Shells

❖ Ritz Analysis

➢ Just consider bending in general,

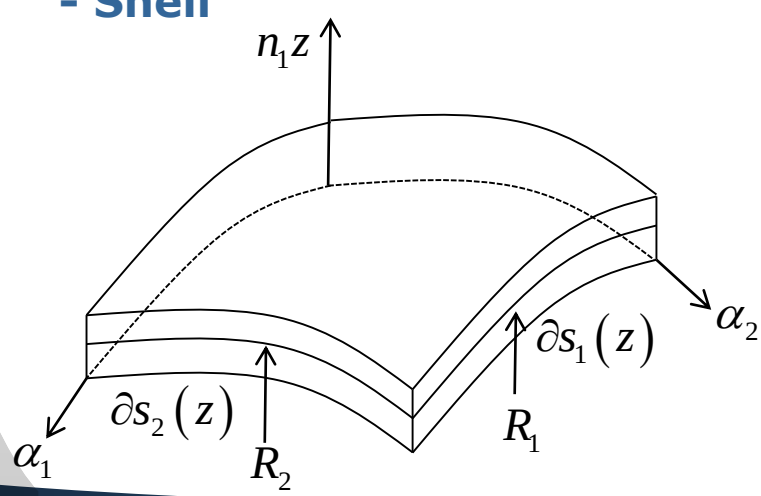
$$u_0 = \sum_{i=1}^{n_u} \varphi_{u_i}(x,y) q_{u_i}(t)$$

$$v_0 = \sum \varphi_{v_i}(x,y) q_{v_i}(t)$$

$$w_0 = \sum \varphi_{w_i}(x,y) q_{w_i}(t)$$

$$[B] = 0$$

- Shell



Assume the following

$$u = u_0(\alpha_1, \alpha_2) + z\beta_1(\alpha_1, \alpha_2)$$

$$v = v_0(\alpha_1, \alpha_2) + z\beta_2(\alpha_1, \alpha_2)$$

$$w = w_0(\alpha_1, \alpha_2)$$

Plug into expression for

$$S_5,\ S_4,\ etc \quad = 0$$

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- to solve for $\beta_1,\ \beta_2$

$$\beta_1 = \frac{u_0}{R_1} - \frac{1}{A_1}\frac{\partial w_0}{\partial \alpha_1}$$

$$\beta_2 = \frac{v_0}{R_1} - \frac{1}{A_2}\frac{\partial w_0}{\partial \alpha_2}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \Delta_1 \begin{bmatrix} S_1^0 \\ S_2^0 \\ S_{12}^0 \\ S_{21}^0 \end{bmatrix} + z\Delta_1 \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \\ \kappa_{21} \end{bmatrix} \qquad \Delta_1 = \begin{bmatrix} \frac{1}{\varphi_1} & & & \\ & \frac{1}{\varphi_2} & & \\ & & \frac{1}{\varphi_1} & \frac{1}{\varphi_2} \end{bmatrix}$$

$$S_3 \approx 0$$

$$S_4 = S_5 = 0$$

$$S_1^0 = \frac{1}{A_1}\frac{\partial u_0}{\partial \alpha_1} + \frac{v_0}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w_0}{R_1}$$

$$S_2^0 = \frac{1}{A_2}\frac{\partial v_0}{\partial \alpha_2} + \frac{u_0}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w_0}{R_2}$$

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$$S_{12}^0 = \frac{1}{A_1}\frac{\partial v_0}{\partial \alpha_1} - \frac{u_0}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$S_{21}^0 = \frac{1}{A_1}\frac{\partial v_0}{\partial \alpha_1} - \frac{v_0}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}$$

$$S_6^0 = S_{12}^0 + S_{21}^0$$

$$S_6 = \frac{1}{\varphi_1 \varphi_2}\left[\left(1 - \frac{z^2}{R_1 R_2}\right) S_6^0 + z\left(1 + \frac{z}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right)\kappa_6^0\right]$$

$$\kappa_6^0 = 2\left(\kappa_{12}^0 + \frac{S_{21}^0}{R_1}\right) = 2\left(\kappa_{21}^0 + \frac{S_{12}^0}{R_2}\right)$$

$$\kappa_1^0 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\kappa_2^0 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}$$

$$\kappa_{12}^0 = \frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\kappa_{21}^0 = \frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_1}$$

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$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \Delta_1' \begin{bmatrix} S_1^0 \\ S_2^0 \\ S_6^0 \end{bmatrix} + z\Delta_1'' \begin{bmatrix} \kappa_1^0 \\ \kappa_2^0 \\ \kappa_6^0 \end{bmatrix}$$

$$\Delta_1' = \begin{bmatrix} \frac{1}{\varphi_1} & 0 & 0 \\ 0 & \frac{1}{\varphi_2} & 0 \\ 0 & 0 & \frac{1}{\varphi_1\varphi_2}\left(1-\frac{z^2}{R_1R_2}\right) \end{bmatrix}, \qquad \Delta_1'' = \begin{bmatrix} \frac{1}{\varphi_1} & 0 & 0 \\ 0 & \frac{1}{\varphi_2} & 0 \\ 0 & 0 & \frac{1}{\varphi_1\varphi_2}\left(1+\frac{z}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right) \end{bmatrix}$$

- Introducing the stress-strain relations for a lamina (plane stress)

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11}^E & c_{12}^E & c_{16}^E \\ c_{12}^E & c_{22}^E & c_{26}^E \\ c_{16}^E & c_{26}^E & c_{66}^E \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} - \underbrace{\begin{bmatrix} e_{13} \\ e_{23} \\ e_{33} \end{bmatrix} E_3}_{\begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix}}$$

→ in force aligned with $\left(\overrightarrow{\alpha_1}, \overrightarrow{\alpha_2}, \vec{n}\right)$

→ plane stress properties

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$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = [c][\Delta_1]\begin{bmatrix} S_1^0 \\ S_2^0 \\ S_{12}^0 \\ S_{21}^0 \end{bmatrix} + z[c]\Delta_1\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \\ \kappa_{21} \end{bmatrix} - \begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix}$$

$$[T] = [c][\Delta_1 \quad z\Delta_1]\begin{bmatrix} S_0 \\ \kappa \end{bmatrix} - [T^E]$$

- Note you also have

$$\begin{bmatrix} S_0 \\ \kappa \end{bmatrix} = \underbrace{[\Omega][\theta]}_{\text{Differential operator for shell theory}}\begin{bmatrix} u^0 \\ v^0 \\ w^0 \end{bmatrix}$$

Refer to "Jia-Rogers"

Shells

- Stress resultants concisely

$$\begin{bmatrix} N_1 & M_1 \end{bmatrix} = \int_{-h/2}^{h/2} T_1 \varphi_2 \begin{bmatrix} 1 & z \end{bmatrix} dz$$

$$\begin{bmatrix} N_2 & M_2 \end{bmatrix} = \int T_2 \varphi_1 \begin{bmatrix} 1 & z \end{bmatrix} dz$$

$$\begin{bmatrix} N_{12} & M_{12} \end{bmatrix} = \int T_6 \begin{bmatrix} 1 & z \end{bmatrix} \varphi_2 dz$$

$$\begin{bmatrix} N_{21} & M_{21} \end{bmatrix} = \int T_6 \begin{bmatrix} 1 & z \end{bmatrix} \varphi_1 dz$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} dz, \qquad \Delta_2 = \begin{bmatrix} \varphi_2 & & \\ & \varphi_1 & \\ & & \varphi_2 \\ & & \varphi_1 \end{bmatrix}$$

$$\Delta_2 \neq \Delta_1^T$$

$$\Delta_2 = \varphi_1 \varphi_2 \Delta_1^T$$

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in contracted notation

$$N_6 = \frac{1}{2}\left(N_{12} - \frac{M_{21}}{R_2} + N_{21} - \frac{M_{12}}{R_1} \right)$$

$$M_6 = \frac{1}{2}\left(M_{12} + M_{21} \right)$$

$$\underbrace{\begin{bmatrix} N' \\ M' \end{bmatrix}}_{6\text{x}1} = \int \begin{bmatrix} \Delta_2' \\ z\Delta_2'' \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} dz$$

$$\Delta_2' = \begin{bmatrix} \varphi_2 & 0 & 0 \\ 0 & \varphi_1 & 0 \\ 0 & 0 & \dfrac{\varphi_2 + \varphi_1}{2} - \dfrac{1}{2}\left(\dfrac{\varphi_1}{R_1} + \dfrac{\varphi_2}{R_2} \right) z \end{bmatrix}$$

$$\Delta_2'' = \begin{bmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_1 & 0 \\ 0 & 0 & \dfrac{\varphi_2 + \varphi_1}{2} \end{bmatrix}$$

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Equivalent Electrically induced stress resultants

$$\underbrace{\begin{bmatrix} N^E \\ M^E \end{bmatrix}}_{6\text{x}1} = \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} \begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix} dz$$

$$\begin{bmatrix} N^{E\prime} \\ M^{E\prime} \end{bmatrix} = \int \begin{bmatrix} \Delta_2' \\ z\Delta_2'' \end{bmatrix} \begin{bmatrix} T_1^E \\ T_2^E \\ T_6^E \end{bmatrix} dz$$

Plugging in expression for stresses

$$\underbrace{\begin{bmatrix} N \\ M \end{bmatrix}}_{8\text{x}1} = \left\{ \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} [c] [\Delta_1 \quad z\Delta_1] dz \right\}_{8\text{x}8} \begin{bmatrix} S^0 \\ \kappa \end{bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} N' \\ M' \end{bmatrix}}_{6\text{x}1} = \left\{ \int \begin{bmatrix} \Delta_2' \\ z\Delta_2'' \end{bmatrix} [c] [\Delta_1' \quad z\Delta_1''] dz \right\}_{6\text{x}6} \begin{bmatrix} S^{0\prime} \\ \kappa^{0\prime} \end{bmatrix} - \begin{bmatrix} N^{E\prime} \\ M^{E\prime} \end{bmatrix}$$

Shells

Shells (continued)

$$\begin{bmatrix} N \\ M \end{bmatrix} = \left\{ \int \begin{bmatrix} \Delta_2 \\ z\Delta_2 \end{bmatrix} [c] [\Delta_1 \quad z\Delta_1] dz \right\} \begin{bmatrix} S^0 \\ \kappa \end{bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

Section properties 8x8 8x1

We can write

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} S^0 \\ \kappa \end{bmatrix} - \begin{bmatrix} N^E \\ M^E \end{bmatrix}$$

$$A = \int \Delta_2 c \Delta_1 = A^T$$

$$B = \int z\Delta_2 c \Delta_1$$

$$D = \int z^2 \Delta_2 c \Delta_1$$

$$\Delta_2 = \varphi_1 \varphi_2 \Delta_1^T$$

$$A = \int \Delta_1^T c \Delta_1 \varphi_1 \varphi_2 dz$$

$$B = \int z\Delta_1^T c \Delta_1 \varphi_1 \varphi_2 dz$$

$$D = \int z^2 \quad \cdots \quad dz$$

Shells

- Total Potential Energy

$$\delta U = \int_{\alpha_1}\int_{\alpha_2}\left\{\begin{bmatrix} N & M \end{bmatrix}\begin{bmatrix} \delta S^0 \\ \delta\kappa \end{bmatrix}\right\} A_1 A_2 d\alpha_1 d\alpha_2$$

$$\delta U = \int_{\alpha_1}\int_{\alpha_2}\left\{\begin{bmatrix} S^0 & \kappa \end{bmatrix}\begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{bmatrix} \delta S^0 \\ \delta\kappa \end{bmatrix} - \begin{bmatrix} N^E & M^E \end{bmatrix}\begin{bmatrix} \delta S^0 \\ \delta\kappa \end{bmatrix}\right\} A_1 A_2 d\alpha_1 d\alpha_2$$

- Kinetic Energy

$$\delta T = \int_V \begin{bmatrix} \dot{u} & \dot{v} & \dot{w} \end{bmatrix} \rho \begin{bmatrix} \delta\dot{u} \\ \delta\dot{v} \\ \delta\dot{w} \end{bmatrix} dV = \int_V \begin{bmatrix} \dot{u}^0 & \dot{v}^0 & \dot{w}^0 \end{bmatrix} \rho \begin{bmatrix} \delta\dot{u}^0 \\ \delta\dot{v}^0 \\ \delta\dot{w}^0 \end{bmatrix} dV$$

$$\int_V \quad = \int\int_{\alpha_1}\int_{\alpha_2} (\qquad\qquad) A_1 A_2 \varphi_1 \varphi_2 d\alpha_1 d\alpha_2 dz$$

If assume $\varphi_1 \varphi_2 \simeq 1$

$$\delta T = \int_{\alpha_1}\int_{\alpha_2} \rho h \begin{bmatrix} \dot{u}^0 & \dot{v}^0 & \dot{w}^0 \end{bmatrix} \begin{bmatrix} \delta\dot{u}^0 \\ \delta\dot{v}^0 \\ \delta\dot{w}^0 \end{bmatrix} A_1 A_2 d\alpha_1 d\alpha_2$$

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If assume $\varphi_1\varphi_2 \neq 1$

Let $h = h' = \int_{-h/2}^{h/2} \rho / \rho_0 \varphi_1 \varphi_2 dz$

- Work term

  equivalent forces/area in α_1, α_2, z direction called F_1, F_2, F_3

  then

$$\delta W = \int_{\alpha_1}\int_{\alpha_1} \left(F_1 \delta u^0 + F_2 \delta v^0 + F_3 \delta w^0 \right) A_1 A_2 d\alpha_1 d\alpha_2$$

- Approximate solution by Rayleigh-Ritz

$$\begin{bmatrix} S^0 \\ \kappa \end{bmatrix} = \underbrace{[\Omega][\theta]}_{\Lambda_u} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \end{bmatrix}$$

Shells

Plugging into Energy,

$$\delta U = \int_{\alpha_1}\int_{\alpha_2}\left\{\begin{bmatrix} u^0 & v^0 & w^0 \end{bmatrix}[\theta]^T[\Omega]^T\begin{bmatrix} A & B \\ B & D \end{bmatrix}[\Omega][\theta]\begin{bmatrix} \delta u^0 \\ \delta v^0 \\ \delta w^0 \end{bmatrix}\right.$$

$$\alpha_1,\ \alpha_2,\ z$$

$$\left.-\begin{bmatrix} N^E & M^E \end{bmatrix}\Omega\theta\begin{bmatrix} \delta u^0 \\ \delta v^0 \\ \delta w^0 \end{bmatrix}\right\}A_1A_2 d\alpha_1 d\alpha_2$$

Ritz solution

$$\begin{bmatrix} u^0 \\ v^0 \\ w^0 \end{bmatrix} = \psi_v(\alpha_1, \alpha_2)\vec{r} = \sum_{i=1}^{n}\psi_{u_i}(\alpha_1, \alpha_2)r_i$$

$$\delta u = r^T\underbrace{[\qquad]}_{\text{Stiffness matrix}}\delta r$$