Chapter 6: Basic Plasticity

Consistent tangent modulus

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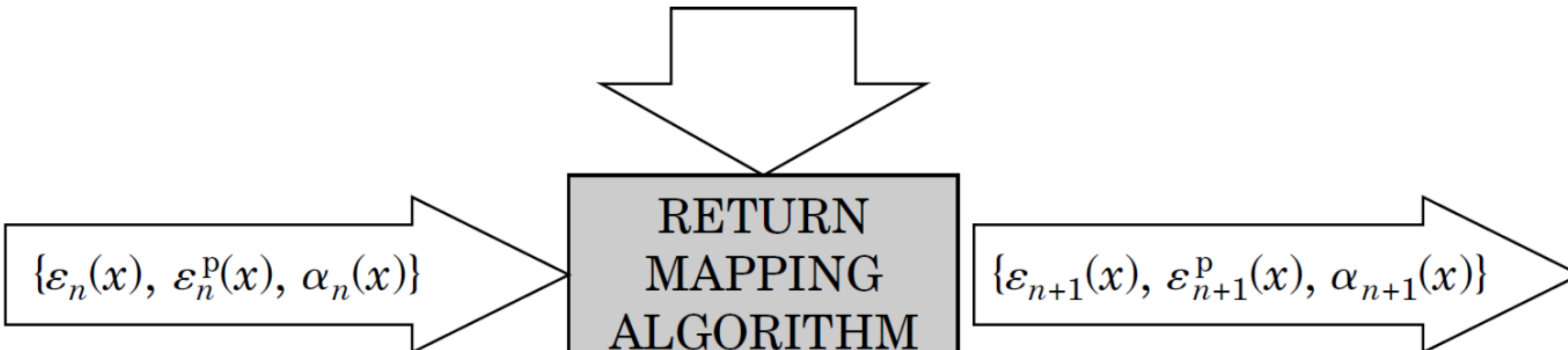
Integration algorithm – Rate independent plasticity

- Stress update or integration in a strain-driven process

Known (or prescribed at the beginning)

Strain Increment $\Delta\varepsilon_n(x)$



Known (or initial condition)

Unknown (or to be updated)

- **Incremental integration at** $[t_n, t_n + \Delta t]$

Integration algorithm – Rate independent plasticity

- Incremental form of rate-independent plasticity

$$\dot{x}(t) = f\left(x(t)\right) \ \& \ x(0) = x_0$$

Integration algorithm in discrete form

$$x_{n+1} = x_n + \Delta t \, f\left(x_{n+\vartheta}\right) \qquad x_{n+1} \Rightarrow x \text{ at } t_{n+1} = t_n + \Delta t$$

$$x_{n+\vartheta} = \vartheta x_{n+1} + \left(1-\vartheta\right) x_n \quad \vartheta \in \left[0,1\right]$$

$\vartheta = 0$ – forward (explicit) Euler

$\vartheta = 1/2$ – midpoint rule

$\vartheta = 1$ – backward (implicit) Euler

Integration algorithm – Rate independent plasticity

- Incremental elasto-plastic initial value problem

$$\varepsilon_{n+1}^{p} = \varepsilon_{n}^{p} + \Delta\gamma\, sign\left(\sigma_{n+1}\right)$$ **Backward Euler**

$$\alpha_{n+1} = \alpha_{n} + \Delta\gamma$$ **with** $\Delta\gamma = \gamma_{n+1}\Delta t \geq 0$

$$\sigma_{n+1} = E\left(\varepsilon_{n+1} - \varepsilon_{n+1}^{p}\right)$$

$$\varepsilon_{n+1} = \varepsilon_{n} + \Delta\varepsilon_{n}$$

$$\left(\sigma_{n+1}, \alpha_{n+1}\right),\ \Delta\gamma$$

Constrained to →

$$f_{n+1} = \left|\sigma_{n+1}\right| - \left(\sigma_{Y} + K\alpha_{n+1}\right) \leq 0$$

$$\Delta\gamma \geq 0$$

$$\Delta\gamma f_{n+1} = 0$$

Discrete form of Kuhn-Tucker conditions

Integration algorithm – Rate independent plasticity

- Return-Mapping Algorithms – Isotropic hardening

- **Trial elastic state or freezing plastic flow**

$$\sigma_{n+1}^{trial} = E\left(\varepsilon_{n+1} - \varepsilon_n^p\right) = \sigma_n + E\Delta\varepsilon_n$$

$$\varepsilon_{n+1}^{p,trial} = \varepsilon_n^p$$

$$\alpha_{n+1}^{trial} = \alpha_n$$

$$f_{n+1}^{trial} = \left|\sigma_{n+1}^{trial}\right| - \left(\sigma_Y + K\alpha_n\right)$$

- **Loading conditions**

$$f_{n+1}^{trial} \leq 0 \Rightarrow \Delta\gamma = 0$$

- Trial state is admissible, and its state is the solution (elastic)

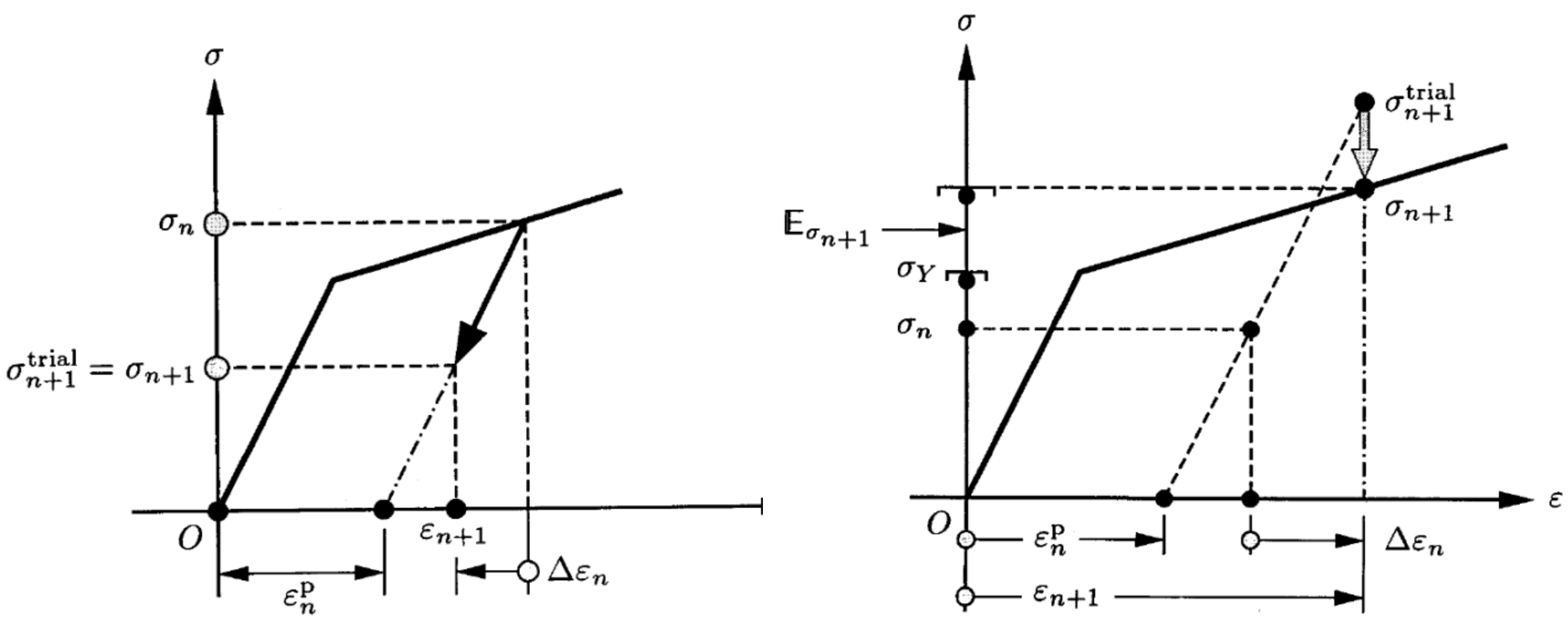
$$f_{n+1}^{trial} > 0$$

- Trial state violates constraint conditions and cannot be the solutions

Integration algorithm – Rate independent plasticity

- Return-Mapping Algorithms – Isotropic hardening

$$f_{n+1}^{trial} > 0 \quad \Rightarrow \quad \Delta\gamma > 0 \;\&\; \Delta\gamma f_{n+1} = 0 \Rightarrow f_{n+1} = 0$$



Integration algorithm – Rate independent plasticity

- Return-Mapping Algorithms – Isotropic hardening

$$f_{n+1}^{trial} > 0 \quad \Rightarrow \quad f\left(\sigma_{n+1}, \alpha_{n+1}\right) = 0 \ \& \ \Delta\gamma > 0$$

To determine the solution $\left\{\sigma_{n+1}, \alpha_{n+1}, \Delta\gamma, \varepsilon_{n+1}^{p}\right\}$

$$\sigma_{n+1} = E\left(\varepsilon_{n+1} - \varepsilon_{n+1}^{p}\right) = E\left(\varepsilon_{n+1} - \varepsilon_{n}^{p}\right) - E\left(\varepsilon_{n+1}^{p} - \varepsilon_{n}^{p}\right)$$

$$= \sigma_{n+1}^{trial} - E\Delta\gamma\, sign\left(\sigma_{n+1}\right) \quad \Rightarrow \quad \left|\sigma_{n+1}\right| sign\left(\sigma_{n+1}\right)$$

$$\varepsilon_{n+1}^{p} = \varepsilon_{n}^{p} + \Delta\gamma\, sign\left(\sigma_{n+1}\right) \qquad = \left|\sigma_{n+1}^{trial}\right| sign\left(\sigma_{n+1}^{trial}\right) - E\Delta\gamma\, sign\left(\sigma_{n+1}\right)$$

$$\alpha_{n+1} = \alpha_{n} + \Delta\gamma$$

$$f_{n+1} = \left|\sigma_{n+1}\right| - \left(\sigma_{Y} + K\alpha_{n+1}\right) = 0$$

$$\Downarrow \quad sign\left(\sigma_{n+1}\right) = sign\left(\sigma_{n+1}^{trial}\right)$$

$$\left|\sigma_{n+1}\right| + E\Delta\gamma = \left|\sigma_{n+1}^{trial}\right|$$

Integration algorithm – Rate independent plasticity

- Return-Mapping Algorithms – Isotropic hardening

$$\left|\sigma_{n+1}\right| + E\Delta\gamma = \left|\sigma_{n+1}^{trial}\right|$$

$$f_{n+1} = \left|\sigma_{n+1}^{trial}\right| - E\Delta\gamma - \left[\sigma_Y + K\alpha_n\right] - K\left[\alpha_{n+1} - \alpha_n\right]$$

$$= f_{n+1}^{trial} - \Delta\gamma\left(E + K\right)$$

$$\boxed{\Delta\gamma = \frac{f_{n+1}^{trial}}{E + K} > 0}$$

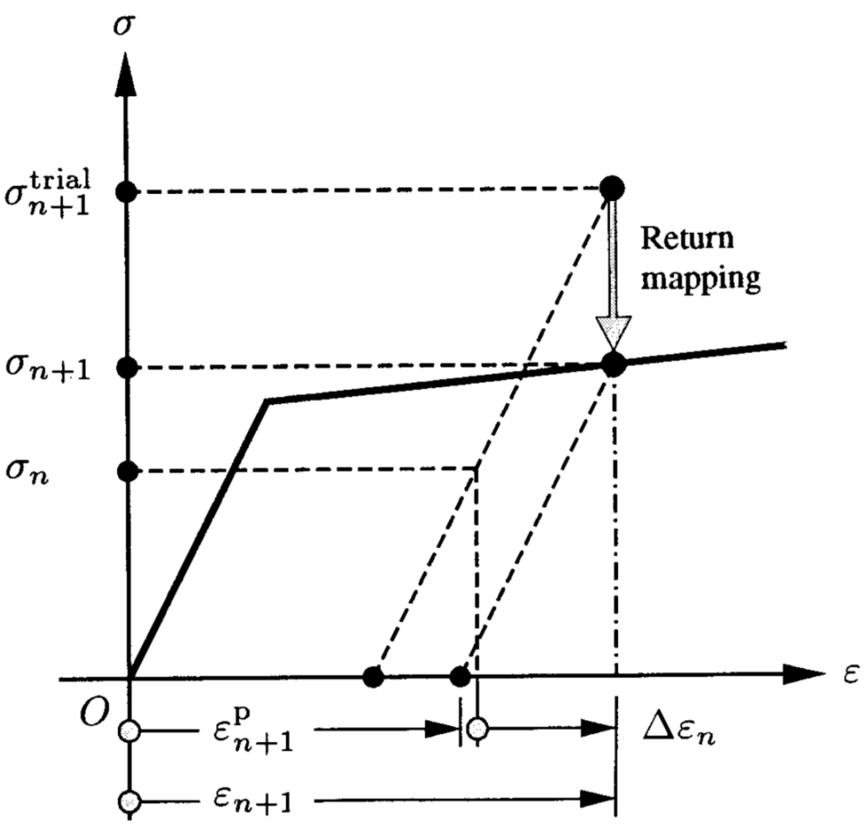
Update →

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E\, sign\left(\sigma_{n+1}^{trial}\right)$$

$$\varepsilon_{n+1}^{p} = \varepsilon_{n}^{p} + \Delta\gamma\, sign\left(\sigma_{n+1}^{trial}\right)$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

Integration algorithm – Rate independent plasticity



BOX 1.4. Return-Mapping Algorithm for 1-D, Rate-Independent Plasticity. Isotropic Hardening.

1. Database at $x \in \mathcal{B} : \{\varepsilon_n^p, \alpha_n\}$.

2. *Given* strain field at $x \in \mathcal{B} : \varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

3. Compute elastic trial stress and test for plastic loading

$$\sigma_{n+1}^{\text{trial}} := E(\varepsilon_{n+1} - \varepsilon_n^p)$$

$$f_{n+1}^{\text{trial}} := \left|\sigma_{n+1}^{\text{trial}}\right| - [\sigma_Y + K\alpha_n]$$

IF $f_{n+1}^{\text{trial}} \le 0$ THEN

Elastic step: set $(\bullet)_{n+1} = (\bullet)_{n+1}^{\text{trial}}$ *& EXIT*

ELSE

Plastic step: Proceed to step **4**.

ENDIF

4. Return mapping

$$\Delta\gamma := \frac{f_{n+1}^{\text{trial}}}{(E + K)} > 0$$

$$\sigma_{n+1} := \left[1 - \frac{\Delta\gamma E}{\left|\sigma_{n+1}^{\text{trial}}\right|}\right] \sigma_{n+1}^{\text{trial}}$$

$$\varepsilon_{n+1}^p := \varepsilon_n^p + \Delta\gamma \,\text{sign}\left(\sigma_{n+1}^{\text{trial}}\right)$$

$$\alpha_{n+1} := \alpha_n + \Delta\gamma$$

Finite element solutions of elasto-plastic IBVP: summary

Proof in the Simo !

Weak form of equilibrium equations

$$\text{Find } u(\cdot,t) \in S_t \text{ such that}:$$

$$\int_B \rho \frac{\partial v}{\partial t} \eta dx + G(\sigma,\eta) = 0 \text{ for all } \eta \in V \text{ and } t \in [0,T]$$

$$\text{where } G(\sigma,\eta) = \int_B \sigma \frac{\partial \eta}{\partial x} dx - \int_B \rho b \eta dx - \bar{\sigma}\eta\big|_{\partial_\sigma B}$$

S_t: admissible displacement solution space

$$S_t = \left\{ u(\cdot,t) : B \rightarrow R \,\middle|\, u(\cdot,t)\big|_{\partial_u B} = \bar{u}(\cdot,t) \right\}$$

V: admissible test function (or equivalently virtual displacement)

$$V = \left\{ \eta : B \rightarrow R \,\middle|\, \eta\big|_{\partial_u B} = 0 \right\}$$

Finite element solutions of elasto-plastic IBVP: summary

Our problem is to find the approximated solution u(x,t) in the finite element context!!

$$\textit{For quasi-static problem},$$
$$\textit{Find } u(\cdot,t)\in S_t \textit{ such that}:$$
$$G(\sigma,\eta)=0 \textit{ for all } \eta\in V$$

Spatial discretization – FE approximation

$B=[0,L]$ is discretized into $B_e=[x_e,x_{e+1}]$, & $B=\bigcup_{e=1}^{n_{el}} B_e$

$h_e=x_{e+1}-x_e$ "Mesh size"

Finite element solutions of elasto-plastic IBVP: summary

Our problem is to find the approximated solution u(x,t) in the finite element context!!

Local approximation of a test function is (linear) interpolation as

$$w_e^h = \sum_{a=1}^{2} N_e^a(x)\, w_e^a$$

$N_e^a(x)$ are the linear shape functions $\quad w_e = \left[w_e^1, w_e^2 \right]^T$ is the vector with nodal values of local element

$$\begin{cases} N_e^1(x) = \dfrac{x_{e+1} - x}{h_e} \\ N_e^2(x) = \dfrac{x - x_e}{h_e} \end{cases}$$

Computation of $G\left(\sigma^h, w^h\right)$ in element-by-element fashion

$$G\left(\sigma^h, w^h\right) = \sum_{e=1}^{n_{el}} G_e\left(\sigma^h, w^h\right)$$

Finite element solutions of elasto-plastic IBVP: summary

Our problem is to find the approximated solution u(x,t) in the finite element context!!

Computation of $G\left(\sigma^h, w^h\right)$ in element-by-element fashion

$$G\left(\sigma^h, w^h\right) = \sum_{e=1}^{n_{el}} G_e\left(\sigma^h, w^h\right)$$

$$\frac{\partial}{\partial x} w_e^h = \left[\frac{\partial}{\partial x} N_e^1 \quad \frac{\partial}{\partial x} N_e^2\right] \mathbf{w}_e = \mathbf{B}_e \mathbf{w}_e \qquad \mathbf{B}_e = \left[-\frac{1}{h_e} \quad \frac{1}{h_e}\right]$$

$$G_e\left(\sigma^h, w^h\right) = \mathbf{w}_e^T \left[f_e^{\text{int}}\left(\sigma^h\right) - f_e^{ext}(t)\right]$$

where

$$f_e^{\text{int}}\left(\sigma^h\right) = \int_{B_e} \mathbf{B}_e^T \sigma^h(x,t)\, dx$$

Internal force vector (elemental)

$$f_e^{ext}(t) = \int_{B_e} \begin{Bmatrix} N_e^1 \\ N_e^2 \end{Bmatrix} \rho b(x,t)\, dx + \left[\bar{\sigma}(t) \begin{Bmatrix} N_e^1 \\ N_e^2 \end{Bmatrix}\right]\Bigg|_{\partial B_e \cap \partial_\sigma B}$$

External load vector (elemental)

Assembled →

Finite element solutions of elasto-plastic IBVP: summary

Our problem is to find the approximated solution u(x,t) in the finite element context!!

Assembled

$$G\left(\sigma^h, w^h\right) = \mathbf{w}^T\left[\mathbf{F}^{\text{int}}\left(\sigma^h\right) - \mathbf{F}^{ext}(t)\right]$$

where

$$\mathbf{F}^{\text{int}}\left(\sigma^h\right) = \mathop{\mathbf{A}}_{e=1}^{n_{el}} f_e^{\text{int}}\left(\sigma^h\right) \;\; \& \;\; \mathbf{F}^{ext}(t) = \mathop{\mathbf{A}}_{e=1}^{n_{el}} f_e^{ext}(t)$$

How to calculate the internal force vector – need numerical integration

$$f_e^{\text{int}}\left(\sigma^h\right) = \sum_{l=1}^{n_{\text{int}}} \mathbf{B}_e^T\, \sigma^h(x,t)\Big|_{x=x_e^l} \omega^l h_e$$

Gauss quadrature !

x_e^l : quadrature point (or integration point)

ω^l : weight

n_{int} : number of quadrature points (or # of integration points)

Finite element solutions of elasto-plastic IBVP: summary

Nonlinear solution procedure – iterative scheme

For a time interval $\left[0,T\right]=\bigcup_{n=1}^{M}\left[t_n,t_{n+1}\right]$

At $t=t_n$ an equilibrium is satisfied $\mathbf{F}^{\text{int}}\left(\sigma_n\right)-\mathbf{F}_n^{ext}=\mathbf{0}$

With an incremental loading at $t=t_{n+1}$ $b_{n+1}=b_n+\Delta b_n$ & $\bar{\sigma}_{n+1}=\bar{\sigma}_n+\Delta\bar{\sigma}_n$

Then, the target problem is to find

$\Delta u_{n+1}^h\in V^h,\ u_{n+1}^h=u_n^h+\Delta u_{n+1}^h,\ \left\{\varepsilon_{n+1}^P,\alpha_{n+1},q_{n+1}\right\}$, and σ_{n+1}^h at $\mathrm{x}_e^l\in B_e$

Such that

$\mathbf{F}^{\text{int}}\left(\sigma_{n+1}^h\right)-\mathbf{F}_{n+1}^{ext}=\mathbf{0}$ & constitutive equation

Finite element solutions of elasto-plastic IBVP: summary

Nonlinear solution procedure – iterative scheme

Step 1: incremental nodal displacement at k-th iteration (start from initial guess)

$$\mathbf{d}_{n+1}^{(k)} = \mathbf{d}_n + \Delta\mathbf{d}_{n+1}^{(k)} \quad \text{or} \quad \boldsymbol{\varepsilon}_{n+1}^{(k)}\Big|_{B_e} = \mathbf{B}_e\,\mathbf{d}_e\Big|_{n+1}^{(k)}$$

Step 2: Compute the stress at the integration points (Stress update algorithm)

$$\boldsymbol{\sigma}_{n+1}^{(k)}$$

Step 3: Evaluate internal force vector and assemble

$$\mathbf{f}_e^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{h}\right)$$

Step 4: Check convergence: if equilibrated $\mathbf{F}^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right) - \mathbf{F}_{n+1}^{ext} = \mathbf{0}$ Then, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{n+1}^{(k)}$

Step 5: Otherwise, find new $\Delta\mathbf{d}_{n+1}^{(k+1)}$ and go to Step 1

Finite element solutions of elasto-plastic IBVP: summary

Nonlinear solution procedure – iterative scheme

How to determine $\Delta\mathbf{d}_{n+1}^{(k+1)}$?

$$\frac{\partial \mathbf{F}^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right)}{\partial \mathbf{d}_{n+1}^{(k)}}\Delta\mathbf{d}_{n+1}^{(k+1)} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \frac{\partial f_e^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right)}{\partial \left.\mathbf{d}_e\right|_{n+1}^{(k)}} \left.\Delta\mathbf{d}_e\right|_{n+1}^{(k+1)}$$

$$= \mathop{\mathbf{A}}_{e=1}^{n_{el}} \int_{B_e} \mathbf{B}_e^T \left[\frac{\partial \boldsymbol{\sigma}_{n+1}^{(k)}}{\partial \boldsymbol{\varepsilon}_{n+1}^{(k)}}\right] \frac{\partial \boldsymbol{\varepsilon}_{n+1}^{(k)}}{\partial \left.\mathbf{d}_e\right|_{n+1}^{(k)}} \left.\Delta\mathbf{d}_e\right|_{n+1}^{(k+1)} dx$$

$$= \mathop{\mathbf{A}}_{e=1}^{n_{el}} \left[\int_{B_e} \mathbf{B}_e^T \left[\frac{\partial \boldsymbol{\sigma}_{n+1}^{(k)}}{\partial \boldsymbol{\varepsilon}_{n+1}^{(k)}}\right] \mathbf{B}_e dx\right] \left.\Delta\mathbf{d}_e\right|_{n+1}^{(k+1)}$$

$$\left.\mathbf{k}_e\right|_{n+1}^{(k)} = \int_{B_e} \mathbf{B}_e^T \left[\frac{\partial \boldsymbol{\sigma}_{n+1}^{(k)}}{\partial \boldsymbol{\varepsilon}_{n+1}^{(k)}}\right] \mathbf{B}_e dx$$

$$\frac{\partial \mathbf{F}^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right)}{\partial \mathbf{d}_{n+1}^{(k)}}\Delta\mathbf{d}_{n+1}^{(k+1)} = \mathbf{K}_{n+1}^{(k)}\Delta\mathbf{d}_{n+1}^{(k+1)},$$

$$\textit{with}\;\; \mathbf{K}_{n+1}^{(k)} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \left.\mathbf{k}_e\right|_{n+1}^{(k)}$$

Linearization

$$\left[\mathbf{F}^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right) - \mathbf{F}_{n+1}^{ext}\right] + \frac{\partial \mathbf{F}^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right)}{\partial \mathbf{d}_{n+1}^{(k)}}\Delta\mathbf{d}_{n+1}^{(k+1)} = \mathbf{0}$$

$$\Delta\mathbf{d}_{n+1}^{(k+1)} = -\left[\mathbf{K}_{n+1}^{(k)}\right]^{-1}\left[\mathbf{F}^{\text{int}}\left(\boldsymbol{\sigma}_{n+1}^{(k)}\right) - \mathbf{F}_{n+1}^{ext}\right]$$

"New guess"

$$\mathbf{C}_{n+1}^{(k)} = \frac{\partial \boldsymbol{\sigma}_{n+1}^{(k)}}{\partial \boldsymbol{\varepsilon}_{n+1}^{(k)}}$$

"Algorithmic tangent modulus"