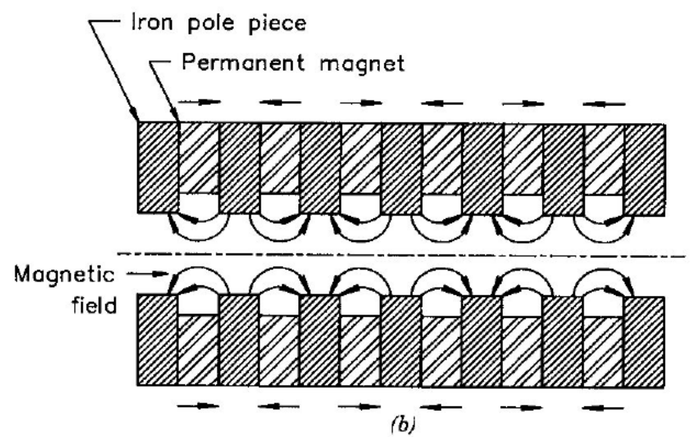
- The paraxial ray equation when considering only periodic forces:

$$R'' = -\left(\frac{\mu_0}{L}\right)^2 R + \frac{\epsilon^2}{R^3} + \frac{K}{R}$$

Einzel lens array



Periodic permanent magnet array



Limiting current for paraxial beams with a uniform solenoid field

- Radial force balance for a cylindrical, paraxial electron beam in a uniform solenoid field B_0.

$$R'' = -\left(\frac{qB_0}{2\gamma m_0 \beta c}\right)^2 R + \frac{\epsilon^2}{R^3} + \frac{K}{R} = 0$$

- The acceptance α is defined as the allowed beam emittance for a given envelope radius when there are no beam-generated forces, i.e. $K = 0$:

$$\alpha^2 = \left(\frac{qB_0}{2\gamma m_0 \beta c}\right)^2 R^4$$

- Using the expression for the generalized perveance, we obtain the matched beam current:

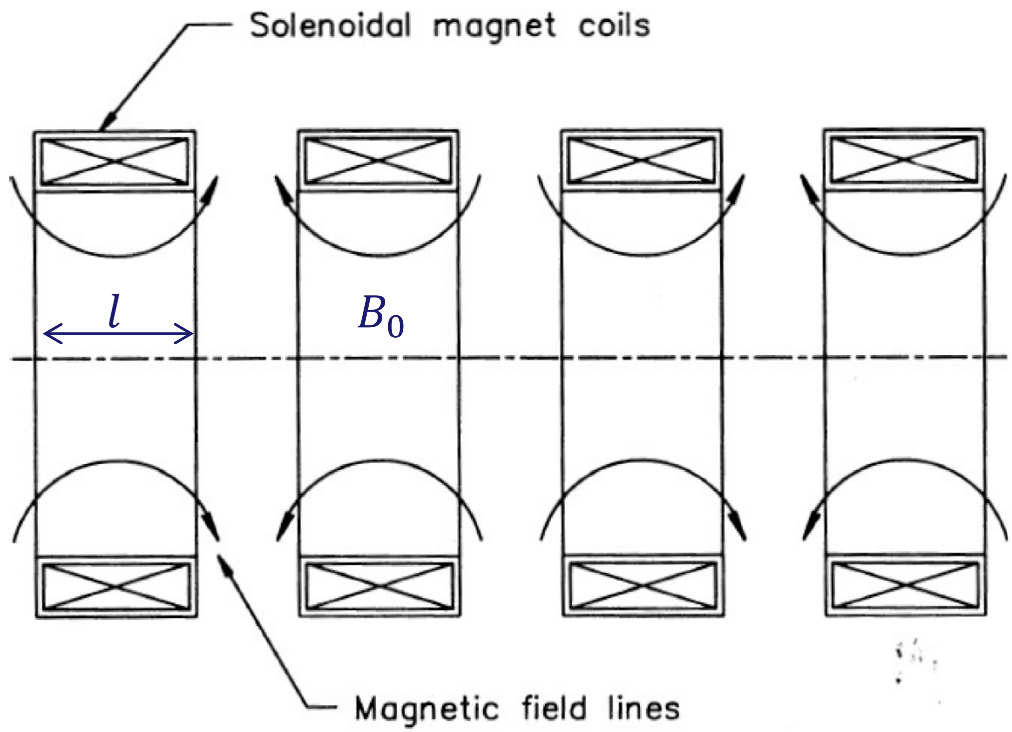
$$K = \frac{\alpha^2}{R^2} - \frac{\epsilon^2}{R^2} = \frac{eI}{2\pi\epsilon_0 m_0 (\beta\gamma c)^3} \qquad I = \left[\frac{\pi\epsilon_0 ec}{2m_0}\right](\beta\gamma)(B_0 R)^2 \left[1 - \frac{\epsilon^2}{\alpha^2}\right]$$

- If there is no emittance, the beam-generated forces exactly balance the focusing force of the axial magnetic field. Here, particle flow is laminar and the allowed current has a maximum value.

Limiting current for paraxial beams with an array of solenoidal lens

- The on-axis magnetic field has variation

$$B_z(0,z) \cong B_0 \sin\left(\frac{\pi z}{l}\right)$$

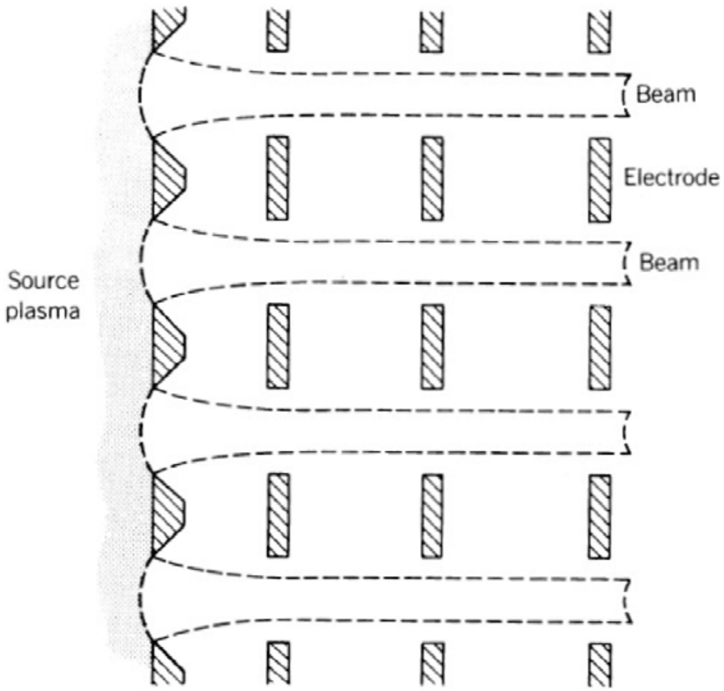


- If the beam envelope oscillations are much smaller than R, the limiting current is given approximately by

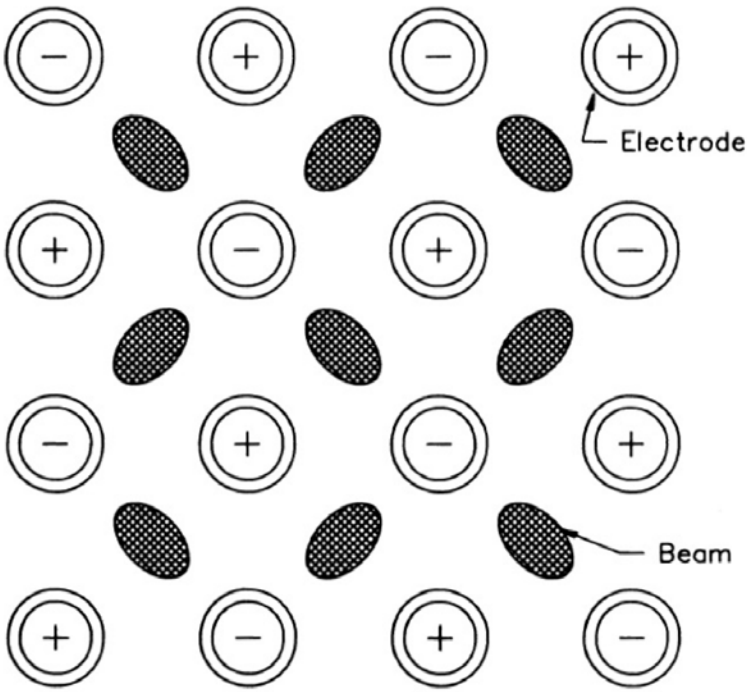
$$I = \left[\frac{\pi\epsilon_0 ec}{2m_0}\right](\beta\gamma)(B_0 R)^2 \left\langle \sin^2\left(\frac{\pi z}{l}\right)\right\rangle \left[1 - \frac{\epsilon^2}{\alpha^2}\right]$$

Multiple-beam ion transport

- One strategy to increase the limiting current in a high-flux ion accelerator is to divide a beam into many segments, each with its own focusing system.
- Electrostatic quadrupole focusing has two advantages for high-current ion beam transport; (1) Electric fields deflect nonrelativistic ions more effectively than magnetic fields. (2) Miniature magnetic quadrupole lenses are difficult to fabricate and to operate because of cooling problems.

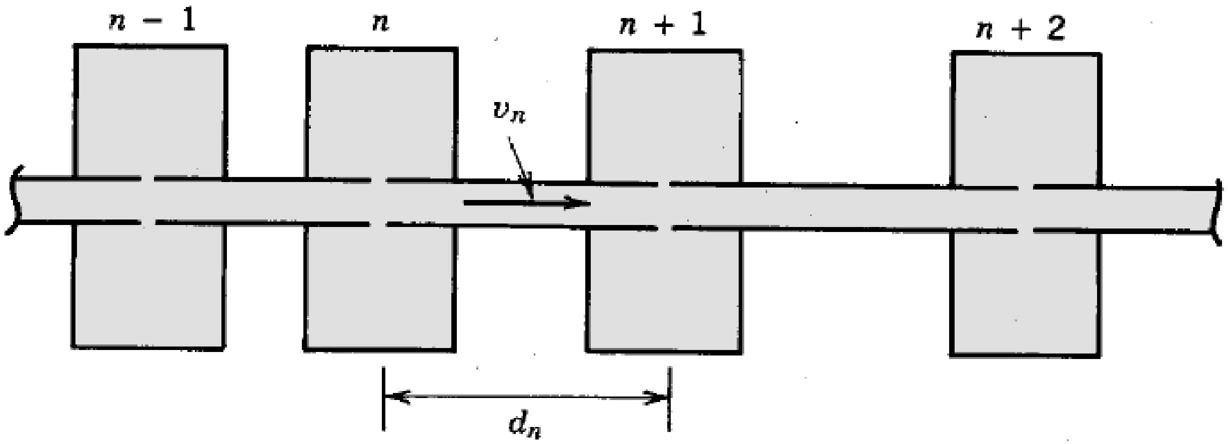


$I = Ni$



Longitudinal space-charge limits in RF accelerators and induction linacs

- Beam-generated axial electric fields can limit the beam current in RF accelerators and induction linacs.



- Ions in RF accelerators must remain in specific phase regions of the accelerating wave. The electric field of a traveling wave can provide stable axial confinement for ions that are localized along z and have a small spread in kinetic energy.
- The wave creates a potential well for ion confinement called an RF bucket. Ions that escape from the bucket quickly lose their synchronization with the wave and are no longer accelerated. Space-charge electric fields can drive ions out of an RF bucket. This process set limits on the current in the accelerator.

Longitudinal space-charge limits in RF accelerators and induction linacs

- The total potential energy for particles in the wave frame:

$$U_t(\Delta z) = \frac{eE_0 v_s}{\omega}\left[1 - \cos\left(\frac{\omega \Delta z}{v_s}\right)\right] + eE_0 \Delta z \sin\phi_s$$

- The depth of the confining potential well:

$$\Delta U_c = \frac{2eE_0 v_s}{\omega}\Psi(\phi_s)$$

- The beam-generated electric potential:

$$e\Delta\phi = \frac{eI_0}{4\pi\epsilon_0 \beta c}\left[1 + 2\ln\left(\frac{r_w}{r_0}\right)\right]$$

- The beam-generated electric force pushes particles out of the bucket if $e\Delta\phi > \Delta U_c$, giving a peak current:

$$I_0 \leq \frac{8\pi\epsilon_0 \beta c \Psi(\phi_s) E_0 v_s}{\omega[1 + 2\ln(r_w/r_0)]}$$