Calculus of Variation

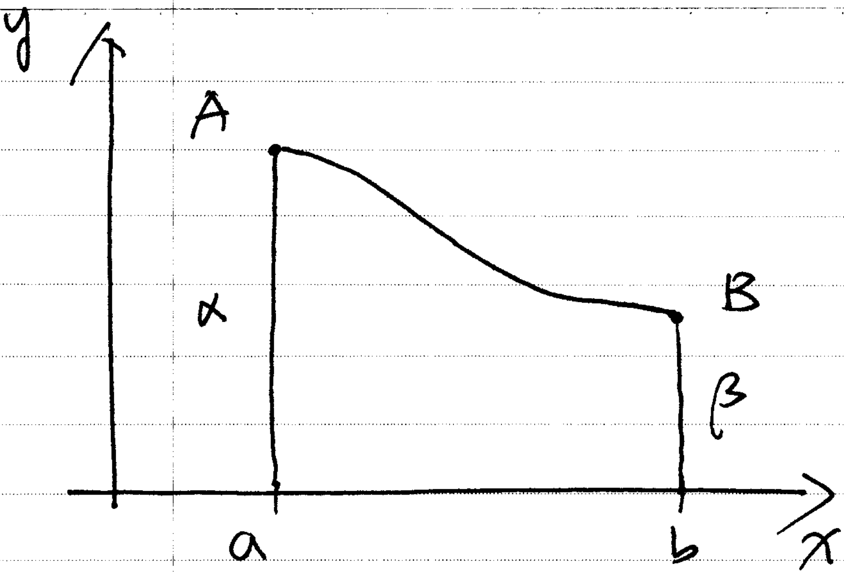
1. The stationary value of a definite integral

- the analytical problems of motion involves a special type of extremum problem
  : the stationary value of a definite integral

  ↑ A branch of mathematics dealing with problems of this nature.

$\boxed{\text{CALCULUS of VARIATION}}$

- Brachistochrone:
  - → the curve of quickest descent
  - John Bernoulli (1696)
  - Find a suitable plane curve along which a particle descents in the shortest time, starting from A and arriving at B.



the unknown curve : $y=f(x)$

$$t=\frac{1}{\sqrt{2g}}\int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{\alpha-y}}\,dx$$

$$f(a)=\alpha,\quad f(b)=\beta$$

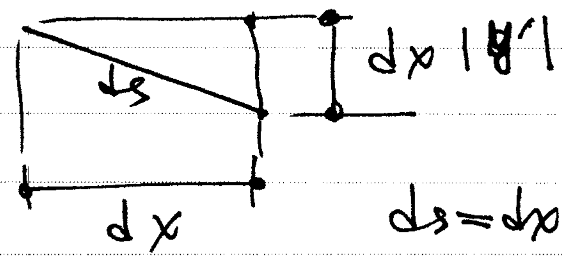
(Derivation)

높이 y에서의 velocity : v

$$mgh=\frac{1}{2}mv^2$$

$$mg(\alpha-y)=\frac{1}{2}mv^2$$

$$v=\sqrt{2g}\sqrt{\alpha-y}$$



$ds = dx\sqrt{1+y'^2}$

$$dt = \frac{\sqrt{1+y'^2}\ dx}{\sqrt{2g}\sqrt{\alpha-y}}$$

$$t = \frac{1}{2g}\int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{\alpha-y}}\,dx$$

Generalization

$$F = F(y, y', x)$$

$$I = \int_a^b F(y, y', x)\,dx$$

$$f(a) = \alpha, \quad f(b) = \beta$$

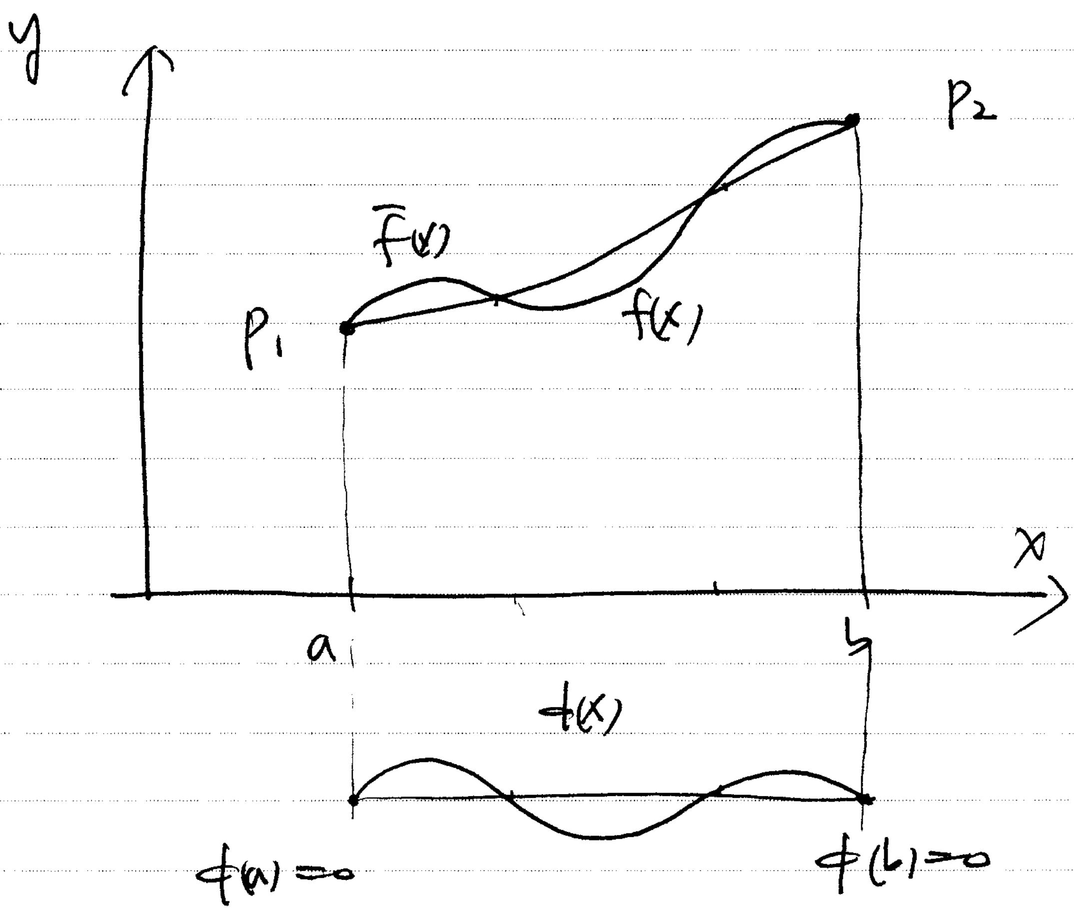
problem statement

Find a function

$$y = f(x)$$

which will make the integral I an extremum or at least a stationary value.

2. The fundamental processes of the calculus of variations



$\phi(x)$: a continuous function

$y = f(x)$: true minimizing function

$y = \bar{y} = \bar{f}(x) = f(x) + \epsilon \phi(x)$

↑ slightly modified function

(version 1)

$$I = \int_a^b F(y, y', x)\, dx$$

$y(a) = f(a) = y_a$: prescribed

$y(b) = f(b) = y_b$: prescribed

$$\bar{F}(y, y', x) = F(\bar{y}, \bar{y}', x)$$

$$= F(y + \epsilon\phi, y' + \epsilon\phi', x)$$

$$I(\epsilon) = \int_a^b F(y + \epsilon\phi, y' + \epsilon\phi', x)\, dx$$

If $\epsilon = 0$, $\bar{y} = y$: minimizing function

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\frac{dI(\epsilon)}{d\epsilon} = \int_a^b \frac{\partial F}{\partial \epsilon}\, dx$$

$$= \int_a^b \frac{\partial}{\partial \epsilon}\left[F(y+\epsilon\phi, y'+\epsilon\phi', x)\right] dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial \bar{y}}\phi + \frac{\partial F}{\partial \bar{y}'}\phi'\right] dx$$

F의 y와 $\bar{y}$ 에 대한 Functional dependence 는 같다.

$\epsilon \to 0$, $\bar{y} \Rightarrow y$, $\bar{y}' \Rightarrow y'$

$$\int_a^b \left[\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right] dx = 0$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\phi\right) = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\phi + \frac{\partial F}{\partial y'}\phi'$$

$$\frac{\partial F}{\partial y'}\phi' = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\phi\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\phi$$

$$\int_a^b \left[\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right]dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\phi - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\phi\right]dx$$

$$+ \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\phi\right)dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\phi\, dx$$

$$+ \left[\frac{\partial F}{\partial y'}\phi(x)\right]_{x=a}^{x=b} = 0 \qquad (3)$$

$\phi(a)=0$, $\phi(b)=0$ 이므로

$$\left[\frac{\partial F}{\partial y'}\phi(x)\right]_{x=a}^{x=b} = 0$$

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]\phi\, dx = 0 \qquad (4)$$

$\phi(x)$는 arbitrary 함수이므로 식 (4)가 항상 성립하기 위해서는 다음이 만족되어야 한다.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad (5)$$

Euler-Lagrange differential Equation

Natural Boundary Conditions

- $y(a)$와 $y(b)$가 prescribed 되지 않았다고 가정하자.
- 그러면 $\epsilon\phi(a)$와 $\epsilon\phi(b)$는 영이 아니어도 된다.
- 그러나 $\epsilon\phi(a) = \epsilon\phi(b) = 0$에 대해서도 식(3)이 만족되어야 하므로

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad (5)$$

는 항상 성립해야 한다.

- 이렇게 되기 위해서는 식(3)에서

$$\left[\frac{\partial F}{\partial y'}\right]_{x=a} = 0, \quad \left[\frac{\partial F}{\partial y'}\right]_{x=b} = 0$$

이 성립해야 한다.

$\downarrow$

Natural Boundary Conditions

(Version 2) Variational Notation

At a certain definite point x

$$\delta y = \bar{f}(x) - f(x) = \epsilon\phi(x)$$

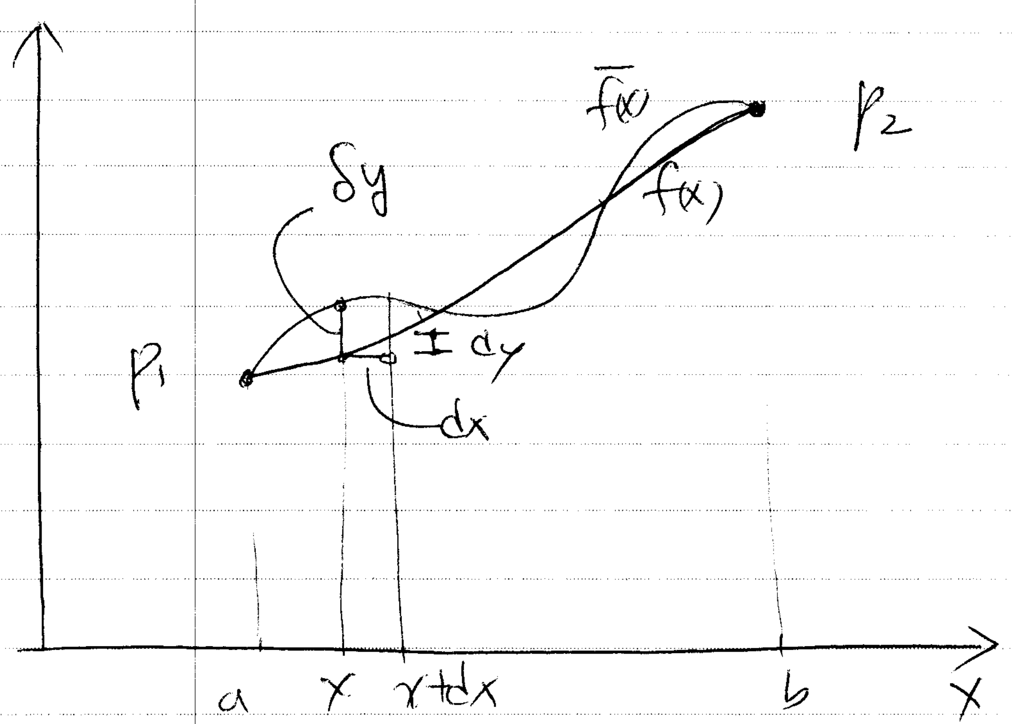
$\uparrow$ Variation

$\uparrow$ infinitesimal change

virtual change

dy : infinitesimal change of a given function $f(x)$ caused by the infinitesimal change ~~f~~ dx of the independent variable

δy : an infinitesimal change of y which produces a new function $y + \delta y$



$$\delta x = 0$$

$$[\delta f(x)]_{x=a} = 0, \quad [\delta f(x)]_{x=b} = 0$$

↑
Variation between definite limits

the commutative properties of the δ-process

$$\frac{d}{dx}\delta y = \frac{d}{dx}\left[\overline{f(x)} - f(x)\right] = \frac{d}{dx}\epsilon\phi(x) = \epsilon\phi'(x)$$

$$\delta\frac{d}{dx}f(x) = \overline{f'(x)} - f'(x) = (y' + \epsilon\phi') - y' = \epsilon\phi'(x)$$

$$\frac{d}{dx}\delta y = \delta \frac{d}{dx} y$$

The derivative of the variation is equal to the variation of the derivative.

$$\delta \int_a^b F(x)dx = \int_a^b \overline{F}(x)dx - \int_a^b F(x)dx$$

$$= \int_a^b [\overline{F}(x) - F(x)]\,dx = \int_a^b \delta F(x)dx$$

$$\delta \int_a^b F(x)dx = \int_a^b \delta F(x)dx$$

the variation of a definite integral is equal to the definite integral of the variation.

(a) Variation and differentiation are permutable processes

(b) Variation and integration are permutable processes

The stationary value of a definite integral treated by the calculus of variations

$$\delta F(y, y', x) = F(y+\epsilon\phi, y'+\epsilon\phi', x) - F(y, y', x)$$

$$= \epsilon\left(\frac{\partial F}{\partial y}\phi + \frac{\partial F}{\partial y'}\phi'\right)$$

Higher order terms of the taylor development can be neglected since ϵ approaches zero.

$$\delta y = \epsilon\phi, \quad \delta y' = \epsilon\phi'$$

$$\delta F = \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\delta y\right)$$

$$= \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y + \frac{\partial F}{\partial y'}\frac{d}{dx}\delta y \quad \to \delta y'$$

$$\frac{\partial F}{\partial y'}\delta y' = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\delta y\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y$$

$$\delta I = \delta\int_a^b F(y, y', x)\,dx$$

$$= \int_a^b \delta F(y, y', x)\,dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]\delta y\,dx$$

$$+ \left[\frac{\partial F}{\partial y'}\delta y\right]_{x=a}^{x=b}$$

$$[\delta y]_{x=a} = 0, \quad [\delta y]_{x=b} = 0$$

$\delta I = 0$ 이기위해서는

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Necessary and sufficient condition for the vanishing of δI.

$$\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1$$

$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}$$

The Euler–Lagrange differential equations for n degrees of freedom

"n degrees of freedom"

~~→~~ n parameters are necessary and sufficient for a unique configuration of the system.

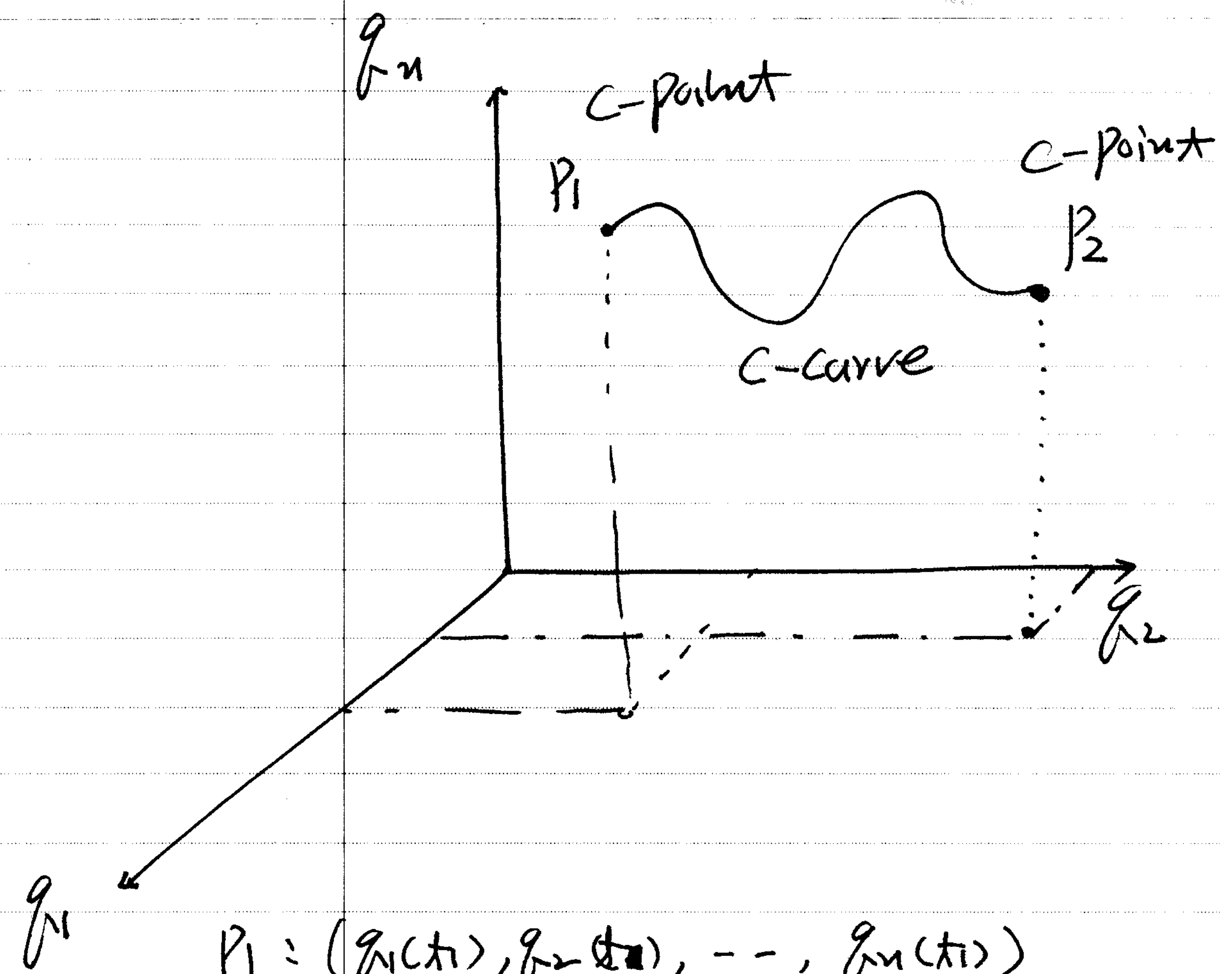
n parameters : $q_1, q_2, --, q_n$
↓
general coordinates of the system

The configuration space

consider $q_1, q_2, \cdots q_n$ as the rectangular coordinates of a "point" P in an n-dimensional space

$$q_1 = q_1(t),\ q_2 = q_2(t), \cdots, q_n = q_n(t)$$

The entire mechanical system is pictured as a single point of a many-dimensional space, called "configuration space".



$P_1 : (q_1(t_1), q_2(t_1), --, q_n(t_1))$

$P_2 : (q_1(t_2), q_2(t_2), ---, q_n(t_2))$

Mapping of the space on itself

$q_1, q_2, ---, q_n, \ \bar{q}_1, \bar{q}_2, \bar{q}_3, ---, \bar{q}_n$

$$\begin{cases} \bar{q}_1 = f_1(q_1, ---, q_n) \\ \vdots \\ \bar{q}_n = f_n(q_1, ---, q_n) \end{cases} \quad \begin{cases} d\bar{q}_1 = \frac{\partial f_1}{\partial q_1} dq_1 + --- + \frac{\partial f_1}{\partial q_n} dq_n \\ \vdots \\ d\bar{q}_n = \frac{\partial f_n}{\partial q_1} dq_1 + --- + \frac{\partial f_n}{\partial q_n} dq_n \end{cases}$$

$$I = \int_{t_1}^{t_2} L(q_1, \dots, q_n ; \dot{q}_1, \dots, \dot{q}_n ; t)\, dt$$

$$[\delta q_k(t)]_{t=t_1} = 0, \quad [\delta q_k(t)]_{t=t_2} = 0$$

$$k = 1, 2, \dots, n$$

$q_1, \dots, q_n$ 은 t에 대한 미지의 함수이고 다음 조건에 의해서 결정되어야 한다.

$$\delta I = 0$$

$$\delta L = \frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \dots + \frac{\partial L}{\partial q_n}\delta q_n$$

$$+ \frac{\partial L}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial L}{\partial \dot{q}_2}\delta \dot{q}_2 + \dots + \frac{\partial L}{\partial \dot{q}_n}\delta \dot{q}_n$$

$$\delta L = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L\, dt$$

$$= \int_{t_1}^{t_2} \left\{ \left[\frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial \dot{q}_1}\delta \dot{q}_1 \right] + \dots + \left[\frac{\partial L}{\partial q_n}\delta q_n + \frac{\partial L}{\partial \dot{q}_n}\delta \dot{q}_n \right] \right\} dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \right) \delta q_1 \, dt + \left[\frac{\partial L}{\partial \dot{q}_1} \delta q_1 \right]_{t_1}^{t_2}$$

$$+ \cdots$$

$$+ \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_n} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} \right) \delta q_n \, dt + \left[\frac{\partial L}{\partial \dot{q}_n} \delta q_n \right]_{t_1}^{t_2}$$

따라서

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0 , \quad k = 1, \cdots, n$$

A System of simultaneous differential equation

$\Updownarrow$

The differential equations of Euler and Lagrange

$\Updownarrow$ mechanics problems

The Lagrangian equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0, \ (k = 1, 2, \cdots, n)$$

$$\ddot{q}_k = \phi_k(q_1, \cdots, q_n, \dot{q}_1, \cdots \dot{q}_n, t)$$

$$q_k = q_k(A_1, \cdots, A_n; B_1, \cdots, B_n; t)$$

$2n$ constants

Variation with auxiliary conditions

$$f_1(q_1, \cdots, q_n, t) = 0$$

$$\vdots$$

$$f_m(q_1, \cdots, q_n, t) = 0$$

q_k 중 m개를 소거 $\rightarrow$ $(n-m)$ DoF system
↑
cumbersome

Method of the Lagrangian multiplier

$$\delta f_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n = 0$$

$$\vdots$$

$$\delta f_m = \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n = 0$$

$$\delta I' = \delta\int_{t_1}^{t_2} L\,dt + \int_{t_1}^{t_2}(\lambda_1\delta f_1 + \cdots + \lambda_m \delta f_m)\,dt = 0$$

$$= \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m}\frac{\partial f_j}{\partial q_i}(\lambda_j)\right]\delta q_i\,dt$$

$\vdots$

$$+\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_n}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n}+\sum_{j=1}^{m}\frac{\partial f_j}{\partial q_n}(\lambda_j)\right]\delta q_n\,dt=0$$

$$\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}+\lambda_1\frac{\partial f_1}{\partial q_k}+\cdots+\lambda_m\frac{\partial f_m}{\partial q_k}=0$$

$$(k=1,\ \cdots,\ n)$$

$$I=\int_{t_1}^{t_2}L\,dt$$

$$I'=\int_{t_1}^{t_2}L'\,dt$$

$$L'=L+\lambda f_1+\cdots+\lambda_m f_m$$

λ_i 는 variation 과정에 대해서는 상수로 간주되어야 한다.

그러나 L' 에는 λ_i 는 t 에 대한 미지의 함수로 포함되어 있다. λ_i 의 Variation 은 Auxiliary conditions 을 도출되게 한다.

Non-holonomic conditions

$$\delta \bar{f}_1 = A_{11}\delta q_1 + A_{12}\delta q_2 + \cdots + A_{1n}\delta q_n = 0$$

$$\vdots$$

$$\delta \bar{f}_m = A_{m1}\delta q_1 + A_{m2}\delta q_2 + \cdots + A_{mn}\delta q_n = 0$$

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{j=1}^{m} \lambda_j A_{jk} = 0$$

Isoperimetric conditions

$$\int_{t_1}^{t_2} f(q_1, \cdots, q_n, t)\,dt = C$$

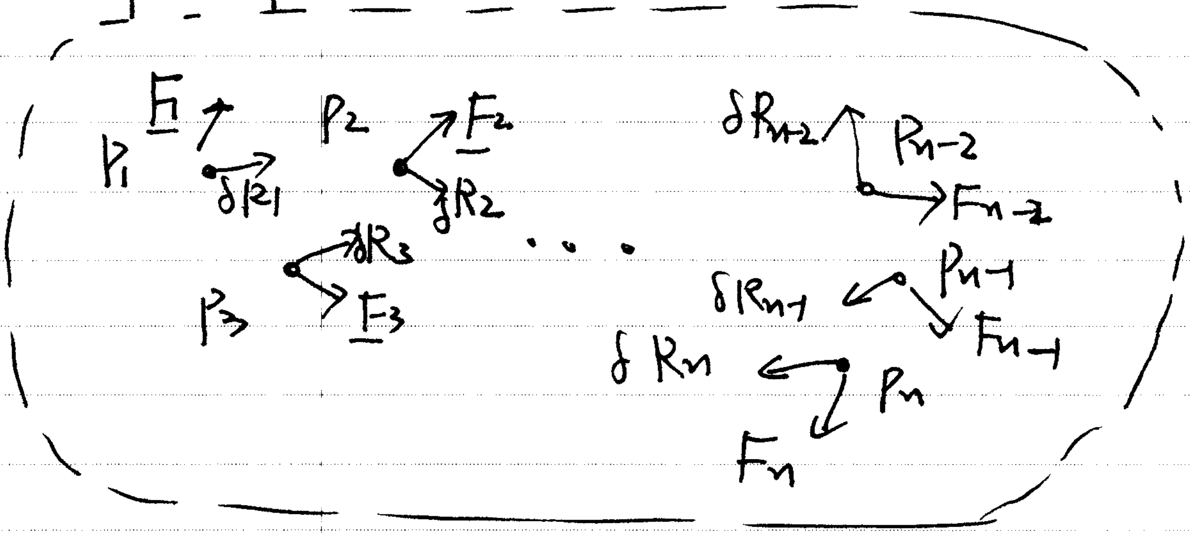
연구: 역사상 기록된 최초의 extremum 문제는 주어진 주변길이에 대해서 둘러싸인 최대 면적을 찾아내는 문제 (Dido's problem)이다. 이 문제에서 상기 조건이 적용된다.

$$\int_{t_1}^{t_2} \left(\frac{\partial f}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f}{\partial q_n}\delta q_n\right) dt = 0$$

$$\delta \int_{t_1}^{t_2} (L + \lambda f)\, dt = 0$$

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \lambda \frac{\partial f}{\partial q_k} = 0$$

The principle of virtual work



+ Points : $P_1, P_2, \cdots, P_n$
+ External forces : $F_1, F_2, \cdots, F_n$
+ virtual displacement : $\delta R_1, \delta R_2, \cdots, \delta R_n$

The virtual displacements must be in harmony with the given kinematic constraints, and we shall assume that they are reversible.

i.e. the given constraints do not prevent us from changing an arbitrary δR_i into $-\delta R_i$

The principle of virtual work

the given mechanical system will be in equilibrium if, and only if, the total virtual work of all the impressed forces vanishes:

$$\delta \overline{W} = F_1 \cdot \delta R_1 + F_2 \cdot \delta R_2 + \cdots + F_n \cdot \delta R_n = 0$$

In generalized coordinates, $q_1, q_2, \cdots, q_n$.

$$\delta \overline{W} = F_1 \delta q_1 + F_2 \delta q_2 + \cdots + F_n \cdot \delta q_n = 0$$

the Force F_i is perpendicular to any possible virtual displacement

$\downarrow$

Requires vanishment of Force F_i

d'Alembert's Principle

Newtonian law of motion

$$mA = F$$

$$F - mA = 0$$

define a vector I

$$I = -mA$$

↑
Force of inertia

$$F + I = 0$$

dynamics is reduced to statics

problem of motion ⟶ problem of Equilibrium

Virtual displacement : a possible but purely mathematical experiment

Virtual displacement can be applied at a certain definite time.
(physically infinite velocity)

→ the actual motion does not enter into account.

d'Alemberts Principle

the total virtual of the impressed forces, augumented by the inertial forces, vanishes for reversible displacements.

$$F_k^e = F_k + I_k$$

effective force

the total virtual work of the effective forces is zero for all reversible variations which satisfy the given kinematical conditions

$$\sum_{k=1}^{N} F_k^e \cdot \delta R_k \equiv \sum_{k=1}^{N} (F_k - m_k A_k) \cdot \delta R_k = 0$$

Hamilton's Principle

$$\int_{t_1}^{t_2} \delta \overline{W}^e dt = \int_{t_1}^{t_2} \sum \left[F_i - \frac{d}{dt}(m_i V_i) \right] \cdot \delta R_i dt$$

$$\int_{t_1}^{t_2} \sum F_i \cdot \delta R_i dt = - \int_{t_1}^{t_2} \delta V dt = -\delta \int_{t_1}^{t_2} V dt$$

$$-\int_{t_1}^{t_2} \frac{d}{dt}(m_i V_i)\cdot \delta R_i\, dt = -\int_{t_1}^{t_2} \frac{d}{dt}(m_i V_i \cdot \delta R_i)\, dt$$

$$+\int_{t_1}^{t_2} m_i V_i \cdot \frac{d}{dt}(\delta R_i)\, dt$$

$$-\left[m_i V_i \cdot \delta R_i\right]_{t_1}^{t_2}$$

$$\int_{t_1}^{t_2} m_i V_i \cdot \frac{d}{dt}\delta R_i\, dt = \int_{t_1}^{t_2} m_i V_i \cdot \delta V_i\, dt$$

$$= \frac{1}{2}\int_{t_1}^{t_2} m_i\, \delta(V_i \cdot V_i)\, dt = \frac{1}{2}\delta \int_{t_1}^{t_2} m_i v_i^2\, dt$$

$$\int_{t_1}^{t_2} \delta \overline{W}^e\, dt = \delta \int_{t_1}^{t_2} \underbrace{\frac{1}{2}\sum m_i v_i^2}_{T}\, dt - \delta \int_{t_1}^{t_2} V\, dt$$

$$L = T - V$$

↑ Lagrangian function

$$\int_{t_1}^{t_2} \delta \overline{W}^e\, dt = \delta \int_{t_1}^{t_2} L\, dt - \left[\sum m_i V_i \cdot \delta R_i\right]_{t_1}^{t_2}$$

Variation between definite limits

$$\delta R_i(t_1) = 0$$
$$\delta R_i(t_2) = 0$$

$$\int_{t_1}^{t_2} \delta \overline{W^e} \, dt = \delta \int_{t_1}^{t_2} L \, dt = \delta A$$

$$A = \int_{t_1}^{t_2} L \, dt$$

$$\delta \overline{W^e} = 0 \longrightarrow \delta A = 0$$

d'Alemberts principle — Hamiltons principle

The definite integral A becomes stationary for arbitrary possible variations of the configuration of the system, provided the initial and final configuration of the system are prescribed

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Lagrangian Equations of motion

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k\,dt$$

$$+ \left[\sum \frac{\partial L}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2}$$

$L = T - V$, (T는 q_k에 종속적이고 V는 $\dot{q}_k$에 독립적이다.)

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum \left(-\frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}\right)\delta q_k\,dt$$

$$+ \left[\sum \frac{\partial T}{\partial \dot{q}_k}\cdot\delta q_k\right]_{t_1}^{t_2}$$

$$\int_{t_1}^{t_2} \delta \overline{W}^e\,dt = \delta \int_{t_1}^{t_2} L\,dt - \left[\sum m_i V_i \cdot \delta R_i\right]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \sum \left(-\frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}\right)\delta q_k\,dt$$

$$+ \left[\sum \frac{\partial T}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \left[\sum m_i V_i \cdot \delta R_i\right]_{t_1}^{t_2}$$