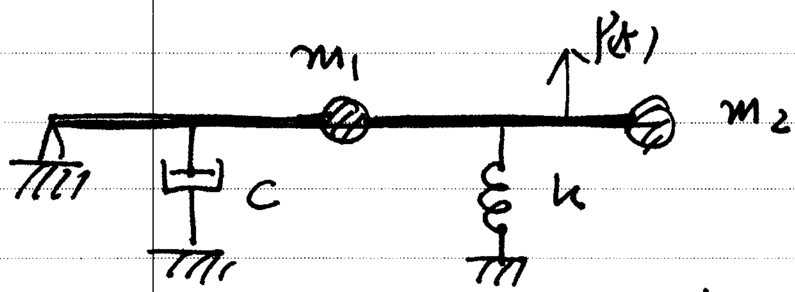
제8장 일반화된 단자유도 시스템

딱 맞는 결과가 아닌 충분히 정확한 결과

8.1 일반화된 단자유도 시스템

Case #1 : 강체 조립체

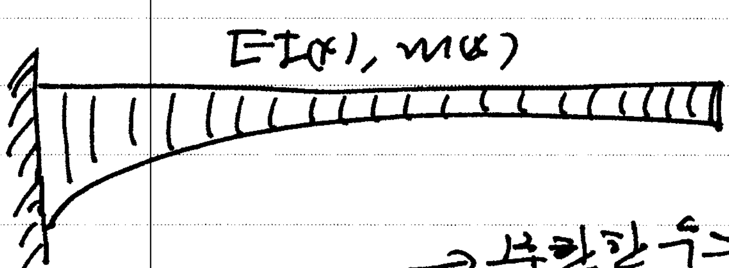


일반화된 변위

변위 $u(x,t) = \psi(x) Z(t)$

형상함수

Case #2 : 분포질량 시스템, 연속체 모델



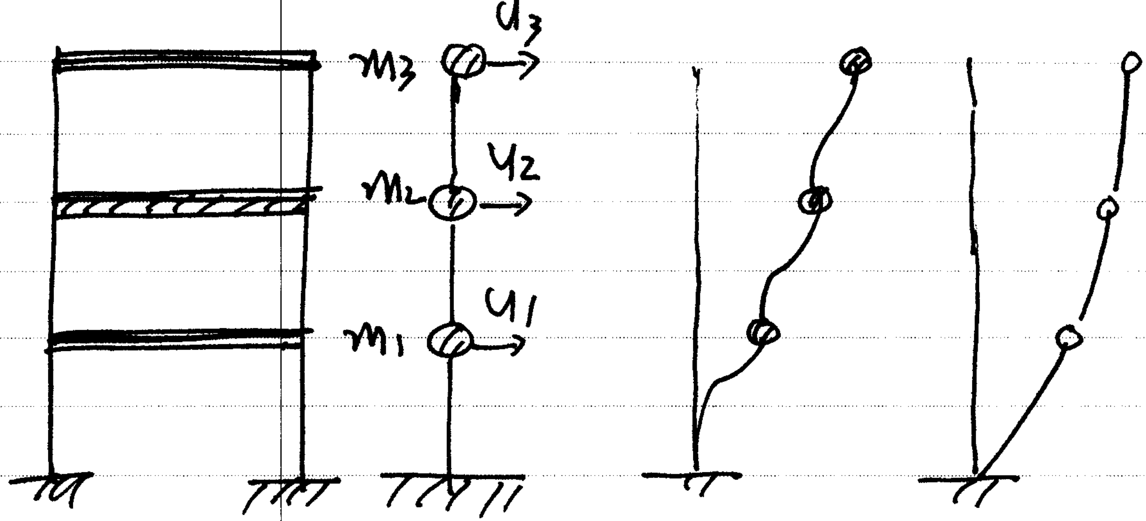
무한한 수의

딱 맞는 해석 : 특성함수를 구해서 계산,

근사해석 : 모드를 가정해서 단자유도 모델링

$u(x,t) = \psi(x) Z(t)$

Case #3: 집중(덩어리) 질량 시스템
(이산모델)



1차-모드 벡터 → ϕ_1 ψ

근사적 형상 vector ←

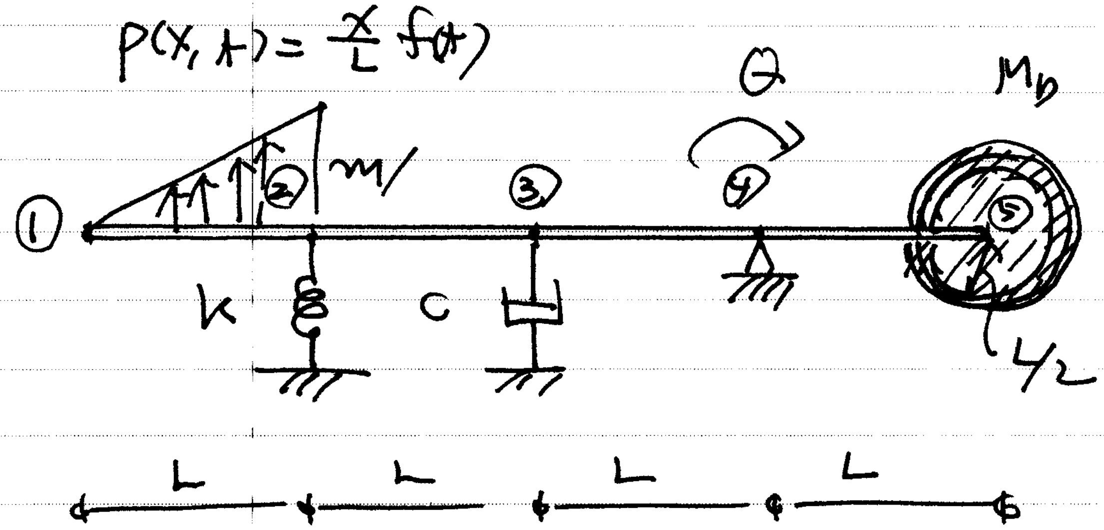
사용 방법: 기본적으로 Energy method

1. Virtual work principle

2. Hamilton's principle

3. Lagrange Equation

8.2. 강체 조립체



절점 ② 에서의 연직변위 u를 DOF로 결정

절점 ④ 에서의 회전각 θ, $\theta = \frac{u}{2L}$

절점 ① 에서의 연직변위, u_1

$$u_1 = (3L)\theta = \frac{3}{2}u$$

$$u(x) = (3L - x)\theta = (3L - x)\frac{u}{2L}$$
$$= \frac{3}{2}u - \frac{x}{2L}u$$

절점 ③ 에서의 수직변위, $u_2 = \frac{u}{2}$

절점 ⑤ 에서의 연직변위 u_5

$$u_5 = \frac{3}{2}u - \frac{4L}{2L}u = -\frac{1}{2}u$$

절편 b에서 Disk는 자유롭게 회전할수 없도록 강체 막대에 연결되어 있다.

Lagrange Equation

$$T_1 = \frac{1}{2}(4mL)\left(\frac{v}{2}\right)^2 = \frac{1}{2}mL(v)^2$$

$$T_2 = \frac{1}{2}\left(\frac{1}{12}4mL\right)(4L)^2\dot{\theta}^2$$

$$= \frac{1}{2}I_0\left(\frac{v}{2L}\right)^2 = \frac{1}{8L^2}I_0 v^2$$

$$T_3 = \frac{1}{2}M_D\left(-\frac{v}{2}\right)^2 = \frac{M_D}{8}v^2$$

$$T_4 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)M_D\left(\frac{L}{2}\right)^2\left(\frac{v}{2L}\right)^2$$

$$= \frac{1}{2}I_D\cdot\frac{v^2}{4L^2} = \frac{I_D}{8L^2}v^2$$

$$I_0 = \frac{1}{12}(4mL)(4L)^2 = \frac{16}{3}mL^3$$

$$I_D = \frac{1}{2}M_D\left(\frac{L}{2}\right)^2 = \frac{1}{8}M_D L^2$$

$$T = T_{①} + T_{②} + T_{③} + T_{④}$$

$$= \left(\frac{mL}{2} + \frac{I_o}{8L^2} + \frac{I_o}{8L^2} + \frac{M_o}{8}\right)\dot{u}^2$$

$$= \frac{1}{2}\left[mL + \frac{M_o}{4} + \frac{I_o}{4L^2} + \frac{I_o}{4L^2}\right]\dot{u}^2$$

$$V = \frac{1}{2}k\dot{u}^2$$

$$F = \frac{1}{2}c\left(\frac{\dot{u}}{2}\right)^2 = \frac{1}{2}\left(\frac{c}{4}\right)\dot{u}^2$$

$$W = p\int_0^L \left(\frac{x}{L}\right)\left(\frac{3}{2} - \frac{x}{2L}\right)u\,dx$$

$$= \frac{pu}{L}\int_0^L \left(\frac{x}{L}\right)\left(\frac{3}{2} - \frac{x}{2L}\right)dx$$

$$= \left(\frac{pu}{L}\right)\left[\frac{3}{4}x^2 - \frac{x^3}{6L}\right]_0^L$$

$$= \left(\frac{pu}{L}\right)\left[\frac{3}{4}L^2 - \frac{L^2}{6}\right] = \frac{17}{12}pu$$

Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial F}{\partial \dot{q}} + \frac{\partial V}{\partial q} = \frac{17}{12}pL$$

$$q = u$$

$$M^* \ddot{u} + C^* \dot{u} + K^* u = P^*$$

$$M^* = \left(mL + \frac{M_0}{4} + \frac{I_0}{4L^2} + \frac{I_0}{4L^2} \right)$$

$$C^* = c/4$$

$$K^* = k$$

8.3. 분포질량과 분포탄성을 가진 시스템

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = p(x,t)$$

Rayleigh-Ritz method (Assumed mode shape method)

$$u(x,t) = \sum_{i=1}^{N} \psi_i(x)\, q_i(t)$$

$$\dot{u}(x,t) = \sum_{i=1}^{N} \psi_i(x)\, \dot{q}_i(t)$$

Kinetic Energy T

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$$T(t) = \frac{1}{2}\int_0^l m(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx$$

Potential Energy, V

$$V(t) = \frac{1}{2}\int_0^l EI(x)\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx$$

External work, We

$$We = \int_0^l p(x,t)\,u(x,t)\,dx$$

Assumed mode shapes

$$T(t) = \frac{1}{2}\int_0^l m(x)\left(\sum_{i=1}^{N}\psi_i(x)\dot{q}_i\sum_{j=1}^{N}\psi_j(x)\dot{q}_j\right)dx$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\dot{q}_i(t)\,\dot{q}_j(t)\left(\frac{1}{2}\int_0^l m(x)\psi_i(x)\psi_j(x)\,dx\right)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{1}{2}\right)m_{ij}\,\dot{q}_i\,\dot{q}_j$$

$$V(t) = \frac{1}{2}\int_0^l EI(x)\left(\sum_{i=1}^{N} \psi_i'' q_i \sum_{j=1}^{N} \psi_j'' q_j\right) dx$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int_0^l EI(x)\psi_i''(x)\psi_j''(x)\,dx\right) q_i q_j$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij}\, q_i q_j$$

$$W_e = \int_0^l p(x,t)\sum_{i=1}^{N} \psi_i(x)\, q_i(t)\, dx$$

$$= \sum_{i=1}^{N}\left(\int_0^l p(x,t)\psi_i(x)\,dx\right) q_i(t)$$

$$= \sum_{i=1}^{N} P_i(t)\, q_i(t)$$

Lagrange Equation

- generalized coordinates $q_n(t)$
- variational approach

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_m} + \frac{\partial V}{\partial q_m} = \frac{\partial W_e}{\partial q_m}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_m}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}m_{ij}\dot{q}_i\dot{q}_j$$

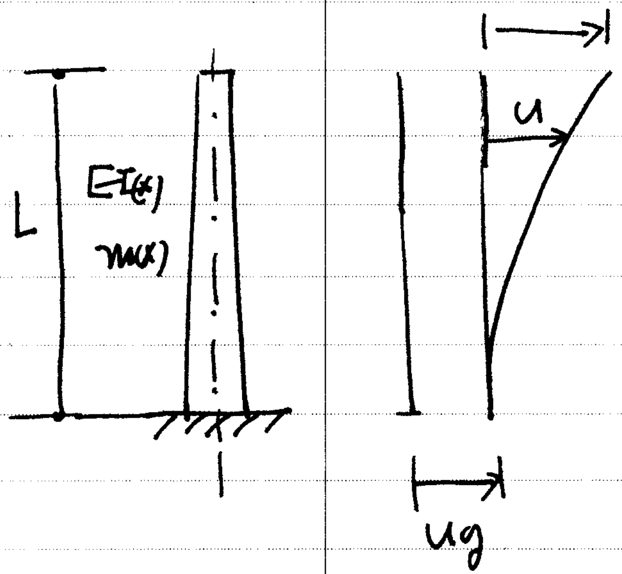
$$+\frac{\partial}{\partial q_m}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}k_{ij}q_iq_j = \frac{\partial}{\partial q_m}\sum_{i=1}^{N}P_i(t)q_i(t)$$

$$\sum_{n=1}^{N} m_{mn}\ddot{q}_n + \sum_{n=1}^{N} k_{mn}q_n = P_m(t)$$

$$m = 1, 2, \cdots, N$$

$$\begin{bmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \\ m_{N1} & \cdots & m_{NN} \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_N \end{Bmatrix} +$$

$$+\begin{bmatrix} k_{11} & \cdots & k_{1N} \\ \vdots & & \vdots \\ k_{N1} & \cdots & k_{NN} \end{bmatrix}\begin{Bmatrix} q_1 \\ \vdots \\ q_N \end{Bmatrix} = \begin{Bmatrix} P_1 \\ \vdots \\ P_N \end{Bmatrix}$$



$$u^t(x,t) = u(x,t) + u_g(t)$$

상대변위

$$u(x,t) = \psi(x)\, q(t)$$

$$\psi(x) = \frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}$$ (균질단면보 처짐곡선)

$$\psi(x) = \frac{x^2}{L^2}$$

$$\psi(x) = 1 - \cos\frac{\pi x}{2L}$$

가능한 예

$$p_{eff} = -m\ddot{u}_g(t)$$

$$\tilde{m} = \int_0^L m(x)\,\psi^2(x)\,dx$$

$$\tilde{k} = \int_0^L EI(x)\left(\psi''(x)\right)^2 dx$$

$$\tilde{L} = \int_0^L m(x)\,\psi(x)\,dx$$

$$\tilde{m}\ddot{q} + \tilde{k}q = -\tilde{L}\ddot{u}_g(t)$$

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}} = \frac{\int_0^L EI(x)\left[\psi''(x)\right]^2 dx}{\int_0^L m(x)\left[\psi(x)\right]^2 dx}$$

내력계산 : 등가정적하중법

$$f_s(x,t) = \left[EI(x)\,u''(x,t)\right]''$$

$$= \left[EI(x)\,\psi''(x)\right]'' q(t)$$

$$\int_0^L \psi(x) f_s(x,t)\,dx = \int_0^L \psi(x)[EI(x)\psi''(x)]'' q(t)\,dx$$

$$= \{\psi(x)[EI(x)\psi''(x)]'\}_0^L \widehat{q(t)} - \{\psi'(x)[EI(x)\psi''(x)]\}_0^L \widehat{q(t)}$$

$$+ \int_0^L EI(x)(\psi'')^2 dx \quad q(t)$$

$\psi(x)$가 경계조건을 만족한다고 가정한다.

$$\int_0^L \psi(x) f_s(x,t)\,dx = \int_0^L EI(x)(\psi'')^2 q(t)\,dx$$

$$= \int_0^L \omega_n^2 [m(x)\,\psi^2(x)]\,dx \quad q(t)$$

$$= \int_0^L \psi(x)\left(\omega_n^2 m(x)\,\psi(x)\,q(t)\right)dx$$

$$\int_0^L \psi(x)\left(f_s(x,t) - \omega_n^2 m(x)\,\psi(x)\,q(t)\right)dx = 0$$

$$f_s(x,t) = \omega_n^2 m(x)\,\psi(x)\,q(t)$$

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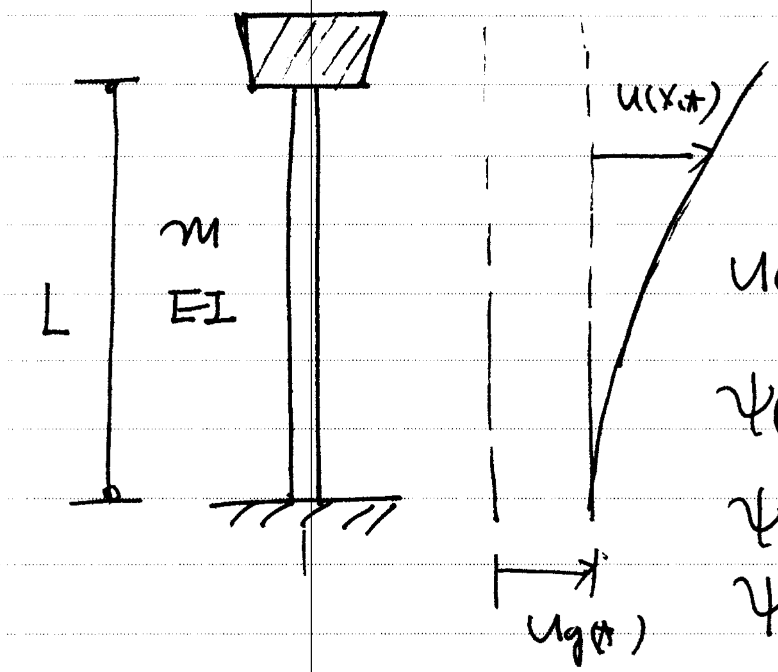
(예) 물탱크



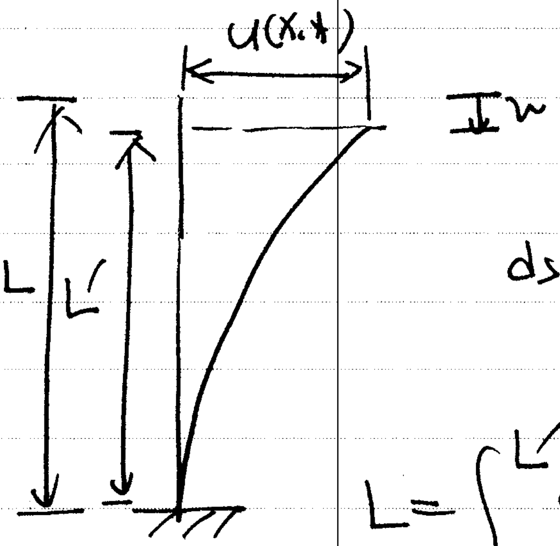
$$u(x,t) = \psi(x)\, q(t)$$

$$\psi(x) = 1 - \cos\frac{\pi x}{L}$$

$$\psi(0) = 0$$

$$\psi(L) = 1$$

$$N = Mg$$



$$ds = \sqrt{dx^2 + du^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$L = \int_0^{L'} ds = \int_0^{L} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\simeq \int_0^{L'} \left(1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right) dx$$

$$L = L' + \int_0^{L'} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$\nu = L - L' = \int_0^{L} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$= q^2(t) \int_0^{L} \left(\frac{1}{2}\right) \psi'^2(x) dx$$

$$T = \frac{1}{2} M (\psi(L)\dot{q} + \dot{u}_g)^2$$
$$+ \frac{1}{2} \int_0^{L} m (\psi(x)\dot{q} + \dot{u}_g)^2 dx$$

$$= \frac{1}{2} M [\dot{q}^2 + 2\dot{u}_g \dot{q} + \dot{u}_g^2]$$
$$+ \frac{1}{2} \int_0^{L} m [\psi^2 \dot{q}^2 + 2\psi \dot{q} \dot{u}_g + \dot{u}_g^2] dx$$

$$V = \frac{1}{2} \int_0^{L} EI \left(\frac{d^2 u}{dx^2}\right)^2 dx$$

$$= \frac{1}{2} \left(\int_0^{L} EI (\psi'')^2 dx\right) q^2$$

$$W_e = N\nu = \frac{1}{2} N \int_0^{L} (\psi')^2 dx \quad q^2$$

Lagrange Equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} + \frac{\partial V}{\partial q} = \frac{\partial W_e}{\partial q}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = M\ddot{q} + M\ddot{u}_g + \left(\int_0^L m\psi^2 dx\right)\ddot{q}$$
$$+\left(\int_0^L m\psi(x)\, dx\right)\ddot{u}_g$$

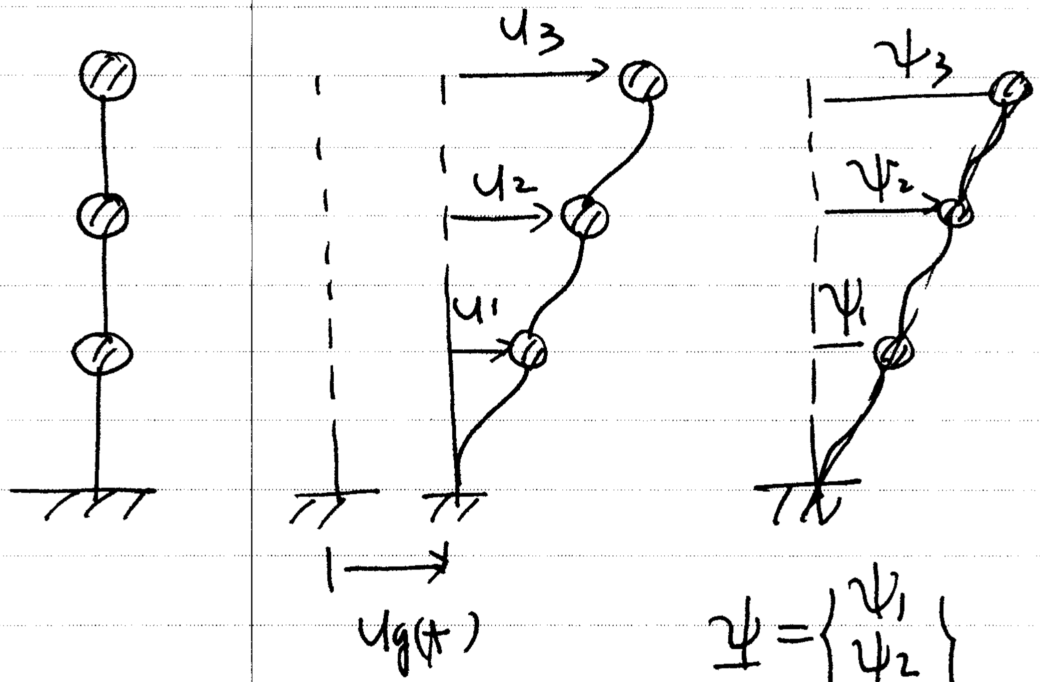
$$\frac{\partial V}{\partial q} = q\int_0^L EI(\psi'')^2 dx$$

$$\frac{\partial W_e}{\partial q} = \left(N\int_0^L (\psi'(x))^2 dx\right) q$$

$$\left(M + \int_0^L m\psi^2 dx\right)\ddot{q} + \left[\int_0^L EI(\psi'')^2 dx\right.$$
$$\left. - N\int_0^L (\psi')^2 dx\right] q = -M\ddot{u}_g - \left(\int_0^L m\psi\, dx\right)\ddot{u}_g$$

$$\tilde{M}\ddot{q} + (\tilde{K} - \tilde{K}_G)\, q = P_{eff}$$

8.4 집중질량 시스템 : 전단건물



$$\underline{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix}$$

가정된 형상함수

$$u_j^t(t) = u_j(t) + u_g(t)$$

$$\underline{u}^t(t) = \underline{u}(t) + \{1\} u_g(t)$$

$$T = \frac{1}{2} (\underline{\dot{u}}^t)^T \underset{\sim}{m} \, \underline{\dot{u}}^t$$

$$= \frac{1}{2} \left(\underline{\dot{u}}(t) + \{1\} \dot{u}_g(t) \right)^T \underset{\sim}{m} \left(\underline{\dot{u}}(t) + \{1\} \dot{u}_g(t) \right)$$

$\underline{u}(t) = \underline{\psi} \, q(t)$ 로 가정한다.

$$T = \frac{1}{2}(\dot{q}\underline{\psi}^T + \{1\}^T \dot{u}_g)\, \underset{\sim}{m}\, (\dot{q}\underline{\psi} + \{1\}\dot{u}_g)$$

$$= \frac{1}{2}(\underline{\psi}^T \underset{\sim}{m} \underline{\psi})\dot{q}^2 + \dot{q}\,\underline{\psi}^T \underset{\sim}{m} \{1\} \dot{u}_g$$

$$+ \frac{1}{2}\left(\{1\}^T \underset{\sim}{m} \{1\} (\dot{u}_g)^2\right)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = (\underline{\psi}^T \underset{\sim}{m} \underline{\psi})\ddot{q} + (\underline{\psi}^T \underset{\sim}{m} \{1\}^T)\ddot{u}_g$$

$$V = \frac{1}{2}\underline{u}^T \underset{\sim}{k}\, \underline{u} = \frac{1}{2} q\, \underline{\psi}^T \underset{\sim}{k} \underline{\psi}\, q$$

$$\frac{\partial V}{\partial q} = (\underline{\psi}^T \underset{\sim}{k} \underline{\psi})\, q$$

$$(\underline{\psi}^T \underset{\sim}{m} \underline{\psi})\ddot{q} + (\underline{\psi}^T \underset{\sim}{k} \underline{\psi})\, q = -\underline{\psi}^T \underset{\sim}{m} \{1\}^T \ddot{u}_g$$

$$\tilde{m}\ddot{q} + \tilde{k}\, q = -\tilde{L}\, q\, \ddot{u}_g$$

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}} = \frac{\underline{\psi}^T \underset{\sim}{k} \underline{\psi}}{\underline{\psi}^T \underset{\sim}{m} \underline{\psi}}$$