Appendix A—Deformational Characteristics of Suspended Cables

A.1 Objectives

The current Chapter discusses the most important formulae that are required for a description of the deformational characteristics of a suspended cable. These include the definition of cable profile under selfweight (e.g. sag and angle of deviation at the anchorages), the evaluation of undeformed and deformed cable length, and the estimation of the installed tension. Simplified formulae and numerical modelling are also discussed, as well as the levels of error attained for the approaches followed.

The intended application is the design and analysis of stays in cable-stayed bridges, although other types of structures whose cables fit the basic assumptions made here, such as suspension bridges and guyed masts, are also considered.

A.2 Static Behaviour

Suspended cables are structural elements characterised by a significant non-linear behaviour. The relatively low level of stress attained by these elements (determined by fatigue considerations) makes this non-linearity predominantly geometric in nature.

A precise description of a cable suspended between two fixed points (*Fig. A.1*) should include the bending and axial deformation, marked by the mechanical stiffnesses EI_0 and EA_0, respectively. It should also take into consideration the installed axial tension T_0 and selfweight (the latter normally constant along the cable length, as long as the cross section remains constant), and finally the end conditions. Given the large displacements caused by the low flexural stiffness, second order effects should also be included.

The evident complexity of the above stated problem is further compounded by the difficulty in a rigorous assessment of the degree of restraint of rotations at the anchorages.

Some simplifications, which enable a more accurate and simple determination of the cable profile $z(x)$ and tension $T(x)$ are, however, possible.

A.2.1 General assumption: Elastic catenary

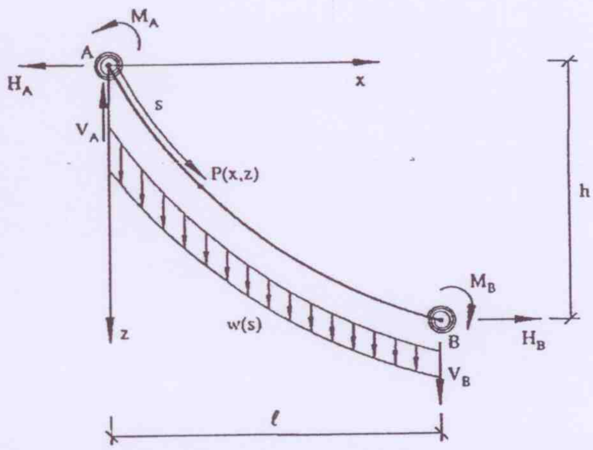

Fig. A.1: Suspended cable subject to selfweight and axial tension

The basic assumption adopted in the study of a suspended cable is that the cable acts as a perfectly flexible elastic structural element. Ignoring the cable bending stiffness is possible in view of its low value when compared with the axial stiffness EA_0. The bending effects can still be assessed locally for the static behaviour at the anchorages or integrated in the dynamic modelling in a simplified form (see Appendix B.4).

Figure A.2 illustrates the flexible cable model resulting from the assumption of null bending stiffness. The equilibrium of a segment with undeformed length s measured from the support A (*Fig. A.2* (b)) allows for the determination of the parametric equations (A.1) and (A.2) of the cable profile, and of equation (A.3) for the evaluation of the tension $T(s)$, given by Irvine [127]

$$x(s) = \frac{H_A s}{EA_0} + \frac{H_A L_0}{W} \cdot \left[\sinh^{-1}\left(\frac{V_A}{H_A}\right) - \sinh^{-1}\left(\frac{V_A - Ws/L_0}{H_A}\right)\right] \tag{A.1}$$

$$z(s) = \frac{Ws}{EA_0}\left(\frac{V_A}{W} - \frac{s}{2L_0}\right) + \frac{H_A L_0}{W}\left\{\left[1 + \left(\frac{V_A}{H_A}\right)^2\right]^{\frac{1}{2}} - \left[1 + \left(\frac{V_A - Ws/L_0}{H_A}\right)^2\right]^{\frac{1}{2}}\right\} \tag{A.2}$$

$$T(s) = \left[H_A^2 + \left(V_A - \frac{W s}{L_0}\right)^2\right]^{\frac{1}{2}} \tag{A.3}$$

In these equations, the Cartesian coordinates x and z of a generic point P are defined as a function of the unstrained length s associated with the cable segment AP, depending on the reactions at the end A, V_A and H_A, on the cable weight $W = mgL_0$, on the unstrained length L_0, and on the axial stiffness EA_0, A_0 being the area of the undeformed cable cross section and E being the elasticity modulus of the cable.

The transcendental equations (A.1) and (A.2) of the cable profile define the so-called *elastic catenary*, and constitute the most precise description of the cable geometry under selfweight. The resolution of these equations requires the knowledge of the reactions H_A and V_A, which are obtained by the introduction of the boundary conditions, resulting in the numerical solution of

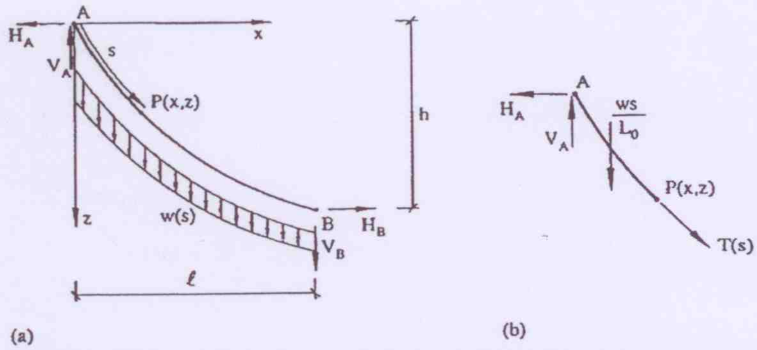

Fig. A.2: Suspended cable subject to selfweight and axial tension: (a) general simplified model; (b) equilibrium of the segment AP

$$\ell = \frac{H_A L_0}{EA_0} + \frac{H_A L_0}{W} \cdot \left[\sinh^{-1}\left(\frac{V_A}{H_A}\right) - \sinh^{-1}\left(\frac{V_A - W}{H_A}\right)\right] \tag{A.4}$$

$$h = \frac{W L_0}{EA_0}\left(\frac{V_A}{W} - \frac{1}{2}\right) + \frac{H_A L_0}{W} \cdot \left\{\left[1 + \left(\frac{V_A}{H_A}\right)^2\right]^{\frac{1}{2}} - \left[1 + \left(\frac{V_A - W}{H_A}\right)^2\right]^{\frac{1}{2}}\right\} \tag{A.5}$$

The knowledge of V_A and H_A allows also for the evaluation of the maximum tension along the cable, T_A, given by

$$T_A = [H_A^2 + V_A^2]^{\frac{1}{2}} \tag{A.6}$$

and for the evaluation of the angle of deviation at the anchorages, ω_A and ω_B (*Fig. A.3*)

$$\omega_A = a\tan\left(\frac{V_A}{H_A}\right) - \alpha \tag{A.7}$$

$$\omega_B = \alpha - a\tan\left(\frac{V_B}{H_B}\right) = \alpha - a\tan\left(\frac{V_A - W}{H_A}\right) \tag{A.8}$$

The cable sag d, defined as the maximum vertical distance to the chord, can be assessed at the point S (*Fig. A.3*), characterised by the Lagrangian coordinate s_S

$$s_S = \frac{L_0}{W}\left(V_A - h\frac{H_A}{\ell}\right) \tag{A.9}$$

and is given by

$$\mathrm{d} = z(s_S) - \frac{h}{\ell}x_S, \quad x_S = x(s_S) \tag{A.10}$$

Finally the deformed cable length L_f can be obtained by

$$L_f = L_0 + \frac{H^2 L_0}{2WEA_0} \cdot \left[\frac{V_A}{H_A} \cdot \sqrt{1 + \left(\frac{V_A}{H_A}\right)^2} + \ln\left(\frac{V_A}{H_A} + \sqrt{1 + \left(\frac{V_A}{H_A}\right)^2}\right) \right.$$

$$\left. - \frac{V_A - W}{H_A} \cdot \sqrt{1 + \left(\frac{V_A - W}{H_A}\right)^2} - \ln\left(\frac{V_A - W}{H_A} + \sqrt{1 + \left(\frac{V_A - W}{H_A}\right)^2}\right) \right] \tag{A.11}$$

Table A.1 presents the geometric and mechanic characteristics of a series of cables from cable-stayed bridges, calculated on the basis of formulae (A.1) to (A.11). These will be used as reference for the next Section, where further simplifications will be introduced. Some useful conclusions can be inferred from the analysis of this table, which covers a wide range of cables:

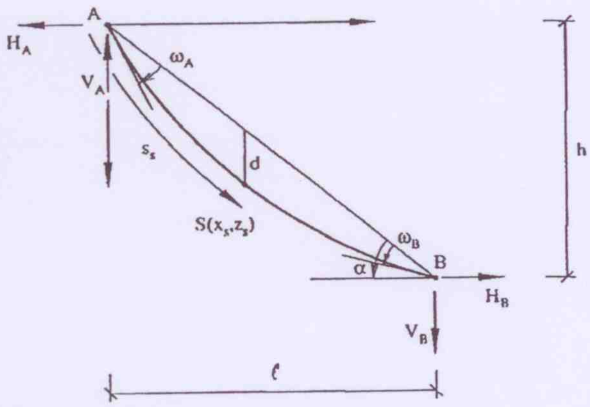

Fig. A.3: Elastic catenary

- Cable-stayed bridge cables are subject to relatively low stresses $\sigma_{\max}$, no greater than 900 *Mpa*. For these cables the variation of tension ΔT along the length does not exceed 2% of the maximum tension;
- The values of the ratio $\frac{mgL_0}{T_{\max}}$ attained for cable stays are also quite small and do not exceed typically 5% of the maximum component of cable tension $T_{\max}$. This ratio increases with cable length, as happens with the longest of the Normandy bridge cables.

An important parameter characteristic of a suspended cable has been introduced by Irvine [127], incorporating both the corresponding geometric and deformational characteristics. This parameter λ^2 is defined as

$$\lambda^2 = \left(\frac{mgL}{T}\right)^2 \cdot \frac{L}{\frac{TL_e}{EA_0}} \tag{A.12}$$

where L and T represent the chord length and the component of tension along the cable chord, respectively, and L_e is a virtual length of cable defined by

$$L_e = \int_0^L \left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^3 \mathrm{d}x \approx L \cdot \left\{1 + 8\left(\frac{\mathrm{d}}{L}\right)^2\right\} \tag{A.13}$$

Typical values attained by stay cables vary in the range 0–1, while for suspension bridges λ^2 is normally greater than 100. Very large stay cables can have a λ^2 value greater than 1, as shown in *Table A.1* for the largest cable of the Normandy bridge. Small values of λ^2 reflect relatively highly stressed and low sagging cables, whose deformation is achieved essentially by extensibility, while large values are typical of very low tensioned and higher sagging cables, whose deformation is mainly of geometric nature, exhibiting therefore a relative inextensibility.

These different characteristics imply different levels of simplification both for static and dynamic analyses.

Cable	Area (cm^2)	EA (kN)	mass (kg/m)	ℓ (m)	h (m)	α (°)	L (m)	L_0 (m)	L_f (m)	d (m)	λ^2
V. Gamma											
HC01	46.5	906 750	42.9	1.81	34.65	87.0098	34.697	34.700	34.778	0.081	0.023
HC24	109.5	2 135 250	100.1	205.96	92.97	24.295	225.972	226.584	227.299	1.074	0.344
HC15	82.5	1 608 750	74.8	126.05	76.52	31.2596	147.457	147.752	148.145	0.569	0.241
Guadiana											
Central 1	77	1 501 500	72.9	152.42	71.94	25.2664	168.544	168.992	169.532	0.467	0.198
Central 16	27	518 700	21.0	18.64	45.84	67.8768	49.485	49.518	49.723	0.198	0.005
Normandy	153	2 907 000	133.0	420	134	17.6952	440.858	441.931	442.962	4.743	3.085
Ikuchi*	85.8	1 716 000	72.5	231.35	84.20	20	246.200	246.671	247.171	1.616	1.254

Table A.1: Geometric, mechanical and deformational characteristics of different cables

Cable	ω_A (°)	ω_B (°)	Ss (m)	Ts (kN)	Tmin (kN)	Tmax (kN)	ΔT (%)	σ_{max} (Mpa)	mgL_0/T_{max} (%)
V. Gamma									
HC01	0.0173	0.0389	28.048	2033.2	2030.4	2045	0.71	439.8	0.71
HC24	0.9234	2.6704	122.157	6738.8	6700.3	6785.5	1.26	619.7	3.28
HC15	0.6836	1.9411	81.924	4275.2	4251.4	4305.5	1.26	521.9	2.52
Guadiana									
Central 1	0.6114	1.9312	79.495	4793.6	4767.9	4817	1.02	625.6	2.51
Central 16	0.1329	0.2360	64.541	2141.7	2144.6	2154	0.44	809.8	0.47
Normandy	2.3281	7.0735	223.881	6778.9	6720.7	6850.5	1.89	447.7	8.42
Ikuchi*	1.3829	4.1450	126.468	3474.5	3450.2	3502.2	1.48	408.2	5.01

* α is estimated.

Table A.1: Continued

A.2.2 Elastic parabola

The use of the transcendental equations presented above to characterise the deformational characteristics of stay cables requires numerical manipulation. Although commercially available software can easily be employed, simplified practical formulae are of interest for a wide range of situations.

The elastic parabola approach applies to shallow cables, i.e. cables with a small sag to span d/L ratio, typically no greater than 1:8. This range covers stays from cable-stayed bridges and most of the cables from suspension bridges. The assumption of a unit ratio between the deformed and undeformed cable length yields the simple formulae for the cable profile defined in cartesian coordinates

$$z(x) = \frac{1}{2}\frac{mg}{H} \cdot \sec\alpha \cdot x \cdot (\ell - x) \cdot \left[1 + \frac{\varepsilon}{6} \cdot \left(1 - 2\frac{x}{\ell}\right)\right] + \frac{h}{\ell} \cdot x \tag{A.14}$$

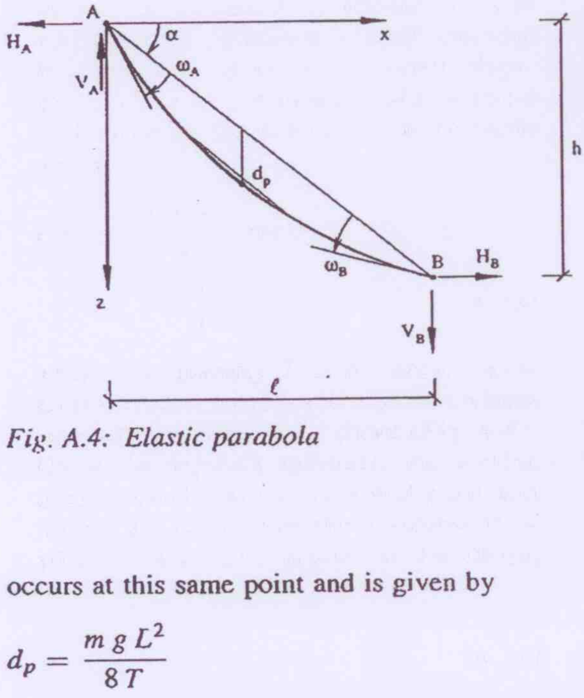

Fig. A.4: Elastic parabola

where the parameter ε, defined by $\varepsilon = mg\ell \sin\alpha/H$ represents a small quantity for a shallow cable with a small slope. Ignoring ε, a simple equation of a parabola is obtained for the description of the cable profile

$$z(x) = \frac{1}{2}\frac{mg}{H} \cdot \sec\alpha \cdot x \cdot (\ell - x) + \frac{h}{\ell} \cdot x \tag{A.15}$$

where the quantity $T = H \cdot \sec\alpha$ represents the cable tension at the section whose tangent is parallel to the chord (*Fig. A.4*). Using the parabola approach, this section occurs actually at the cable mid-point and the sag d_p, i.e. the maximum vertical deviation of the cable profile to the chord, occurs at this same point and is given by

$$d_p = \frac{m\,g\,L^2}{8\,T} \tag{A.16}$$

The angles of deviation at the anchorages A and B, ω_A and ω_B, are

$$\omega_A = \omega_B = a\tan\left(\frac{mgL}{2H} + \frac{h}{\ell}\right) - \alpha \tag{A.17}$$

and the deformed cable length L_{fp} can be approximated by

$$L_{fp} = L \cdot \left[1 + \frac{8}{3} \cdot \left(\frac{d_p}{L}\right)^2 - \frac{32}{5} \cdot \left(\frac{d_p}{L}\right)^4\right] \tag{A.18}$$

Table A.2 presents the deformational characteristics of the above described stay cables based on the parabolic approach expressions (A.15) to (A.18).

Cable	ω_A (°)= ω_B (°)	T (kN)	d_p (m)	L_{fp} (m)	ε_{dp} (%)	ε_T (%)	ε_{Lfp} (%)
V. Gama							
HC01	0.0173	2056.8	0.031	34.697	–61.8	1.2	–0.2
HC24	1.6239	6834.0	0.917	225.982	–14.6	1.4	–0.6
HC15	1.1278	4336.4	0.460	147.461	–19.2	1.4	–0.5
Guadiana							
Central 1	1.1365	4841.0	0.524	168.548	12.4	1.0	–0.6
Central 16	0.1401	2166.3	0.029	49.485	–85.3	1.1	–0.5
Normandy	4.3590	6933.6	4.572	440.985	–3.6	2.3	–0.4
Ikuchi*	2.5525	3531.9	1.526	246.225	5.6	1.7	–0.4

Table A.2: Deformational characteristics of cables resultant from parabolic approach

It can observed from *Table A.2* that the parabolic approach provides significant errors in the description of the static behaviour of the cable, in local quantities, like the angles of deviation to the chord at the anchorages and sag (ε_{dp}). The error committed increases both with the angle of inclination of the cable to the horizontal and with the chord length. And even though the error in the deformed cable length (ε_{Lfp}) is low, the fact is that a very small error in the initial cable length may result in a high error in the sag evaluation. Although very practical and useful for an approximate analysis during the design phase, the parabolic approach is not convenient whenever a precise description of the static behaviour of a stay cable is required, namely for installation purposes.

A.2.3 Numerical modelling

The integration of the cable stay behaviour in the numerical description of a cable-stayed bridge requires further simplifications whose effects should be acknowledged.

A.2.3.1 Linear model: Truss element

The simplest and also most common approach employed in the numerical modelling of a stay cable is based on the idealisation of the so-called truss element.

The truss element is a two node elastic finite element characterised by null bending stiffness and an axial stiffness EA_0/L, whose weight is concentrated at the nodes (*Fig. A.5*). These characteristics correspond actually to the treatment of the cable as a spring element, not accounting for geometric effects and providing

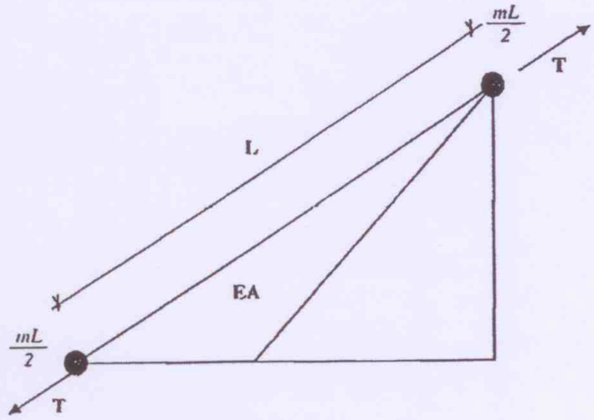

Fig. A.5: Truss element

naturally a poor description of the local deformational characteristics: both the sag and angles of deviation at anchorages are nil, the cable undeformed length is equal to the chord length, and the tension is assumed constant along the cable.

Despite the poor local characteristics, the linear model is of great interest for a global analysis of the bridge behaviour, allowing for a good estimation of the force distribution in the cable-stayed bridge, and providing therefore important information for the design of the stay cables. The major source of error associated with the linear model results from geometric effects. So, for taut stay cables, with a low λ^2 value, small errors are expected, while for less tensioned or very long cables, with high values of λ^2, the errors may be significant.

A.2.3.2 Linear model refinement: Equivalent modulus of elasticity

It is still possible to introduce the non-linear geometric behaviour in a simplified form. This is achieved through the approximation of the cable profile by a parabola and determination of the axial stiffness as a function of both the cable tension and sag. An Equivalent Modulus of Elasticity E_{eq} is obtained which incorporates these quantities through the parameter λ^2 and is given by

$$E_{eq} = \frac{E}{1 + \frac{\lambda^2}{12}} \tag{A.19}$$

Another equivalent formula is given by Ernst [128]

$$E_{eq} = \frac{E}{1 + \frac{\gamma^2 L^2}{12\sigma^3} E} \tag{A.20}$$

where γ is the specific weight and σ is the tensile stress of the cable. The variation of E_{eq} with λ^2 is represented in *Fig. A.6*, showing that for standard taut stay cables ($\lambda^2 < 1$) the correction is actually very small ($\lambda^2 = 1$, $E_{eq} = 0.92\,E$), while for very long stay cables the correction becomes significant (for the largest of the Normandy stay cables, $\lambda^2 = 3.1$, $E_{eq} = 0.79\,E$). *Figure A.7* shows the variation of the ratio E_{eq}/E with the cable chord length for different levels of stress. Using information from *Table A.1*, one can understand that this ratio is greater than 0.90 for all of the analysed stay cables, except the largest of Normandy bridge.

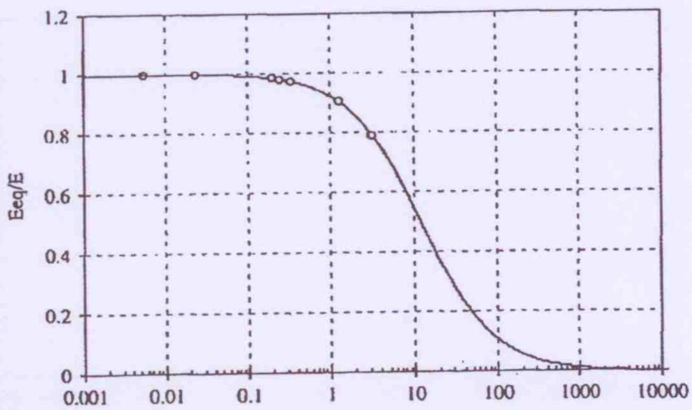

Fig. A.6: Variation of the ratio E_{eq}/E with λ^2

The use of an Equivalent Modulus of Elasticity E_{eq} in the description of cable stay behaviour provides an improved distribution of forces throughout the bridge cables and a better approximation of the global cable deformability, and therefore an improved shape of the bridge profile under permanent loads. It should be noted however that second order effects associated with other bridge elements, namely the deck and towers, which suffer significant compressions, may be more relevant in terms of the global bridge behaviour. This means that the use of a non-linear geometric formulation for global static analysis of the bridge may be more relevant than the introduction of corrections to sag/tension of individual cables.

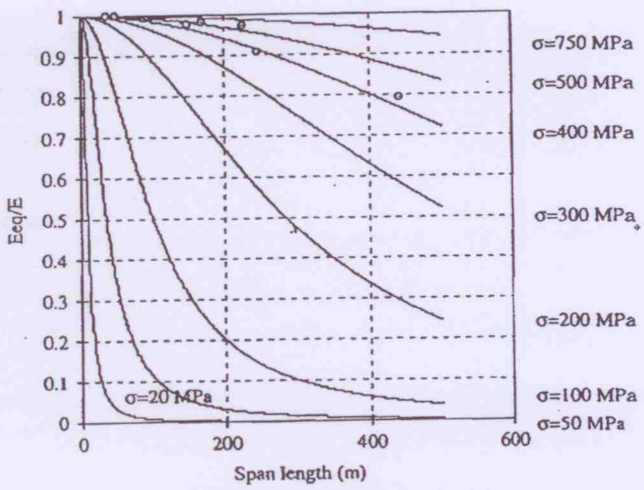

Fig. A.7: Variation of the ratio E_{eq}/E with the cable span for different levels of cable stress

A.2.3.3 Linear model refinement: Multi-link approach

A natural extension of the idealisation of the stay cable as a simple truss element to a series of truss elements (*Fig. A.8*) has been proposed in the past [129] as a computational improvement that allows for the accounting of geometric effects, as long as the discretisation is complemented with a geometric non-linear analysis. Owing to the resulting large dimension of numerical models, and to computational limitations, the implementation of this modelling technique has not been a current trend in the global modelling of a cable-stayed bridge. It should be noticed however that currently available commercial software and computer memory allow for reasonable computing times in the face of the advantages obtained: using an adequate number of elements to discretise a stay cable, the corresponding weight, applied at the nodes, approximates the distributed weight of the cable, and therefore the corresponding profile approximates the elastic catenary profile. As for the number of necessary elements to represent adequately the deformational behaviour of a stay cable, it is relevant to analyse the error associated with the discretisation for several physical quantities of interest. For this purpose, two stay cables from Vasco da Gama bridge were considered, the longest, HC024, and the shortest, HC01, with properties listed in *Table A.1*. The cables were discretised into a successively growing number of truss elements. Taking the 100 link discretisation as a basis, the relative errors obtained for various physical quantities were calculated and are plotted in *Fig. A.9*. The analysed quantities are: the minimum axial force, T_{min}; the sag d; the relative rotation to the chord at one end, ω_B; and also the first five natural frequencies, f_1 to f_5. The relative errors, represented for each quantity in *Fig. A.9*, are designated respectively as *ETmin, Edv, Ewb, Ef1, . . ., Ef5*.

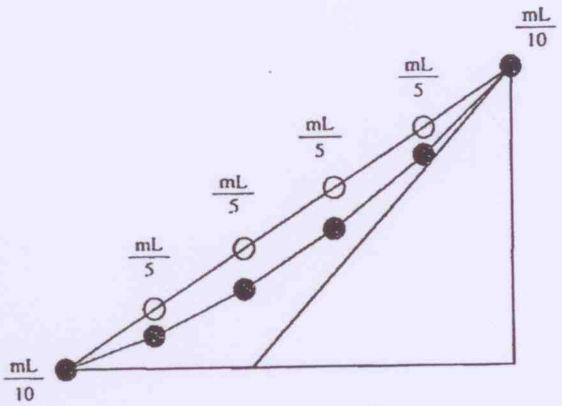

Fig. A.8: Multi-link approach: undeformed and deformed mesh under selfweight

The analysis of *Fig. A.9*: shows that a discretisation in 20 truss elements provides relative errors of less than 5% in all the mentioned quantities for the two cables. But if, for example, only the first three vibration modes of the cable are of interest and local effects, like the end rotations, are not relevant, then a discretisation into 9 elements is sufficient, both for the shortest and longest cables:

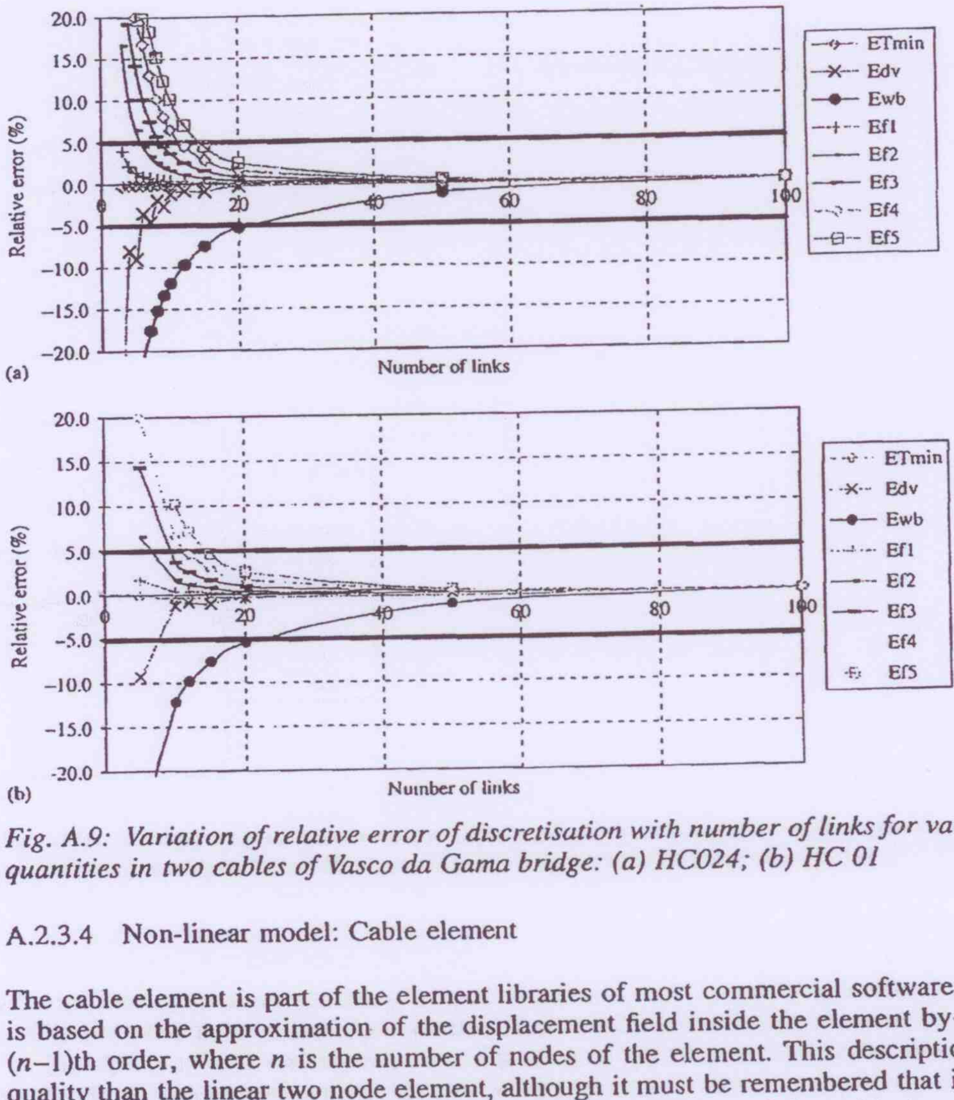

Fig. A.9: Variation of relative error of discretisation with number of links for various physical quantities in two cables of Vasco da Gama bridge: (a) HC024; (b) HC 01

A.2.3.4 Non-linear model: Cable element

The cable element is part of the element libraries of most commercial software packages and is based on the approximation of the displacement field inside the element by a parabola of $(n-1)$th order, where n is the number of nodes of the element. This description is of better quality than the linear two node element, although it must be remembered that it is still based on a linear elastic formulation. Therefore, only the division of the cable into various finite elements combined with a geometric non-linear formulation provides a good approximation of the corresponding static and dynamic behaviour.

A.2.3.5 Comparative analysis for global study of a cable-stayed bridge

In order to understand the relevance of the chosen numerical formulation of the stay cables in the global behaviour of a cable-stayed bridge, some results of a study developed on a conventional medium size bridge of this type are presented.

The bridge [130], represented in *Fig. A.10*, is composed of a central span of 324 m, two lateral spans of 135 m and two transition spans of 36 m, having a total length of 666 m. The 18 m wide deck has a prestressed concrete box section that is partially supported at the concrete 100 m high A-shaped towers and suspended by stay cables. These are arranged in a semi-fan, in a total of 32 pairs anchored at the top of each tower.

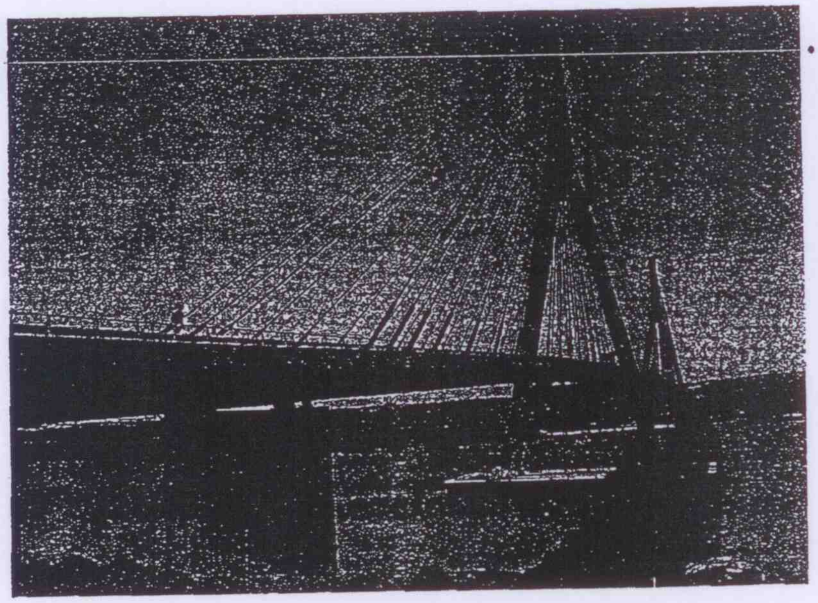

Fig. A.10: General view of cable-stayed bridge under analysis

The stay cables' chord length varies in the range of 49.5–170.2m, and the stress measured on the site varies in the range of 340–680 MPa. The Irvine parameter λ^2 varies, for the set of cables, in the range of 0.0162–0.4795, meaning that the Equivalent Modulus of Elasticity E_{eq}, defined by (A.19), is in the range of $0.962 \cdot E_a$ to $0.999 \cdot E_a$, E_a being the modulus of elasticity of steel. The fundamental natural frequency of the stay cables is in the interval between 0.78–2.97 Hz, while the first bridge natural frequencies of vertical, lateral and torsional modes were calculated as 0.38, 0.52 and 1.49 Hz, respectively.

The studies developed on this bridge consisted in the analysis of the static response to selfweight, followed by an evaluation of natural frequencies and modal shapes. This analysis was performed introducing the various degrees of simplification referred along this Chapter, i.e., (i) linear analysis based on idealisation of the cables as simple truss elements; (ii) linear analysis based on idealisation of the cables as truss elements with correction of the Modulus of Elasticity to take into account local effects (Equivalent Modulus of Elasticity approach); (iii) non-linear geometric analysis, based on idealisation of the cables as simple truss elements and (iv) non-linear geometric analysis based on the idealisation of each stay cable as a series of ten truss elements (multi-link approach).

It should be noted that for simplification purposes, the equivalent Modulus of Elasticity of stay cables was made uniform and equal to the average value of $0.975 \cdot E_a$.

Table A.3 systematises some of the most important results obtained, expressed in terms of maximum displacement at the lateral span, midspan and top of one tower, maximum

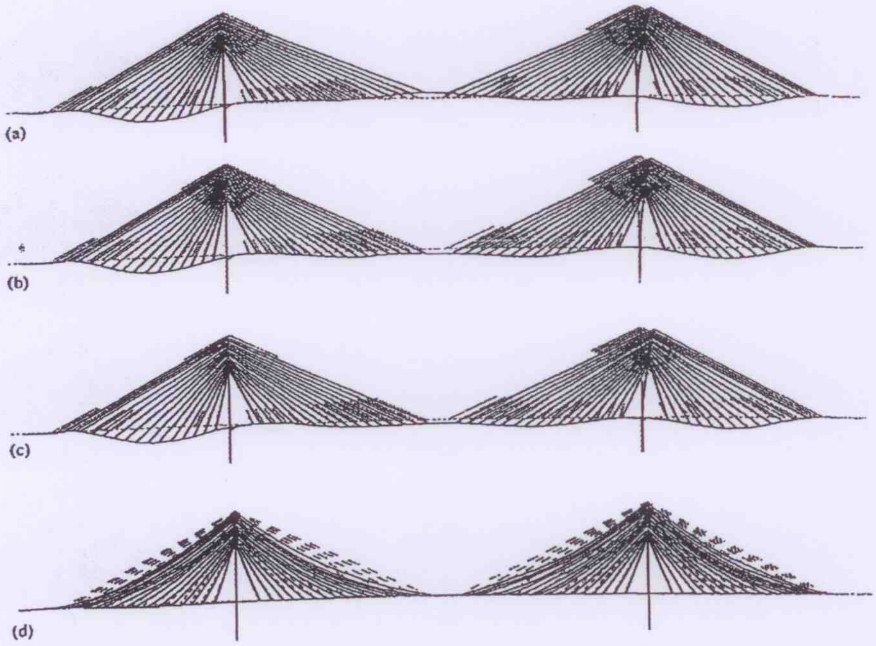

Fig. A.11: Dead load configuration of cable-stayed bridge for the analysis: (a) linear, $E_a =$ 200 GPa; (ii) Linear, $E_{eq} = 195$ GPa; (c) geometric non-linear, one truss element per cable; (d) geometric non-linear, 10 truss elements per cable

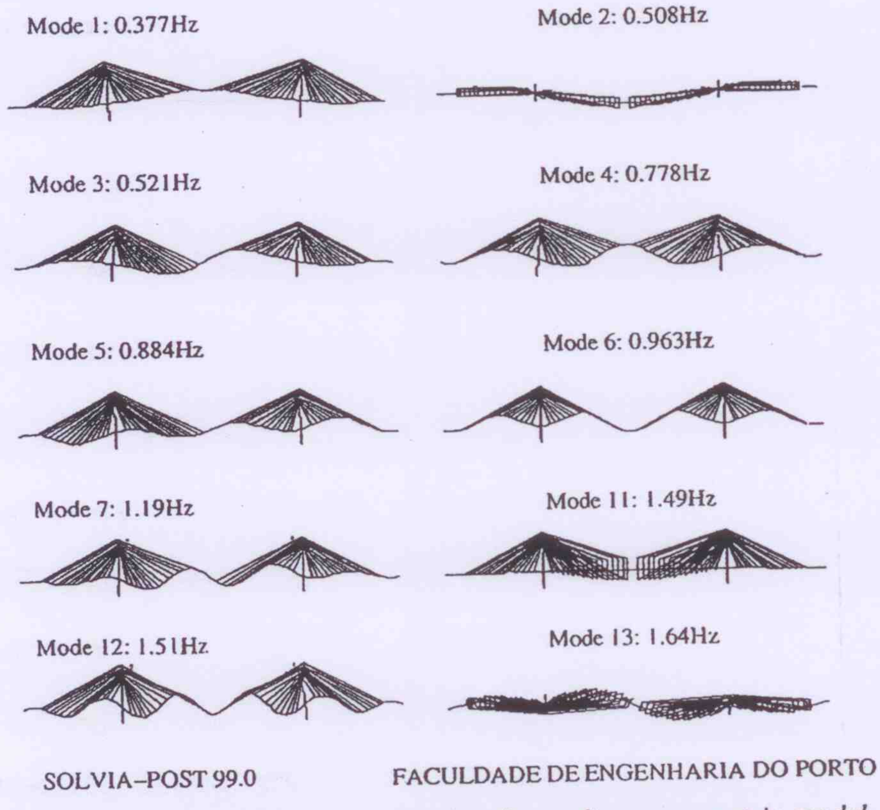

Fig. A.12: Modal shapes associated with non-linear geometric model, idealising each stay cables as one truss element compression force in the deck and towers, maximum bending moment in deck and tower, and most relevant natural frequencies.

The analysis of this table shows that, although not very significant in absolute value, the introduction of the equivalent modulus of elasticity leads to a substantial modification of the deformed structure under selfweight. *Figure A.11*, representing the undeformed and deformed finite element mesh obtained for each analysis, illustrates this aspect, showing that the reduction of cable stiffness leads to higher mid-span displacements. Note that the deformed configurations are represented at the same scale for the first three cases of analysis. The comparison between deformed configurations obtained under linear or geometric non-linear analyses shows very slight differences while the stay cables are idealised as single truss elements (model (iii)). However, the idealisation of each stay cable as a series of ten truss elements allows for the consideration of the cable geometric non-linearity, which significantly changes both deformations and internal forces.

With respect to the dynamic behaviour of the bridge, it can be noticed from *Table A.3* that the natural frequencies of the most relevant modes (modal configurations represented in *Fig. A.12*) have very slight variations for the various models, meaning that geometric non-linear effects have no significant influence in the global dynamic behaviour of the bridge. It is important however to notice that the multi-link model leads to the introduction of cable dynamics in the global bridge behaviour. As a consequence, significant coupling between cable and bridge vibration occurs, leading to numerous vibration modes, here called multiple modes, characterised by very close natural frequencies

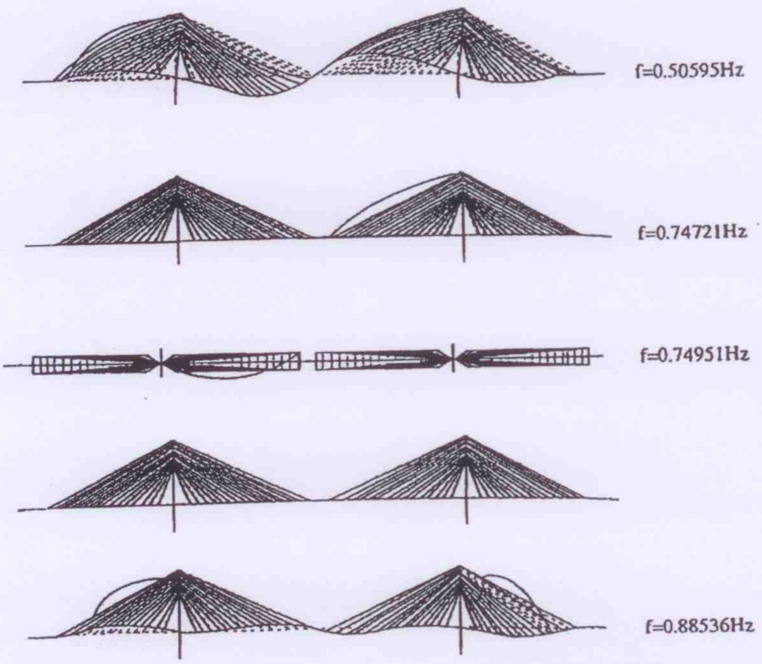

Fig. A.13: Modal shapes associated with non-linear geometric model, idealising each stay cable as a series of ten truss elements and similar configurations of deck and towers, but involving the participation of different cables [85]. *Figure A.13* shows typical configurations of vibration modes involving the participation of cables with different degrees. It is thought that the parametric excitation phenomenon described in Chapter 5 is enhanced by this coupling effect.