

## Chapter 18

# Bose-Einstein Gases

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# 18.1 Black Body Radiation

- Black body

**Black body** is an idealized physical body that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence.

- Black body radiation

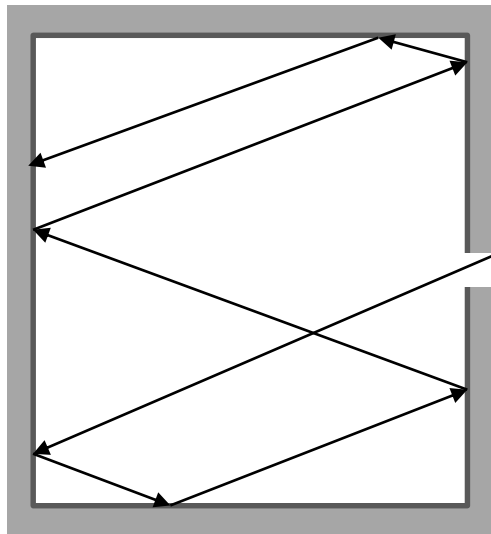
A black body in thermal equilibrium emit black body radiation(electromagnetic waves) whose spectrum is only regarded with temperature.

# 18.1 Black Body Radiation

- Black body and Photon gas

Consider a volume,  $V$  enclosed by insulated wall with small hole.

Photons injected from the hole nearly re-emitted so that the inner surface of the volume can be regarded as a black body while inner space is treated to be filled with photon gas.



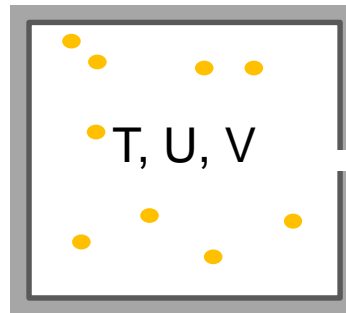
$\sum N_i \neq N$ , because photons continue to be absorbed and emitted.  
 $\sum N_i \epsilon_i = U$ , because the wall is isolated.

# 18.1 Black Body Radiation

- Photon gas with Bose Einstein statistics

Photon gas enclosed with black body surface follows Boson statistics while having no constraint about particle numbers.

Photons are bosons of spin 1 and obey Bose-Einstein statistics.



# 18.1 Black Body Radiation

- The number of photons per quantum state

$$f_j = \frac{N_j}{g_j} = \frac{1}{e^{\varepsilon_j/kT} - 1}$$

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{\varepsilon/kT} - 1}$$

$$f(\nu) = \frac{N(\nu)}{g(\nu)} = \frac{1}{e^{h\nu/kT} - 1}$$

- The number of quantum states with frequencies in the range  $\nu$  to  $\nu + d\nu$

$$g(\nu)d\nu = 2 \times \frac{4\pi V}{c^3} \nu^2 d\nu \quad c : \text{the speed of light}$$

- The energy in the range  $\nu$  to  $\nu + d\nu$

$$\begin{aligned} u(\nu)d\nu &= N(\nu)d\nu \times h\nu \\ &= g(\nu)f(\nu)d\nu \times h\nu \\ &= \frac{8\pi V \nu^2 d\nu}{c^3} \cdot \frac{h\nu}{e^{h\nu/kT} - 1} \end{aligned}$$

**Plank radiation formula**

# 18.1 Black Body Radiation

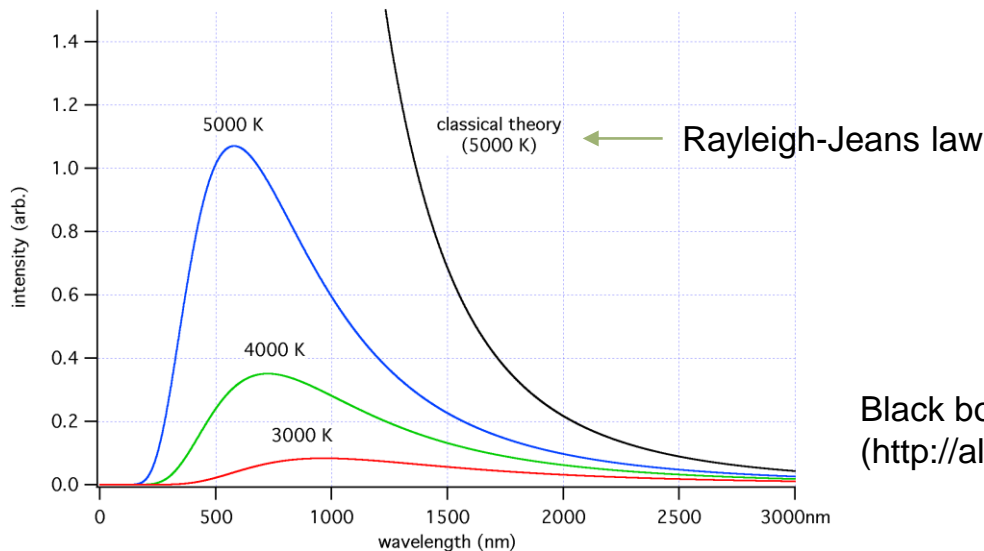
- The blackbody spectrum is often expressed in terms of the wavelength.

Then,  $u(\nu)d\nu \propto u(\lambda)d\lambda$

$$\nu = \frac{c}{\lambda} \quad d\nu = -\frac{c}{\lambda^2}d\lambda \quad |d\nu| = \frac{c}{\lambda^2}|d\lambda|$$

$$u(\lambda)d\lambda = \frac{8\pi V \frac{c^2}{\lambda^2} \frac{c}{\lambda^2} d\lambda}{c^3} \frac{hc}{\lambda} \frac{1}{e^{hc/\lambda kT} - 1} = 8\pi hcV \frac{d\lambda}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

The energy per unit wavelength in the range  $\lambda$  to  $\lambda + d\lambda$  (wavelength spectrum)



Black body radiation spectrum  
(<http://alfalfasurvey.wordpress.com>)

# 18.1 Black Body Radiation

- The total energy density

$$U = \int_0^{\infty} u(\lambda) d\lambda$$

$$\frac{U}{V} = 8\pi hc \int_0^{\infty} \frac{d\lambda}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

$$x = \frac{hc}{\lambda kT}$$

$$\frac{U}{V} = \frac{8\pi h}{h^3 c^3} (kT)^4 \int_0^{\infty} \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

$$\text{Thus, } \frac{U}{V} = aT^4 \quad a = \frac{8\pi^5 k^4}{15h^3 c^3} = 7.55 \times 10^{-16} \text{ J}/(\text{m}^3 \text{K}^4)$$

- The energy flux

$$e = \frac{c}{4} \left( \frac{U}{V} \right) = \sigma T^4 \quad \sigma = \frac{ca}{4} = 5.67 \times 10^{-8} \text{ W}/\text{m}^2 \text{K}^4 \quad \text{Stefan - Boltzmann law}$$

# 18.1 Black Body Radiation

- The wavelength  $\lambda_{max}$  at which  $u(\lambda)$  is a **maximum** satisfies a relation known as Wien's displacement law.

$$u(\lambda) = 8\pi hcV \frac{1}{\lambda^5 (e^{hc/\lambda kT} - 1)}$$

$$\frac{d}{d\lambda} [\lambda^5 (e^{hc/\lambda kT} - 1)] = 0$$

$$x = \frac{hc}{\lambda kT} \quad \frac{x}{5} = 1 - e^{-x}$$

$$\frac{1}{x^2} \frac{d}{dx} [x^{-5} (e^x - 1)] = \frac{1}{x^2} [x^{-5} e^x - 5x^{-6} (e^x - 1)] = 0$$

$$\therefore x = 4.96$$

$$\frac{hc}{\lambda_{max} kT} = 4.96$$

$$\lambda_{max} T = \frac{hc}{4.96k} = 2.90 \times 10^{-3} \text{ mK}$$

*Wien's displacement law*



# 18.1 Black Body Radiation

- For long wavelengths,  $\frac{hc}{\lambda kT} \ll 1$

$$e^{hc/\lambda kT} \approx 1 + \frac{hc}{\lambda kT} \quad \text{Taylor series}$$

$$\begin{aligned} u(\lambda)d\lambda &= 8\pi hcV \frac{1}{\lambda^5 \left(\frac{hc}{\lambda kT}\right)} \\ &= V \frac{8\pi kT}{\lambda^4} d\lambda \quad \text{Rayleigh - Jeans Formula} \end{aligned}$$

- For short wavelengths,  $e^{\frac{hc}{\lambda kT}} \gg 1$

$$u(\lambda)d\lambda = 8\pi hcV \frac{e^{-hc/\lambda kT}}{\lambda^5} d\lambda \quad \text{Wien's law}$$

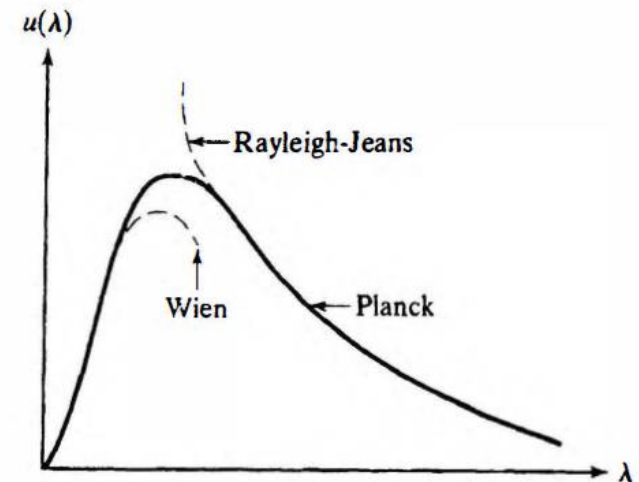


Fig. Sketch of Planck's law, Wien's law and the Rayleigh-Jeans law

# 18.2 Properties of a Photon Gas

- The heat capacity

$$\frac{U}{V} = aT^4 = \frac{8\pi^5 k^4}{15h^3 c^3} T^4 \quad \left(a = \frac{8\pi^5 k^4}{15h^3 c^3}\right)$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{32\pi^5 k^4}{15h^3 c^3} T^3 V$$

- The absolute entropy

$$S = \int_0^T \frac{C_V}{T} dT = \frac{32\pi^5 k^4 V}{15h^3 c^3} \cdot \frac{1}{3} T^3$$

- The Helmholtz function  $H = U - TS$

$$F = U - TS = aT^4 V - \frac{4}{3} aT^4 V = -\frac{1}{3} aT^4 V$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{1}{3} aT^4 = \frac{1}{3} \left(\frac{U}{V}\right) \quad \text{cf. Ideal gas } P = \frac{2U}{3V}$$

# 18.3 Bose Einstein Condensation

- The gas of noninteracting particles of large mass such that quantum effects only become important at very low temperatures. → **Ideal Bose-Einstein gas**
- $^4\text{He}$  undergoes a remarkable phase transition known as **Bose-Einstein condensation**.
- The Bose Einstein continuum distribution

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

Chemical potential  $\mu(T) = ?$

# 18.3 Bose Einstein Condensation

- For Maxwell-Boltzmann distribution (Dilute gas)

$$\text{BE: } f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}}$$

$$\mu = -kT \ln \frac{Z}{N}$$

$$Z = \left( \frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} V$$

$$\therefore \frac{\mu}{kT} = -\ln \left[ \left( \frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} \frac{V}{N} \right]$$

- As an example, for  $^4\text{He}$  at standard temperature and pressure,

$$\frac{\mu}{kT} = -12.43, \frac{\varepsilon}{kT} = 1.5, \frac{\varepsilon-\mu}{kT} = 13.9 \text{ and } f(\varepsilon) = 9 \times 10^{-7}$$

$$g(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V}{h^3} m^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon \quad \text{from Eq. (12.26)}$$

$$N = N_0 + N_{ex}$$

$$N_{excited} = \int N(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V m^{\frac{3}{2}}}{h^3} \int \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}$$

## 18.3 Bose Einstein Condensation

$$N_{excited} = \int N(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi Vm^{\frac{3}{2}}}{h^3} \int \frac{\varepsilon^{\frac{1}{2}}d\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1}$$

- For the ground state (at very low temperature, this ground state becomes significant as bosons condense into this lowest state),  $T \rightarrow 0$ ,  $\varepsilon = 0$  and  $N(\varepsilon) \rightarrow N$

$$\frac{1}{e^{-\mu/kT} - 1} \cong N$$

$$-\frac{\mu}{kT} = \ln\left(1 + \frac{1}{N}\right) \cong \frac{1}{N} \sim 0$$

- For low temperature,  $\exp\left(-\frac{\mu}{kT}\right) \sim 1$

$$x = \frac{\varepsilon}{kT}$$

$$\begin{aligned} N_{ex} &= V \frac{2}{\sqrt{\pi}} \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} \int_0^{\infty} \frac{x^{\frac{1}{2}}dx}{e^x - 1} = 2.612V \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} \\ &= 2.612 \frac{\sqrt{\pi}}{2} \end{aligned}$$

# 18.3 Bose Einstein Condensation

- Bose temperature  $T_B$  is the temperature above which all the bosons should be in excited states. Thus  $N_{ex} = N$  and  $T = T_B$ .

$$N = 2.612V \left( \frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}}$$

$$T_B = \frac{h^2}{2\pi mkT} \left( \frac{N}{2.612V} \right)^{\frac{2}{3}}$$

- For  $T > T_B$ , all the bosons are in excited states.

For  $T < T_B$ , increasing number of bosons occupy the ground state until at  $T = 0$ .

$$N = N_0 + N_{ex}$$

$$\frac{N_{ex}}{N} = \left( \frac{T}{T_B} \right)^{\frac{3}{2}} \quad \frac{N_0}{N} = 1 - \left( \frac{T}{T_B} \right)^{\frac{3}{2}}$$

→  $6.02 \times 10^{23}$   $^4\text{He}$  atoms confined to a volume of  $22.4 \times 10^{-3} \text{m}^3$ ,  $T_B \sim 0.036 \text{K}$

# 18.3 Bose Einstein Condensation

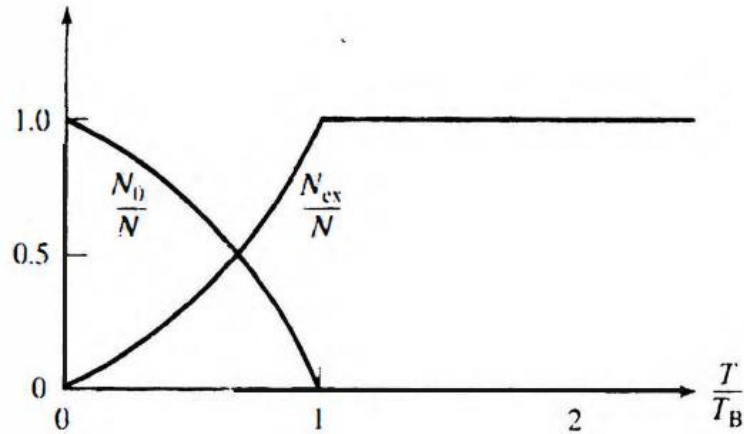


Fig. Variation with temperature of  $N_0/N$  and  $N_{ex}/N$  for a boson gas

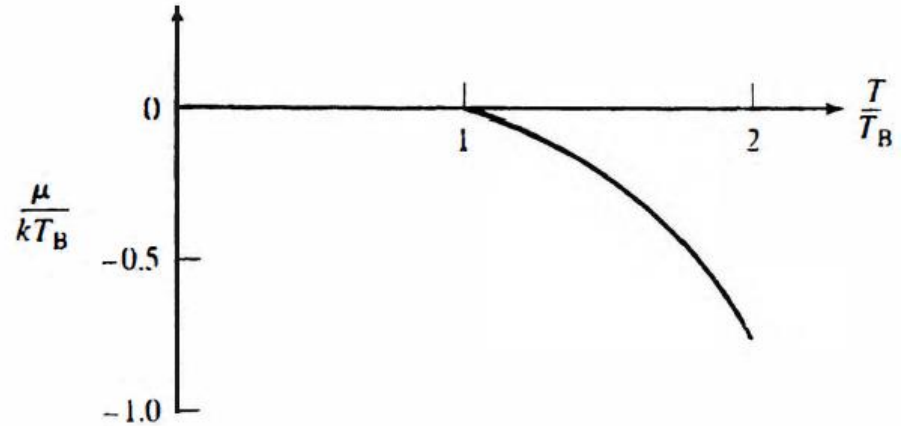


Fig. Variation with temperature of  $\mu/kT_B$  versus  $T/T_B$ .

# 18.4 Properties of a boson gas

- The internal energy

$$U = \sum N_j \varepsilon_j = N_0 \varepsilon_0 + N_{ex} \varepsilon_{ex}$$

$$\text{For } T > T_B, \quad N_0 = 0 \quad U = \frac{3}{2} NkT$$

$$T_B = \frac{h^2}{2\pi mk} \left( \frac{N}{2.612V} \right)^{\frac{2}{3}}$$

$$\text{For } T < T_B, \quad N_0 \gg 1, \varepsilon_0 = 0 \quad \frac{N_{ex}}{N} = \left( \frac{T}{T_B} \right)^{\frac{3}{2}} \quad U = ?$$



# 18.4 Properties of a boson gas

- For  $T < T_B$ ,

$$\begin{aligned}
 U &= \int_0^\varepsilon \varepsilon N(\varepsilon) d\varepsilon \\
 &= \int_0^\varepsilon \varepsilon \cdot \frac{1}{e^{(\varepsilon-\mu)/kT} - 1} \cdot \frac{4\sqrt{2}\pi V}{h^3} m^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon = g(\varepsilon) d\varepsilon \\
 &= 2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{\varepsilon^{\frac{3}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} - 1} \\
 \mu = 0 \ (T < T_B), x = \varepsilon/kT &\quad \curvearrowright \\
 &= -2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{x^{\frac{3}{2}} dx}{e^x - 1} \cdot (kT)^{\frac{5}{2}} \\
 &= \frac{3}{2} \times 1.33 kT \left(\frac{2\pi m kT}{h^2}\right)^{\frac{3}{2}} V = \frac{3\sqrt{\pi}}{4} \times 1.34 \\
 &= 0.770 N kT \left(\frac{T}{T_B}\right)^{\frac{3}{2}} \qquad T_B = \frac{h^2}{2\pi m k} \left(\frac{N}{2.612V}\right)^{\frac{2}{3}}
 \end{aligned}$$

# 18.4 Properties of a boson gas

- The heat capacity

$$C_V = \frac{dU}{dT} = 1.92Nk \left( \frac{T}{T_B} \right)^{\frac{3}{2}}$$

- The absolute entropy

$$S = \int_0^T \frac{C_V dT}{T} = 1.28Nk \left( \frac{T}{T_B} \right)^{\frac{3}{2}} \quad T \rightarrow 0, S \rightarrow 0$$

- The Helmholtz function  $H = U - TS$

$$F = U - TS = -0.51NkT \left( \frac{T}{T_B} \right)^{\frac{3}{2}} \quad T < T_B$$

$$= -1.33kT \left( \frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} V$$

$$P = 1.33kT \left( \frac{2\pi mkT}{h^2} \right)^{\frac{3}{2}} \quad T < T_B$$

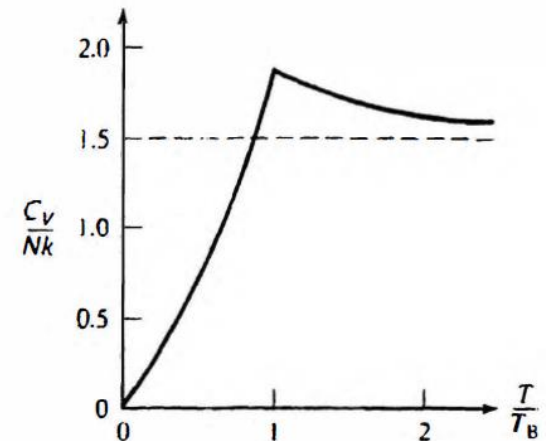


Fig. Variation with temperature of the heat capacity of a boson gas.