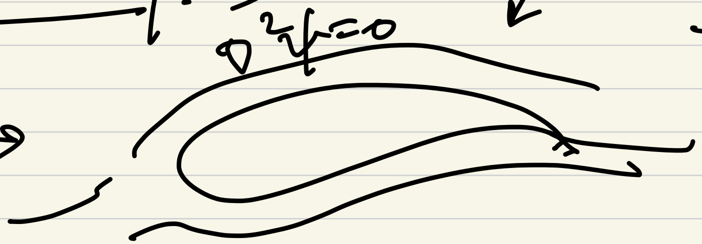
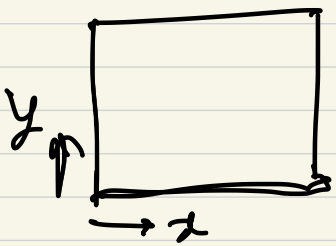
Panel method

HW3 TA 목 강의 upload

8.9 Numerical analysis

$$\bigstar \begin{cases} \nabla^2\psi = 0 \\ \nabla^2\phi = 0 \end{cases}$$





separation of variable

$\psi = X(x)Y(y)$

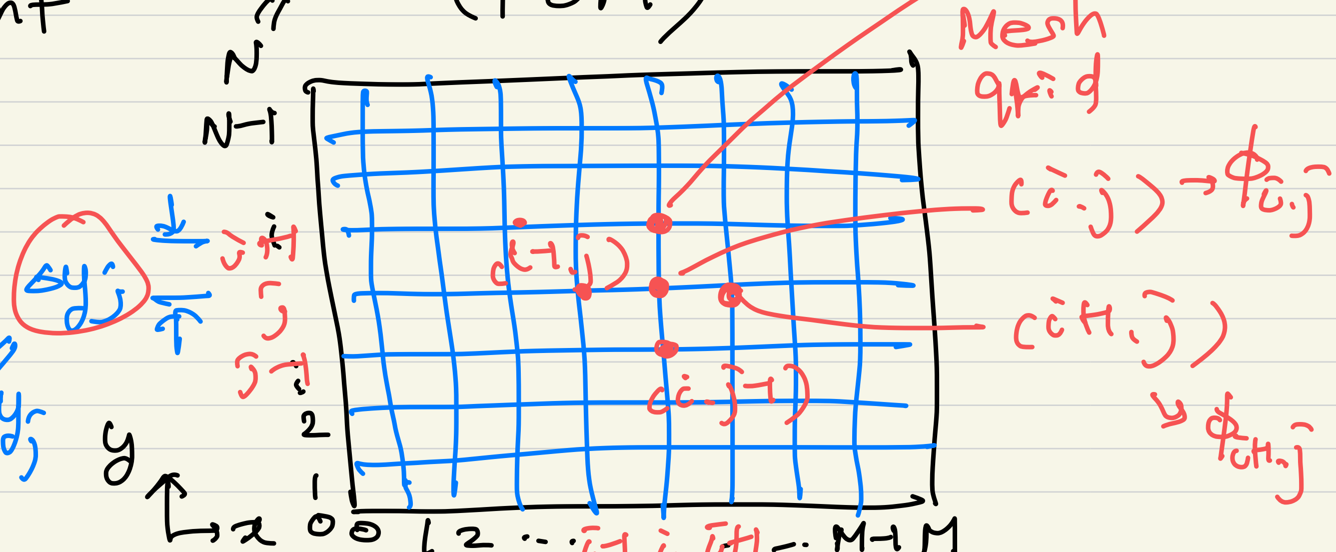
$\nrightarrow$ $\nabla^2\phi=0$ numerical method

① finite difference method (FDM) ← ☆ 유한차분법

② " volume " (FVM) ← " 체적 "

③ " element " (FEM) ← " 요소 "

$$\boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0}$$

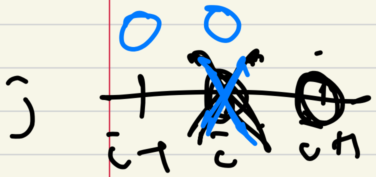


grid spacings Δx_{i-1}, $\Delta x_i = x_{i+1} - x_i$

FDM ← Taylor series expansion

① — $\psi_{i+1,j} = \psi_{i,j} + \Delta x_i \dfrac{\partial\psi}{\partial x}\Big|_{i,j} + \dfrac{1}{2}\Delta x_i^2 \dfrac{\partial^2\psi}{\partial x^2}\Big|_{i,j} + \dfrac{1}{6}\Delta x_i^3 \dfrac{\partial^3\psi}{\partial x^3}\Big|_{i,j} + \cdots$

$$\Rightarrow \boxed{\frac{\partial\psi}{\partial x}\Big|_{i,j} = \frac{\psi_{i+1,j} - \psi_{i,j}}{\Delta x_i}} \underbrace{- \frac{1}{2}\Delta x_i \frac{\partial^2\psi}{\partial x^2}\Big|_{i,j}}_{\text{leading truncation error}} - \frac{1}{6}\Delta x_i^2 \frac{\partial^3\psi}{\partial x^3}\Big|_{i,j} + \cdots$$



forward FDM

leading truncation error $O(\Delta x)$

first-order accuracy

② — $\psi_{i-1,j} = \psi_{i,j} - \Delta x_{i-1} \dfrac{\partial\psi}{\partial x}\Big|_{i,j} + \dfrac{1}{2}\Delta x_{i-1}^2 \dfrac{\partial^2\psi}{\partial x^2}\Big|_{i,j} - \dfrac{1}{6}\Delta x_{i-1}^3 \dfrac{\partial^3\psi}{\partial x^3}\Big|_{i,j} + \cdots$

$$\Rightarrow \boxed{\frac{\partial\psi}{\partial x}\Big|_{i,j} = \frac{\psi_{i,j} - \psi_{i-1,j}}{\Delta x_i}} \underbrace{+ \frac{1}{2}\Delta x_i \frac{\partial^2\psi}{\partial x^2}\Big|_{i,j}}_{} - \frac{1}{6}\Delta x_i^3 \frac{\partial^3\psi}{\partial x^3}\Big|_{i,j} + \cdots$$

backward FDM

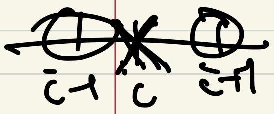
$O(\Delta x)$

Δx

↑

($\Delta x_i = \Delta x_{i-1}$)

① − ② : $\psi_{i+1,j} - \psi_{i-1,j} = 2\Delta x \dfrac{\partial\psi}{\partial x}\Big|_{i,j} + \dfrac{1}{3}\Delta x^3 \dfrac{\partial^3\psi}{\partial x^3}\Big|_{i,j} + \cdots$



$$\Rightarrow \boxed{\left.\frac{\partial\psi}{\partial x}\right|_{i,j} = \frac{\psi_{i+1,j}-\psi_{i-1,j}}{2\Delta x}} \underbrace{-\frac{1}{6}\Delta x^{2}\left.\frac{\partial^3\psi}{\partial x^3}\right|_{i,j}} + \cdots$$

central difference method

leading trunc. error

$O(\Delta x^2)$ second-order accuracy

$$①+② : \boxed{\left.\frac{\partial^2\psi}{\partial x^2}\right|_{i,j} = \frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{\Delta x^2}} \underbrace{-\frac{1}{12}\Delta x^{2}\left.\frac{\partial^4\psi}{\partial x^4}\right|_{i,j}} + \cdots$$

central difference method

leading trunc. error

$O(\Delta x^2)$ second-order accuracy

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

차분 FDM 유한

central

$$\boxed{\frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{\Delta x^2}+\frac{\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1}}{\Delta y^2}=0}$$

($i=1,2,\cdots,M-1$ $\quad j=1,2,\cdots,N-1$

set of difference eqs. $(M-1)\times(N-1)$

$i=1,\ j=1$: $\dfrac{\psi_{2,1} - 2\psi_{1,1} + \psi_{0,1}}{\Delta x^2} + \dfrac{\psi_{1,2} - 2\psi_{1,1} + \psi_{1,0}}{\Delta y^2} = 0$

boundary terms.

$\vdots$ — — — -

$i=M-1,\ j=N-1$: . — — — —

$$
\begin{bmatrix}
-4 & 1 & 0 & \cdots & 0 & 1 & & 0 \\
1 & -4 & 1 & & & 1 & & \\
 & \ddots & \ddots & \ddots & & & \ddots & \\
1 & & & \ddots & \ddots & \ddots & & 1 \\
 & 1 & 0 & & \ddots & \ddots & \ddots & \\
0 & & \ddots & & & \ddots & \ddots & \\
 & & & 1 & & & 1 & -4
\end{bmatrix}
\begin{bmatrix}
\psi_{1,1} \\ \psi_{2,1} \\ \vdots \\ \psi_{M-1,1} \\ \vdots \\ \psi_{i,j} \\ \vdots \\ \psi_{M-1,N-1}
\end{bmatrix}
=
\begin{bmatrix}
- \\ 0 \\ 0 \\ 0 \\ - \\ \vdots \\ - \\ \vdots \\ -
\end{bmatrix}
$$

sparse matrix

$(M-1)(N-1) \times (M-1)(N-1)$

$\boxed{A\psi = b} \rightarrow \psi = A^{-1}b$ X

expensive

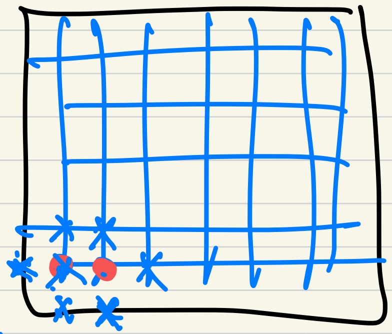
→ iterative method (기계공학수치해석)

$$\frac{\psi^{k}_{i+1,j} - 2\psi^{k+1}_{i,j} + \psi^{k}_{i-1,j}}{\Delta x^2} + \frac{\psi^{k}_{i,j+1} - 2\psi^{k+1}_{i,j} + \psi^{k}_{i,j-1}}{\Delta y^2} = 0$$

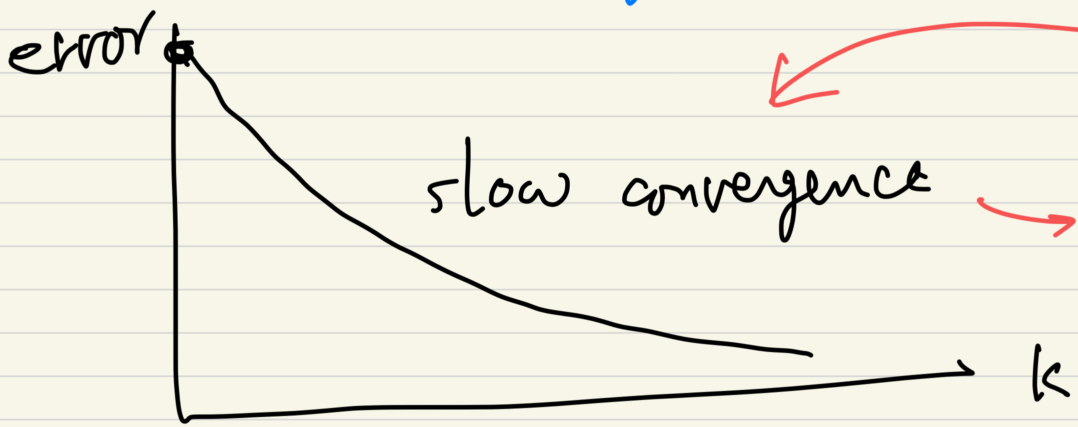
→ Jacobi iteration

k ← iteration index

$k \quad k+1 \quad \psi^{k+1}_{i-1,j} \qquad k \quad k+1 \quad k$



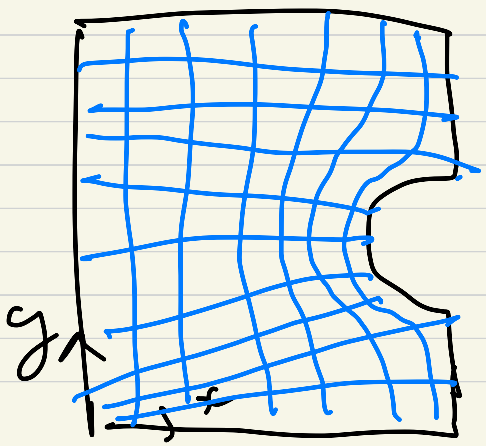
G-S iteration



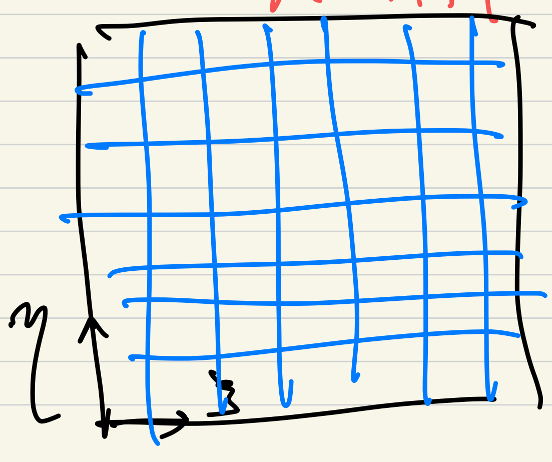
→ Speed-up

SOR

multigrid method



coord. transf. →



- Navier-Stokes eq. (incomp. flow), $\rho = \text{const}$

$$\left\{\begin{aligned} &\cancel{\rho\frac{\partial u}{\partial t}} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u \\ &\cancel{\rho\frac{\partial v}{\partial t}} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu\nabla^2 v \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{aligned}\right.$$

$\Rightarrow$ 전산유체역학

CFD

computational fluid dynamics

2-D steady, u, v, p

FDM

FDM central difference

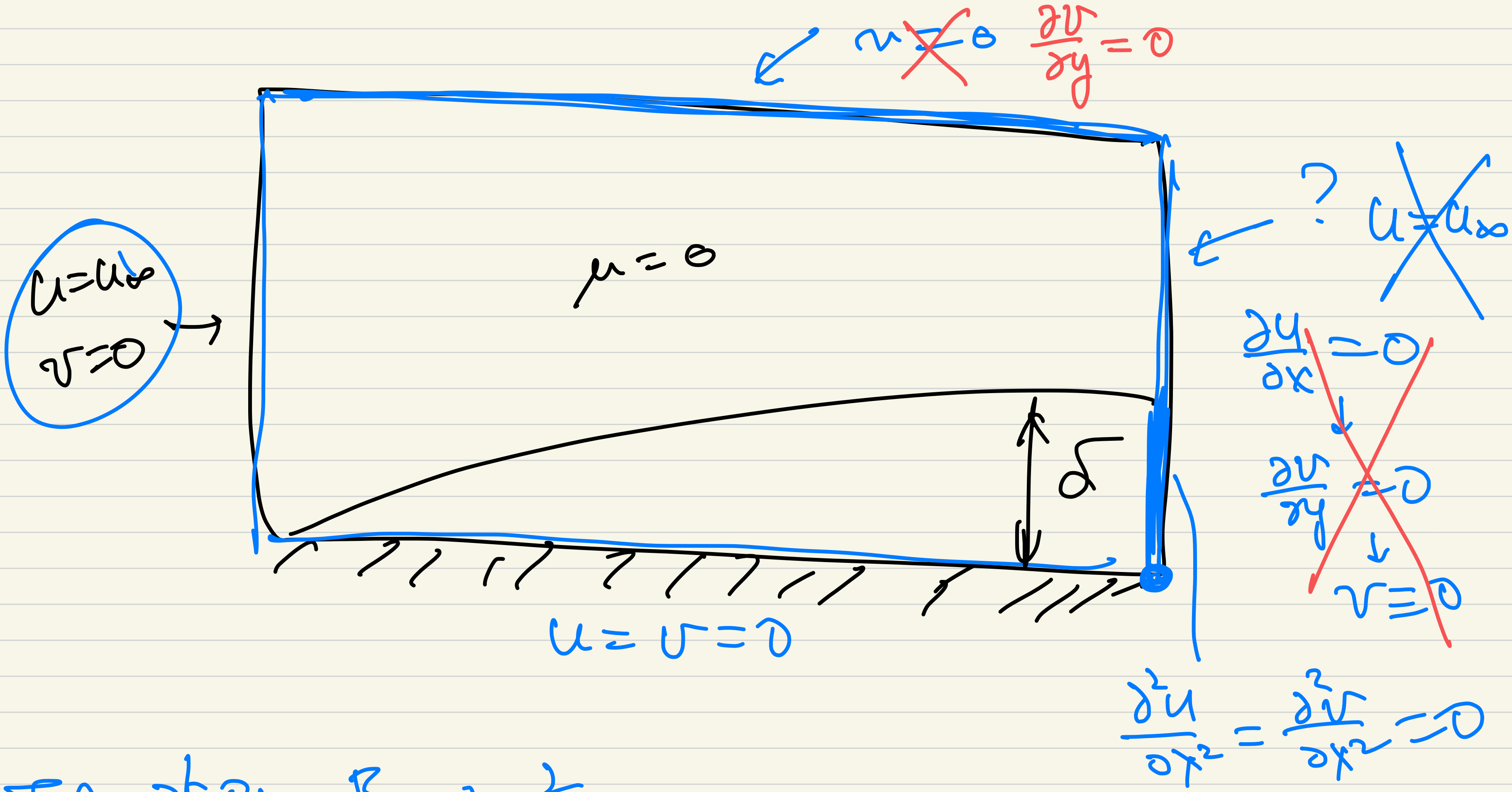
$$\rho u_{i,j}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \rho v_{i,j}\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} = -\frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} + \mu\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \mu\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

⋮

Complicated!.. $\Rightarrow$ develop numerical method to solve N-S eqs.!

Commercial package

* <u>boundary condition</u> ?



TA 강의 특 or 눈