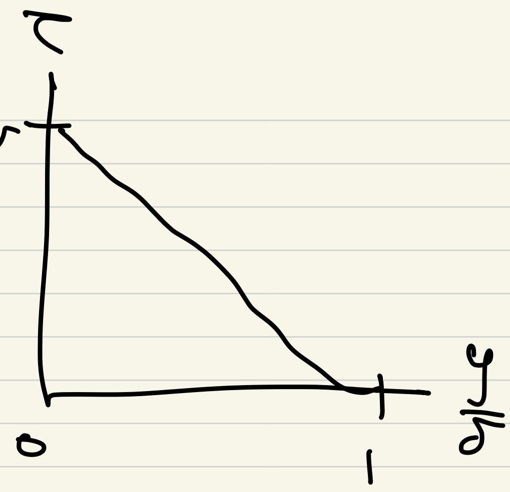
$$\tau = \mu \frac{d\langle U \rangle}{dy} - \rho \langle uv \rangle$$ total shear stress



$$\tau(y) = \tau_w \left(1 - \frac{y}{\delta}\right)$$

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$: wall-shear velocity
friction

$$\delta_\nu = \frac{\nu}{u_\tau}$$: viscous length scale

$$Re_\tau = \frac{u_\tau \delta}{\nu} = \frac{\delta}{\delta_\nu}$$: friction Reynolds number

$$\left.\begin{aligned} y^+ &= \frac{y u_\tau}{\nu} \\ u^+ &= \frac{\langle U \rangle}{u_\tau} \end{aligned}\right)$$

+ : wall units

viscous wall region : $y^+ < 50$ direct effect of molecular viscosity on the shear stress

outer layer : $y^+ > 50$ direct effect of viscosity is negligible.

viscous sublayer : $y^+ < 5$ Reynolds shear stress is negligible

As $Re \uparrow$, $\frac{\delta_\nu}{\delta} = \frac{\nu/u_\tau}{\delta} = \frac{\nu}{u_\tau \delta} = \frac{1}{Re_\tau} \downarrow$

fraction of viscous wall region decreases

- Mean velocity profiles

Channel flow is completely specified by

ρ, ν, δ and $-dp_w/dx$

or " " $u_\tau = \sqrt{\tau_w/\rho} = \left(-\frac{\delta}{\rho}\frac{dp_w}{dx}\right)^{\frac{1}{2}}$

$\Rightarrow \frac{\langle U \rangle}{u_\tau} = F_0\left(\frac{y}{\delta}, Re_\tau\right)$ from dimensional analysis

[universal non-dimensional ft.

$$\frac{d\langle U\rangle}{dy} \Big/ \frac{u_\tau}{y} = \Phi\left(\frac{y}{\delta}, \frac{y}{\delta_\nu}\right) \qquad Re_\tau = \frac{\delta}{\delta_\nu} \;//\; \frac{\delta}{\delta_\nu}$$

$$\Phi\left(\frac{y}{\delta}, Re_\tau\right) \rightarrow \Phi\left(\frac{y}{\delta}, \frac{y}{\delta_\nu}\right)$$

- Law of the wall

Prandtl (1925): at high Re #, close to the wall ($y/\delta \ll 1$), there is an inner layer in which $\langle U\rangle$ is determined by the viscous scales.

$$\rightarrow \frac{d\langle U\rangle}{dy} \Big/ \frac{u_\tau}{y} = \Phi_1\left(\frac{y}{\delta_\nu}\right) \quad \text{for } \frac{y}{\delta} \ll 1$$

$$\left(u^+ = \langle U\rangle / u_\tau \,,\; y^+ = y u_\tau / \nu\right)$$

universal ft.

$$\rightarrow \frac{du^+}{dy^+} = \frac{1}{y^+}\Phi_1(y^+) \xrightarrow[\text{integration}]{} \boxed{u^+ = f_w(y^+)} \; \boxed{\text{for } \frac{y}{\delta} \ll 1}$$

inner layer

* viscous sublayer $u^+ = f_w(y^+)$

@ wall, $(y=0)$, $\langle U\rangle = 0 \rightarrow 0 = f_w(0) \rightarrow \underline{f_w(0) = 0}$

$$u^+ = f_w(y^+) \rightarrow \langle U\rangle = u_\tau f_w(y^+)$$

$$\left.\frac{\partial\langle U\rangle}{\partial y}\right|_{y=0} = u_\tau f_w'(\overset{0}{y^+}) \cdot \frac{u_\tau}{\nu}$$

$$\underbrace{\rho\nu \left.\frac{\partial\langle U\rangle}{\partial y}\right|_{y=0}}_{\tau_w} = \underbrace{\rho u_\tau^2}_{\tau_w} f_w'(0) \Rightarrow \underline{f_w'(0) = 1}$$

$$\underset{u^+}{f_w(y^+)} = \underbrace{f_w(0)}_{0} + y^+ \underbrace{f_w'(0)}_{1} + \frac{1}{2} y^{+2} f_w''(0) + \cdots$$

$$\Rightarrow \boxed{u^+ = y^+} + O(y^{+2}) \quad \boxed{\text{for } y^+ < 5}$$

viscous sublayer

* Log law

At high Re, outer part of inner layer $\rightarrow$ large y^+

$y^+ = \frac{y u_\tau}{\nu} = 0.1\frac{u_\tau \delta}{\nu}$ i.e. $y^+ \simeq 0.1 \frac{\delta}{\delta_\nu} = 0.1\, Re_\tau \gg 1$

$\searrow \frac{y}{\delta} = 0.1$

Then, the viscosity has little effect.

$\rightarrow \Phi_1\left(\frac{y}{\delta_\nu}\right)$ does not depend on ν

$\rightarrow \Phi_1$ is const. $\Phi_1 = \frac{1}{\kappa}$ for $\frac{y}{\delta} \ll 1$ and $y^+ \gg 1$

$$\therefore \frac{du^+}{dy^+} = \frac{1}{\kappa y^+} \rightarrow \boxed{u^+ = \frac{1}{\kappa} \ln y^+ + B}$$

log law

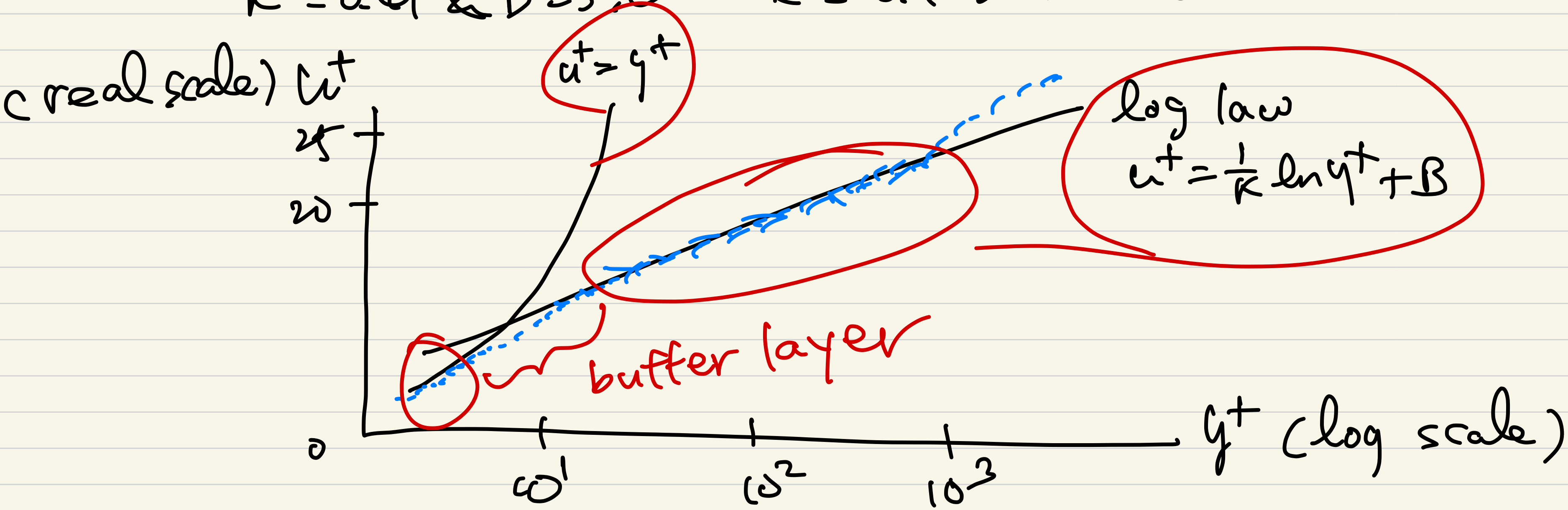
logarithmic law of the wall

von Karman (1930)

κ : von Karman constant

$\kappa = 0.41$, $B = 5.2$

$\kappa = 0.41$ & $B = 5.0$ $\quad$ $\kappa = 0.4$ & $B = 5.5$



y/δ: 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 0.3, 1

outer layer: direct effect of viscosity on $\langle U \rangle$ is negligible

overlap layer (region): between inner ($y/\delta < 0.1$) and outer ($y^+ > 50$) layers

log-law region

$$u^+ = \frac{1}{\kappa} \ln y^+ + B$$

$\langle U \rangle$ determined by u_τ & y^+, indep. of U_0 & δ

Inner layer

viscous wall region

significant contribution of viscosity on shear stress

buffer layer: region between viscous sublayer ($y^+ < 5$) and log-law region ($y^+ > 30$)

viscous sublayer

$$y^+ = \frac{y u_\tau}{\nu} = \frac{y}{\delta} \cdot \frac{\delta u_\tau}{\nu} = \frac{y}{\delta} \cdot Re_\tau$$

y^+: 1, 5, 10, 30, 50, 100, 1000, 10000

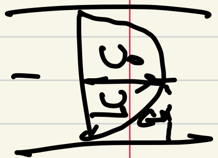
$u^+ = y^+$

Reynolds shear stress is negligible

* Velocity-defect law

in outer layer ($y^+ > 50$), $\Phi\left(\frac{y}{\delta}, \frac{y}{\delta_\nu}\right)$ is indep. of ν.

$$\rightarrow \quad \frac{d\langle U\rangle}{dy} \Big/ \frac{u_\tau}{y} = \Phi_0\left(\frac{y}{\delta}\right)$$



$\longrightarrow$ integrate from y to δ

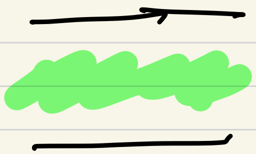
$$\boxed{\frac{U_0 - \langle U\rangle}{u_\tau} = F_D\left(\frac{y}{\delta}\right)}$$

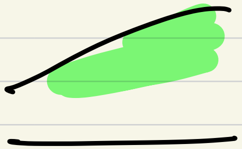
velocity-defect law

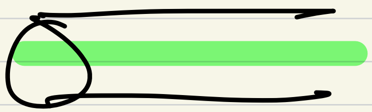
$y^+ > 50$

not universal ft.

is different for diff. flows







At sufficiently high Re #, there is an overlap region between inner layer ($y/\delta < 0.1$) and outer layer ($y^+ > 50$).

In this region, $\frac{d\langle U\rangle}{dy} \Big/ \frac{u_\tau}{y} = \Phi_1\left(\frac{y}{\delta_\nu}\right)$ inner layer

$= \Phi_0\left(\frac{y}{\delta}\right)$ outer layer

$= \text{const}$

$$\rightarrow \frac{d\langle U\rangle}{dy} \Big/ \frac{u_\tau}{y} = \frac{1}{\kappa} \quad \text{for } \delta_\nu \ll y \ll \delta$$

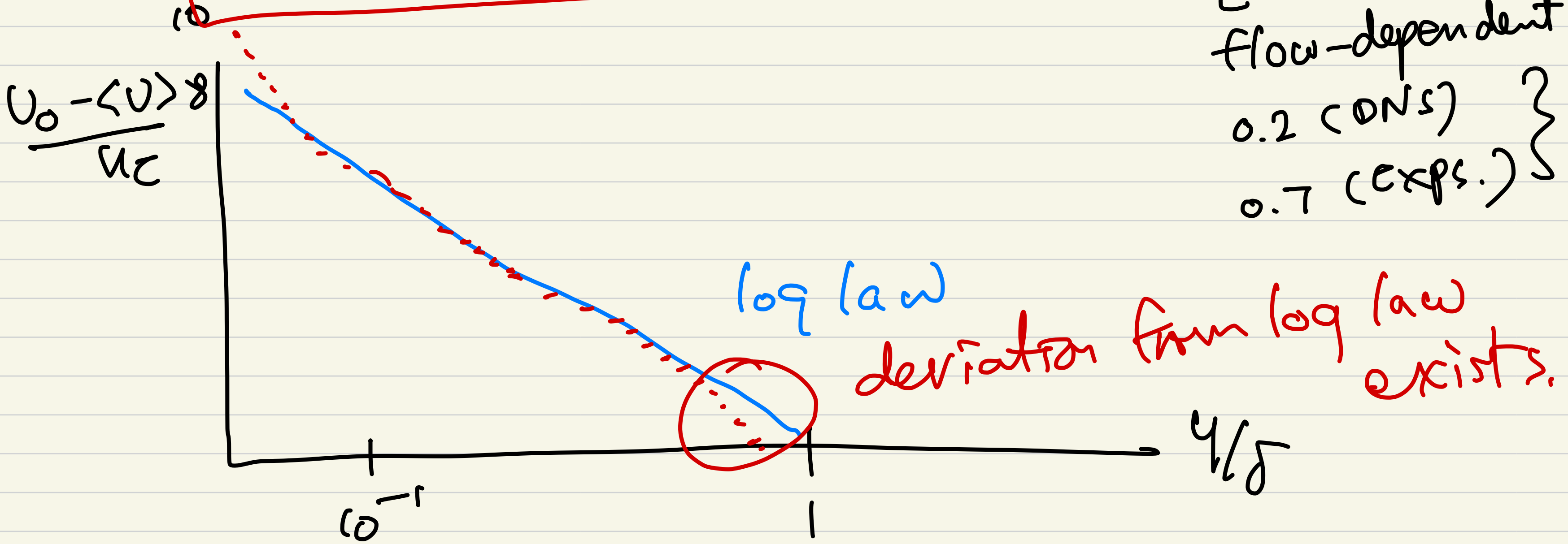
Millikan (1938)

↳ log-law again $\rightarrow u^+ = \frac{1}{\kappa} \ln y^+ + B$

Or, $\boxed{\frac{U_0 - \langle U\rangle}{u_\tau} = F_D\left(\frac{y}{\delta}\right) = -\frac{1}{\kappa} \ln \frac{y}{\delta} + B_1}$ for $\frac{y}{\delta} \ll 1$

B_1: flow-dependent const.

0.2 (DNS)
0.7 (exps.) } small



⑦ Friction law & Reynolds number

Relationships among U_0, $\bar{U}$ and u_τ

- bulk velocity $\bar{U}$ from log law

$$\frac{U_0 - \langle U \rangle}{u_\tau} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + B_1$$

@ $y = \delta$, $\langle U \rangle = U_0 \rightarrow B_1 = 0$

$$\frac{U_0 - \bar{U}}{u_\tau} = \frac{U_0 - \frac{1}{\delta}\int_0^\delta \langle U \rangle dy}{u_\tau} = \frac{1}{\delta}\int_0^\delta \frac{U_0 - \langle U \rangle}{u_\tau} dy$$

$$\simeq \frac{1}{\delta}\int_0^\delta \left(-\frac{1}{\kappa}\ln\frac{y}{\delta}\right) dy = \frac{1}{\kappa} \simeq 2.4$$

agrees well with exp. data.

.. inner layer $\frac{\langle U \rangle}{u_\tau} = \frac{1}{\kappa} \ln \frac{y}{\delta_\nu} + B$

outer layer $\frac{U_0 - \langle U \rangle}{u_\tau} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + B_1$

$$+ \quad \frac{U_0}{u_\tau} = \frac{1}{\kappa} \ln \frac{\delta}{\delta_\nu} + B + B_1$$

$$\frac{\delta}{\delta_\nu} = \frac{\delta}{\nu / u_\tau} = \frac{\delta u_\tau}{\nu} = \underbrace{\frac{\delta U_0}{\nu}}_{=Re_0} \cdot \frac{u_\tau}{U_0}$$

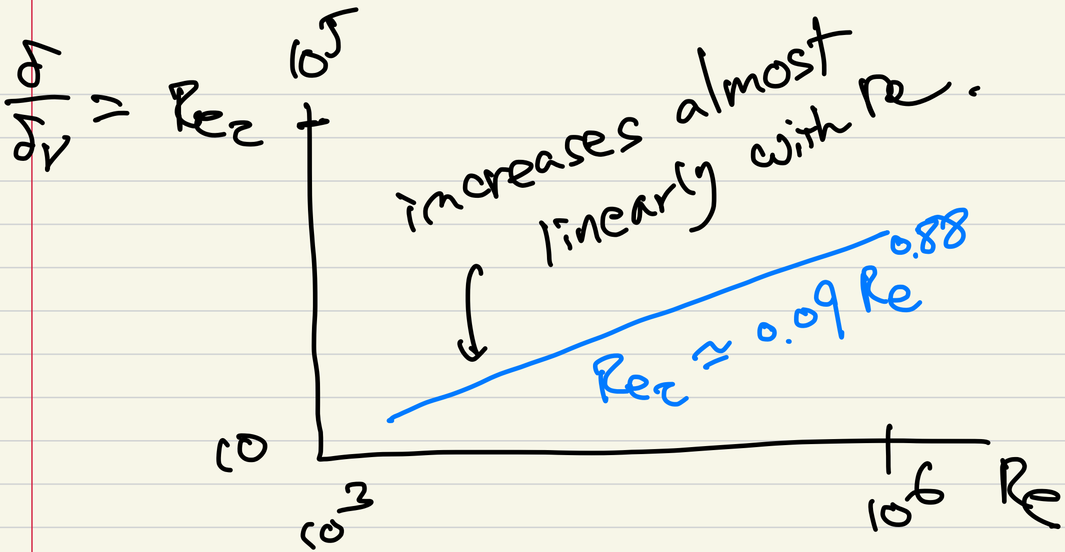
$$\boxed{\frac{U_0}{u_\tau} = \frac{1}{\kappa} \ln\left[Re_0 \left(\frac{U_0}{u_\tau}\right)^{-1} \right] + B + B_1}$$

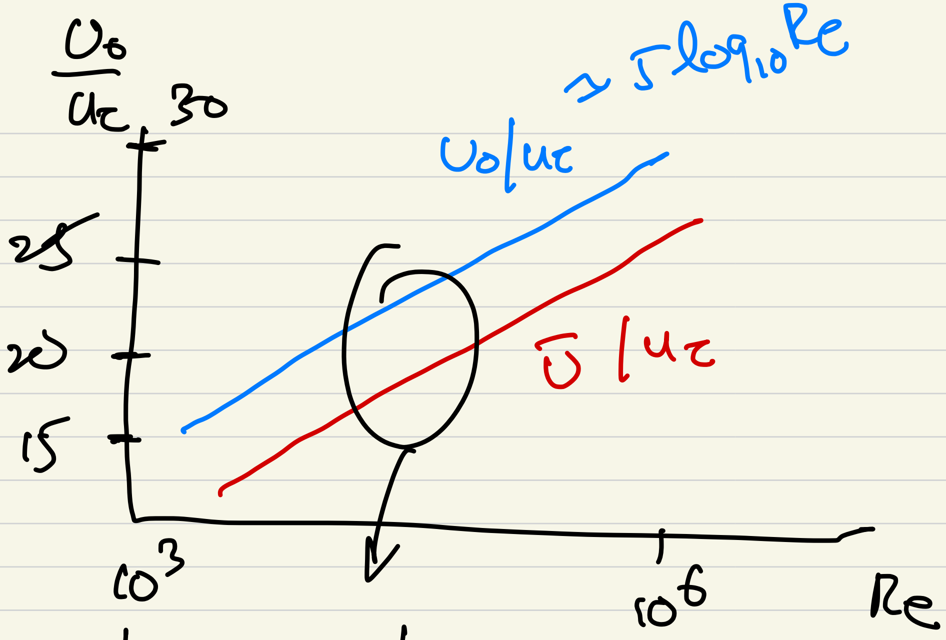
$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = 2\left(\frac{u_\tau}{U_0}\right)^2 \rightarrow \frac{U_0}{u_\tau} = \sqrt{\frac{C_f}{2}}$$

$$\boxed{\sqrt{\frac{2}{C_f}} = \frac{1}{\kappa} \ln\left(Re_0 \sqrt{\frac{C_f}{2}}\right) + B + B_1}$$

Given Re_0, C_f (or $\frac{U_0}{u_\tau}$) can be obtained.

good agreement at $Re > 3000$





velocity ratios increase very slowly with Re #.

- At high Re, viscous length scale is very small.

  e.g., $\delta = 2\,\text{cm}$, $Re = 10^5$

  $$Re_\tau = 0.09\, Re^{0.88} \simeq 2260 = \frac{\delta}{\delta_\nu} \rightarrow \delta_\nu = 0.88 \times 10^{-5}\,\text{m} \simeq 10^{-5}\,\text{m}$$

  $\underline{y^+ = 100} \rightarrow y = 1\,\text{mm}!$

significant fraction of the increase in mean velocity bet wall & centerline occurs in the viscous wall region.

δ 2cm

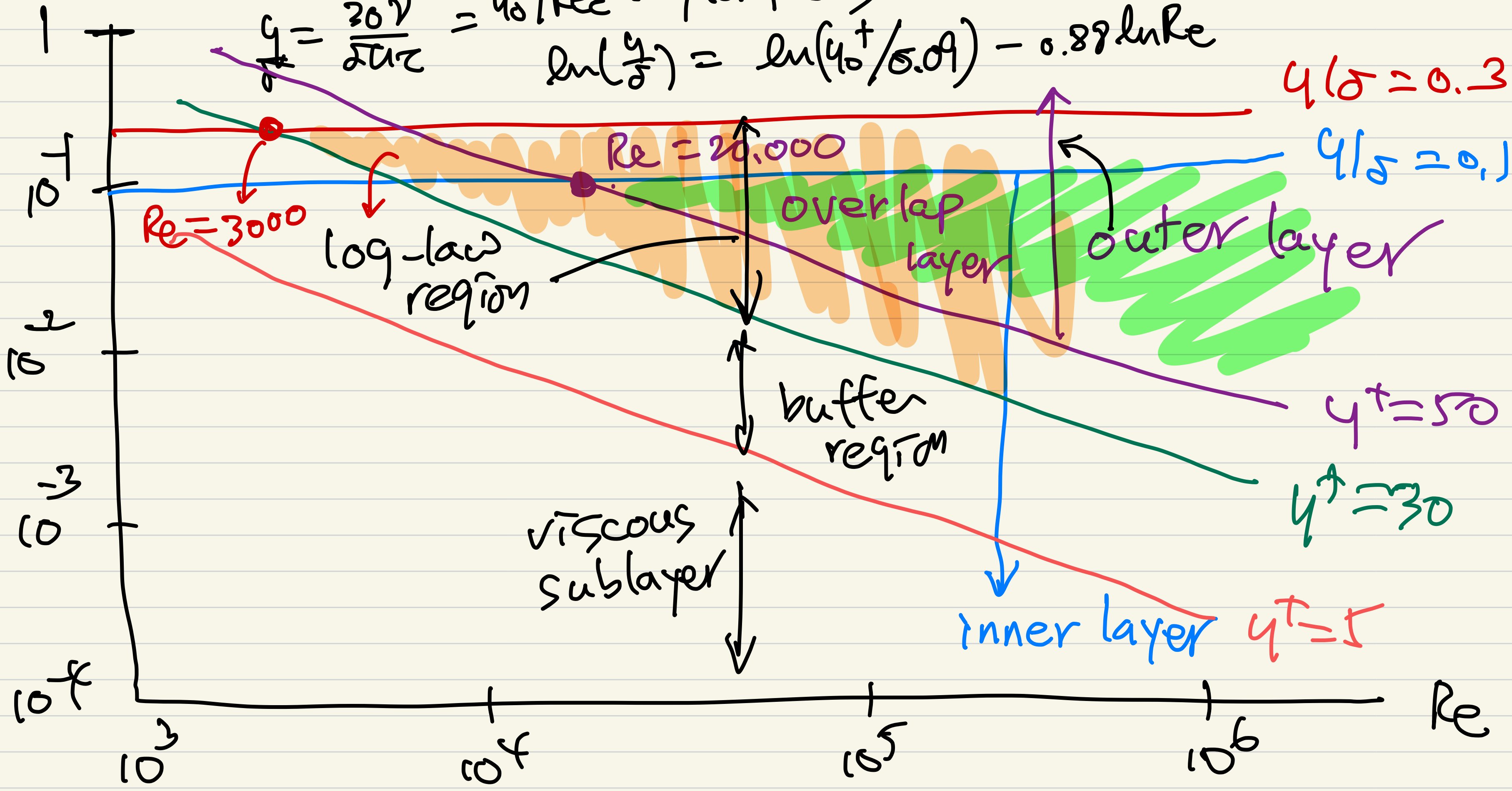
e.g., $\delta = 2\,\mathrm{cm}$, $Re = 10^5 \rightarrow y^+ = 10 \rightarrow y = 0.1\,\mathrm{mm}$

$\hookrightarrow \langle U \rangle_{y^+=10} \doteq 0.3\,U_0$!

$y^+ = \frac{y u_\tau}{\nu} = y_0^+$

$\frac{y}{\delta} = \frac{30\nu}{\delta u_\tau} = y_0^+ / Re_\tau = y_0^+ / (0.09\,Re^{0.88})$

$\ln\left(\frac{y}{\delta}\right) = \ln\left(y_0^+/0.09\right) - 0.88 \ln Re$



log-law persists beyond the region suggested by the overlap argument.