제 10장 자유진동

$m\ddot{u} + c\dot{u} + ku = p(t)$

$\underset{\sim}{m}$: symmetric positive definite matrix

$$\underline{u}^T \underset{\sim}{m} \underline{u} > 0, \quad \underline{u} \neq \{0\}$$

↑ Kinetic energy와 관련됨

$\underset{\sim}{k}$: symmetric positive matrix

$$\underline{u}^T \underset{\sim}{k} \underline{u} \geq 0, \quad \underline{u} \neq \{0\}$$

↑ potential energy

$\underline{u}^T \underset{\sim}{k} \underline{u} = 0 \Rightarrow$ rigid body motion

positive definite system

$\underset{\sim}{m}$: positive definite

$\underset{\sim}{k}$: positive definite

positive semi-definite system

$\underset{\sim}{m}$: positive definite

$\underset{\sim}{k}$: positive

MOOKEUK

Undamped system (무감쇠 시스템)

$\underline{m}\ddot{\underline{u}} + \underline{k}\underline{u} = \underline{0}$

$u(0)$, $\dot{u}(0)$ at $t=0$

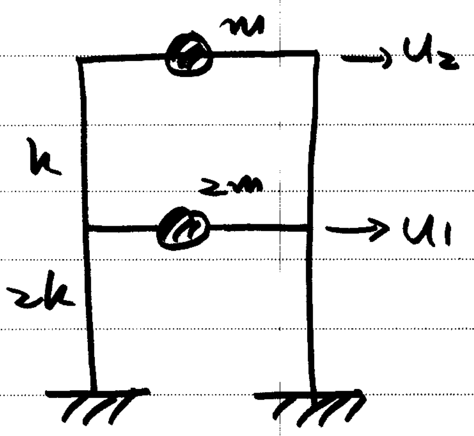


그림 10.1.1

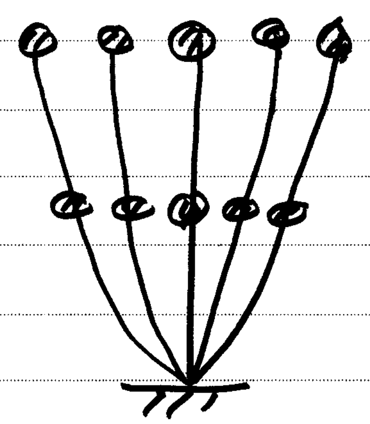


그림 10.1.2

∨ 일반적으로 $u_1(t)$, $u_2(t)$는 시간에 대해서 조화운동이 아니다.

∨. 그러나 특정한 변위들의 $u_1(0)$, $u_2(0)$에 대해서 $u_1(t)$, $u_2(t)$는 조화 진동을 하고 $\underline{u}(t)$는 형상이 변하지 않고 진폭만 시간에 따라서 달라진다.

⟶ 고유모드

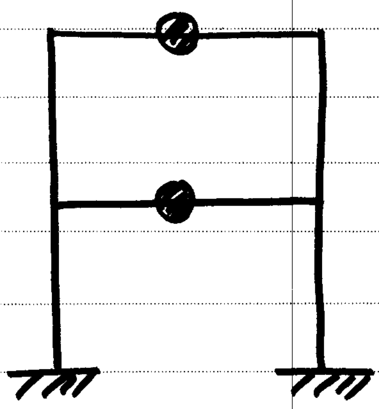
↓

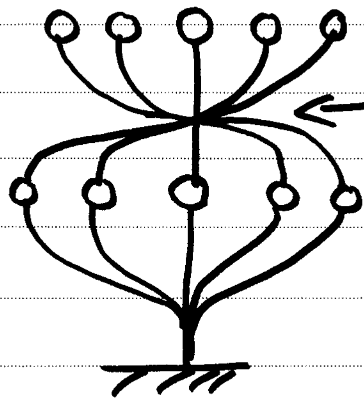
· 고유 진동주기 (고유주기) T_n

· 고유 각진동수 ω_n

· 고유 진동수 f_n

MOOKEUK





10.2. 고유진동수, 고유진동모드

$$\underset{\sim}{m}\,\ddot{\underline{u}} + \underset{\sim}{k}\,\underline{u} = \underline{0} \qquad (1)$$

separation of variables

$$\underline{u}(t) = q_n(t)\,\underline{\phi}_n$$

$$\dot{\underline{u}}(t) = \dot{q}_n(t)\,\underline{\phi}_n \qquad (2)$$

$$\ddot{\underline{u}}(t) = \ddot{q}_n(t)\,\underline{\phi}_n$$

$(2) \rightarrow (1)$

$$\underset{\sim}{m}\,\underline{\phi}_n\,\ddot{q}_n(t) + \underset{\sim}{k}\,\underline{\phi}_n\,q_n(t) = \underline{0} \qquad (3)$$

$\underline{\phi}_n^T$ 를 앞에 곱한다.

MOOKEUK

$$\underline{\phi}_n^T \underset{\sim}{M} \underline{\phi}_n \ddot{q}_n(t) + \underline{\phi}_n^T \underset{\sim}{K} \underline{\phi}_n q_n(t) = 0 \qquad (4)$$

$$\frac{\underline{\phi}_n^T \underset{\sim}{K} \underline{\phi}_n}{\underline{\phi}_n^T \underset{\sim}{M} \underline{\phi}_n} = -\frac{\ddot{q}_n(t)}{q_n(t)} = \lambda_n \qquad (5)$$

$$\ddot{q}_n(t) + \lambda_n q_n(t) = 0 \qquad (6)$$

- $\lambda > 0$: positive definite system

$$q_n(t) = A e^{i\sqrt{\lambda_n} t} + B e^{-i\sqrt{\lambda_n} t}$$

: Time Harmonic motion

- $\lambda = 0$: rigid body motion

$$q_n(t) = At + B$$

: $t \to \infty$, $q_n(t)$ diverges

- $\lambda < 0$: energy is not finite

$$q_n(t) = A e^{\sqrt{-\lambda_n} t} + B e^{-\sqrt{-\lambda_n} t}$$

(6) $\longrightarrow$ (3)

$\underset{\sim}{K}\,\underset{\sim}{\phi}_n\, q_n(t) - \lambda_n\, \underset{\sim}{M}\, \underset{\sim}{\phi}_n\, q_n(t) = \underset{\sim}{0}$

$$\left(\underset{\sim}{K}\,\underset{\sim}{\phi}_n - \lambda_n\, \underset{\sim}{M}\, \underset{\sim}{\phi}_n\right) q_n(t) = \underset{\sim}{0} \qquad (7)$$

항상 성립하기 위해서는

(eigenvalue problem) $\rightarrow$ $\left(\underset{\sim}{K} - \lambda_n \underset{\sim}{M}\right) \underset{\sim}{\phi}_n = \underset{\sim}{0}$ (8)

자명해가 아닌경우로 해가 존재하기 위해서는

$$\left|\underset{\sim}{K} - \lambda_n \underset{\sim}{M}\right| = 0$$

or

$$\det\left(\underset{\sim}{K} - \lambda_n \underset{\sim}{M}\right) = 0 \qquad (9)$$

: characteristic determinant
characteristic equation

characteristic equation의 근들
characteristic value 또는 eigenvalue

λ_n: 고유값
$\omega_n = \sqrt{\lambda_n}$: 고유각 진동수
$f_n = \omega_n / 2\pi$: 고유 진동수
$T_n = 2\pi / \omega_n = 1/f_n$: 고유주기

MOOKEUK

$$T_1 \geq T_2 \geq \cdots \geq T_n$$
$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$$

Eigenvector, characteristic vector
normal mode, $\underline{\phi}_n$

$$(\underline{k} - \lambda_n \underline{m})\underline{\phi}_n = 0 \qquad (10)$$

$$\lambda_n \Longleftrightarrow \phi_n$$

방정식 (10)의 비자명해 $\underline{\phi}_n$

각 모드는 형상은 변하지 않으면서 그 고유주기를 가지고 일정하게 진동한다.

10.3. 모드행렬과 스펙트럴 행렬

$$\underline{\Phi} = [\underline{\phi}_1\ \underline{\phi}_2 \cdots \underline{\phi}_N]$$

$$= \begin{bmatrix} \phi_{11} & \phi_{12} & & \phi_{1N} \\ \vdots & & & \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{bmatrix} \text{: 모드행렬}$$

$$\underline{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2 \end{bmatrix} \text{: 스펙트럴 행렬}$$

17/10

Generalized eigenvalue problem

- Hildebrand, "methods of applied mathematics"

$$\underset{\sim}{A}\,\underline{X} = \lambda\,\underset{\sim}{B}\,\underline{X}$$

A, B are real, symmetric, positive definite matrices.

① All the eigenvalues are real and positive.

② For each eigenvalue λ_n or natural frequency ω_n of multiplicity 1, there is an associated linearly independent eigenvector (mode shape vector)

③ If λ_n is a root of multiplicity r, then there are r linearly independent eigenvectors associated with λ_n.

④ If λ_m and λ_n are two distinct eigenvalues, then the corresponding eigenvectors $\underline{\phi}_m$, and $\underline{\phi}_n$ are orthogonal with respect to A and B matrices.

MOOKEUK

$$\underline{\phi}_m = \begin{Bmatrix} \phi_{1m} \\ \vdots \\ \phi_{Nm} \end{Bmatrix}, \quad \underline{\phi}_n = \begin{Bmatrix} \phi_{1n} \\ \vdots \\ \phi_{Nn} \end{Bmatrix}$$

$$\underline{\phi}_n^T \underset{\sim}{A} \underline{\phi}_m = 0, \quad \underline{\phi}_m^T \underset{\sim}{A} \underline{\phi}_n = 0$$

$$\underline{\phi}_n^T \underset{\sim}{B} \underline{\phi}_m = 0, \quad \underline{\phi}_m^T \underset{\sim}{B} \underline{\phi}_n = 0$$

(증명)

$$\underset{\sim}{A} \underline{\phi}_m = \lambda_m \underset{\sim}{B} \underline{\phi}_m$$

$$\underset{\sim}{A} \underline{\phi}_n = \lambda_n \underset{\sim}{B} \underline{\phi}_n$$

$$\underline{\phi}_n^T \underset{\sim}{A} \underline{\phi}_m = \lambda_m \underline{\phi}_n^T \underset{\sim}{B} \underline{\phi}_m$$

$$\parallel$$

$$\underline{\phi}_m^T \underset{\sim}{A} \underline{\phi}_n = \lambda_n \underline{\phi}_m^T \underset{\sim}{B} \underline{\phi}_n$$

$$(\lambda_m - \lambda_n) \underline{\phi}_m^T \underset{\sim}{B} \underline{\phi}_n = 0$$

$$\lambda_m \neq \lambda_n \rightarrow \underline{\phi}_m^T \underset{\sim}{B} \underline{\phi}_n = 0$$

$$\downarrow$$

$$\underline{\phi}_m^T \underset{\sim}{A} \underline{\phi}_n = 0$$

⑤ If λ_n ~~are~~ is an eigenvector of multiplicity r, the eigenvectors associated with λ_n can always be chosen so that they are mutually orthogonal with respect to $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

: Gram-Schmidt orthogonalization procedure

→ Hilderbrand

⑥ the eigenvectors are normalized with respect to B, if $\underline{\phi}_n^T \underset{\sim}{B} \underline{\phi}_n = 1$.

⑪ the set of N eigenvectors is complete. Any vector of dimension N can be expressed as a linear combination of the N eigenvectors.

$$\underline{X} = \sum_{i=1}^{N} a_i \underline{\phi}_i$$

$$\underset{\sim}{B}\underline{X} = \sum_{i=1}^{N} a_i \underset{\sim}{B} \underline{\phi}_i$$

$$\underline{\phi}_j^T \underset{\sim}{B} \underline{X} = \sum_{i=1}^{N} a_i \underline{\phi}_j^T \underset{\sim}{B} \underline{\phi}_i = a_j \underline{\phi}_j^T \underset{\sim}{B} \underline{\phi}_j$$

$$a_j = \frac{\underline{\phi}_j^T \underset{\sim}{B} \underline{X}}{\underline{\phi}_j^T \underset{\sim}{B} \underline{\phi}_j}$$

Dynamic System

$$\underset{\sim}{m}\ddot{\underline{u}} + \underset{\sim}{k}\underline{u} = \underline{0}$$

Restrained structure

- $\underset{\sim}{m}$, $\underset{\sim}{k}$ are symmetric positive definite
- All eigenvalues are real positive.

Unrestrained structure

- $\underset{\sim}{m}$: positive definite
- $\underset{\sim}{k}$: positive semi-definite
- All the eigenvalues are real and greater than or equal to zero.
- At least one eigenvalue is zero that is associated with rigid body motion

Orthogonality

$$\underline{\phi}_n^T \underset{\sim}{m} \underline{\phi}_m = \begin{cases} 0 & m \neq n \\ \text{nonzero} & m = n \end{cases}$$

$$\underset{\sim}{k}\underline{\phi}_n = \omega_n^2 \underset{\sim}{m}\underline{\phi}_n$$

$$\underline{\phi}_n^T \underset{\sim}{k}\underline{\phi}_n = \omega_n^2 \underline{\phi}_n^T \underset{\sim}{m}\underline{\phi}_n$$

10.5 모드 직교성의 의미

<의미 1>

r 번째 모드의 변위를 통해서 n 번째 모드의 관성력이 하는 일은 영(zero)이다.

$$\underline{u}_n(t) = q_n(t)\,\underline{\phi}_n$$
$$\ddot{\underline{u}}_n(t) = \ddot{q}_n(t)\,\underline{\phi}_n$$

$$(\underline{f}_I)_n = -\underline{m}\,\ddot{\underline{u}}_n(t) = -\underline{m}\,\underline{\phi}_n\,\ddot{q}_n(t)$$

$$\underline{u}_r(t) = q_r(t)\,\underline{\phi}_r$$

$$(\underline{f}_I)_n^T\,\underline{u}_r = -(\underline{\phi}_n^T\,\underline{m}\,\underline{\phi}_r)\,\ddot{q}_n(t)\,q_r(t)$$
$$= 0,\ \text{if}\ n \neq r$$

<의미 2>

r 번째 모드의 변위를 따라가면서 n 번째 변위와 관련된 등가 정적하중이 하는 일은 영(zero)이다.

$$(\underline{f}_S)_n = \underline{k}\,\underline{u}_n(t) = \underline{k}\,\underline{\phi}_n\,q_n(t)$$

$$(\underline{f}_S)_n^T\,\underline{u}_r = (\underline{\phi}_n^T\,\underline{k}\,\underline{\phi}_r)\,q_n(t)\,q_r(t)$$
$$= 0,\ \text{if}\ n \neq r$$

12/10

10.6 모드의 정규화.

$$M_n = \underline{\phi}_n^T \underset{\sim}{m} \underline{\phi}_n = 1$$

$$\underset{\sim}{\Phi}^T \underset{\sim}{m} \underset{\sim}{\Phi} = \underset{\sim}{I}$$

$$K_n = \underline{\phi}_n^T \underset{\sim}{k} \underline{\phi}_n = \omega_n^2 M_n = \omega_n^2$$

$$\underset{\sim}{K} = \underset{\sim}{\Phi}^T \underset{\sim}{k} \underset{\sim}{\Phi} = \underset{\sim}{\Omega}^2$$

10.7. 변위의 모드전개

$$\underline{u} = \sum_{r=1}^{N} \underline{\phi}_r q_r = \underset{\sim}{\Phi} \underline{q}$$

$$\underline{\phi}_n^T \underset{\sim}{m} \underline{u} = \sum_{r=1}^{N} \underline{\phi}_n^T \underset{\sim}{m} \underline{\phi}_r q_r$$

$$= (\underline{\phi}_n^T \underset{\sim}{m} \underline{\phi}_n) q_n$$

$$q_n = \frac{\underline{\phi}_n^T \underset{\sim}{m} \underline{u}}{\underline{\phi}_n^T \underset{\sim}{m} \underline{\phi}_n}$$

$$q_r = \frac{\underline{\phi}_r^T \underset{\sim}{m} \underline{u}}{\underline{\phi}_r^T \underset{\sim}{m} \underline{\phi}_r}$$

16.8. 비감쇠 시스템의 자유진동

$$\underset{\sim}{m}\ddot{\underline{u}} + \underset{\sim}{k}\underline{u} = 0$$

$$\underline{u} = \underline{u}(0), \quad \dot{\underline{u}} = \dot{\underline{u}}(0), \quad t = 0$$

$$\underline{u}(t) = \sum_{r=1}^{N} \underline{\phi}_r q_r(t), \quad \ddot{\underline{u}}(t) = \sum_{r=1}^{N} \underline{\phi}_r \ddot{q}_r(t)$$

$$\underset{\sim}{m}\sum_{r=1}^{N} \underline{\phi}_r \ddot{q}_r(t) + \underset{\sim}{k}\sum_{r=1}^{N} \underline{\phi}_r q_r(t) = 0$$

$$\underline{\phi}_n^T \underset{\sim}{m}\sum_{r=1}^{N} \underline{\phi}_r \ddot{q}_r(t) + \underline{\phi}_n^T \underset{\sim}{k}\sum_{r=1}^{N} \underline{\phi}_r q_r(t) = 0$$

$$(\underline{\phi}_n^T \underset{\sim}{m} \underline{\phi}_n)\ddot{q}_n(t) + (\underline{\phi}_n^T \underset{\sim}{k} \underline{\phi}_n) q_n(t) = 0$$

$$\boxed{M_n \ddot{q}_n(t) + K_n q_n(t) = 0}$$

$$\ddot{q}_n(t) + \left(\frac{K_n}{M_n}\right) q_n(t) = 0$$

$$\boxed{\ddot{q}_n(t) + \omega_n^2 q_n(t) = 0}$$

초기조건

$$\underline{u}(0) = \sum_{r=1}^{N} \underline{\phi}_r\, q_r(0), \quad \underline{\dot{u}}(0) = \sum_{r=1}^{N} \underline{\phi}_r\, \dot{q}_r(0)$$

$$\underline{\phi}_n^T \underset{\sim}{m}\, \underline{u}(0) = \sum_{r=1}^{N} \underline{\phi}_n^T \underset{\sim}{m}\, \underline{\phi}_r\, q_r(0)$$

$$q_n(0) = \frac{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{u}(0)}{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\phi}_n}$$

$$\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\dot{u}}(0) = \sum_{r=1}^{N} \underline{\phi}_n^T \underset{\sim}{m}\, \underline{\phi}_r\, \dot{q}_r(0)$$

$$\dot{q}_n(0) = \frac{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\dot{u}}(0)}{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\phi}_n}$$

Decoupled normal coordinate equations

$$M_n \ddot{q}_n(t) + K_n q_n(t) = 0$$

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = 0$$

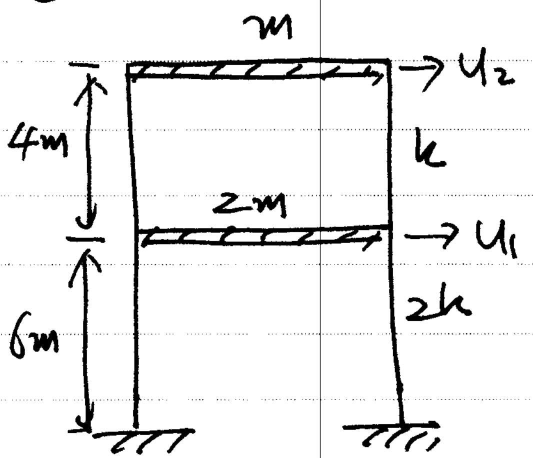
$$\left\{ \begin{aligned} q_n(0) &= \frac{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{u}(0)}{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\phi}_n} \\ \dot{q}_n(0) &= \frac{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\dot{u}}(0)}{\underline{\phi}_n^T \underset{\sim}{m}\, \underline{\phi}_n} \end{aligned} \right.$$

(Solution)

$$q_n(t) = q_n(0)\cos\omega_n t + \frac{1}{\omega_n}\dot{q}_n(0)\sin\omega_n t$$

$$\underline{u}(t) = \sum_{n=1}^{N} \phi_n q_n(t)$$

(예)



$m = 98\times10^3$ kg

$k = 720\times10^3$ kg-f/m

초기조건

$u_1(0) = 0.1$ m

$u_2(0) = 0.1$ m

$\dot{u}_1(0) = \dot{u}_2(0) = 0$

$$\underset{\sim}{m} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}, \quad \underset{\sim}{k} = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}$$

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

힘의 단위로서 ton-f 를 사용한다

$$\underset{\sim}{m} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}, \quad \underset{\sim}{k} = \begin{bmatrix} 3\times720 & -720 \\ -720 & 720 \end{bmatrix}$$

힘의 단위로서 k-Newton-f 를 사용한다

$$\underset{\sim}{m} = \begin{bmatrix} 2\times9.8 & 0 \\ 0 & 9.8 \end{bmatrix}, \quad \underset{\sim}{k} = \begin{bmatrix} 3\times720\times9.8 & -720\times9.8 \\ -720\times9.8 & 720\times9.8 \end{bmatrix}$$

지배방정식

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 3\times72 & -72 \\ -72 & 72 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

특성 방정식

$$\left| \underset{\sim}{k} - \overset{\lambda}{\omega^2} \underset{\sim}{m} \right| = 0$$

$$\begin{vmatrix} 3\times72 - 2\lambda & -72 \\ -72 & 72-\lambda \end{vmatrix} = 0$$

$$(72-\lambda)(3\times72 - 2\lambda) - 72^2 = 0$$

$$(2\lambda - 72)(\lambda - 2\times72) = 0$$

$$\lambda_1 = 36, \quad \lambda_2 = 144$$

17/60

$$\omega_1 = \sqrt{\lambda_1} = 6 \quad , \quad \omega_2 = \sqrt{\lambda_2} = 12$$

$$f_1 = 6/2\pi \ Hz, \quad f_2 = (12/2\pi)\ Hz$$

$$T_1 = \frac{2\pi}{6} \ sec, \quad T_2 = \frac{\pi}{6} \ sec$$

ω_1 에 대응하는 특성벡터, 고유벡터 $\underline{\phi}_1$

$$\begin{pmatrix} 3\times 72 - 2\lambda_1 & -72 \\ -72 & 72-\lambda_1 \end{pmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$(-72)\phi_{11} + (72-36)\phi_{21} = 0$$

$$\phi_{21} = 1, \quad \phi_{11} = \frac{1}{2}$$

$$\{\underline{\phi}_1\} = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix}$$

ω_2 에 대응하는 고유벡터 $\underline{\phi}_2$

$$\begin{pmatrix} 3\times 72 - 2\lambda_2 & -72 \\ -72 & 72-\lambda_2 \end{pmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

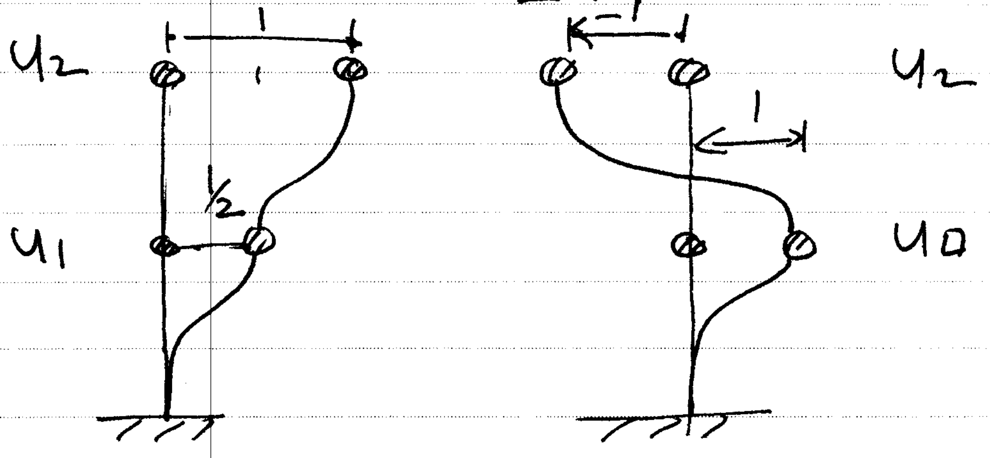
$$(-72)\phi_{12} + (72-144)\phi_{22} = 0$$

$$\phi_{12} = 1, \quad \phi_{22} = -1$$

$$\underline{\phi}_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

18/10

$$\Phi = [\phi_1, \phi_2] = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix}$$



제1모드　　　　제2모드

$$M_1 = \phi_1^T \underset{\sim}{m} \phi_1 = (1/2 \;\; 1)\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}$$

$$= (2)(\tfrac{1}{2})^2 + (1)(1)^2 = 1.5$$

$$M_2 = \phi_2^T \underset{\sim}{m} \phi_2 = (1 \;\; -1)\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= (2)(1)^2 + (1)(-1)^2 = 3.0$$

$$K_1 = \omega_1^2 M_1 = (36)(1.5)$$

$$K_2 = \omega_2^2 M_2 = (144)(3.0)$$

초기조건

$$\underline{\phi}_1^T \underset{\sim}{M} \underline{u}(0) = \left(\tfrac{1}{2} \quad 1\right) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0.1 \end{Bmatrix}$$

$$= 0.2$$

$$\underline{\phi}_2^T \underset{\sim}{M} \underline{u}(0) = \left(1 \quad -1\right) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0.1 \end{Bmatrix}$$

$$= 0.1$$

$$q_1(0) = \frac{\underline{\phi}_1^T \underset{\sim}{M} \underline{u}(0)}{M_1} = \frac{0.2}{1.5} = \frac{2}{15}$$

$$q_2(0) = \frac{\underline{\phi}_2^T \underset{\sim}{M} \underline{u}(0)}{M_2} = \frac{0.1}{3} = \frac{1}{30}$$

Decoupled equation

$$\left.\begin{array}{l} \ddot{q}_1(t) + 36\, q_1(t) = 0 \\ q_1(0) = \frac{2}{15}, \quad \dot{q}_1(0) = 0 \end{array}\right) \text{모드 1}$$

$$\left.\begin{array}{l} \ddot{q}_2(t) + 144\, q_2(t) = 0 \\ q_2(0) = \frac{1}{30}, \quad \dot{q}_2(0) = 0 \end{array}\right) \text{모드 2}$$

$$q_1(t) = q_1(0)\cos 6t + \frac{1}{6}\dot{q}_1(0)\sin 6t$$

$$= \left(\frac{2}{15}\right)\cos 6t$$

$$q_2(t) = q_2(0)\cos 12t + \frac{1}{12}\dot{q}_2(0)\sin 12t$$

$$= \left(\frac{1}{30}\right)\cos 12t$$

$$\underline{u}(t) = \sum_{n=1}^{2} \underline{\phi}_n q_n(t) = \sum_{n=1}^{2} \underline{u}_n$$

$$= \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}\left(\frac{2}{15}\right)\cos 6t + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\left(\frac{1}{30}\right)\cos 12t$$

$$= \begin{Bmatrix} \frac{1}{15}\cos 6t + \frac{1}{30}\cos 12t \\ \frac{2}{15}\cos 6t - \frac{1}{30}\cos 12t \end{Bmatrix}$$

등가 정적하중

$$(f_s) = (f_s)_1 + (f_s)_2$$

$$(f_s)_1 = \underline{k}\,\underline{u}_1 = \omega_1^2 \underline{M}\,\underline{\phi}_1 q_1(t)$$

$$(f_s)_2 = \underset{\sim}{k}\,\underline{u}_2 = \omega_2^2 \underset{\sim}{M}\,\underline{\phi}_2\, q_2(t)$$

$$(\underline{f_s})_1 = (36)\begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}\begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix}\left(\frac{2}{15}\right)\cos 6t$$

$$= \begin{Bmatrix} 48 \\ 48 \end{Bmatrix}\cos 6t \quad (\text{ton-f})$$

$$(\underline{f_s})_2 = (144)\begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\left(\frac{1}{30}\right)\cos 12t$$

$$= \begin{Bmatrix} 96 \\ -48 \end{Bmatrix}\cos 12t \quad (\text{ton-f})$$

$t=0$ 에서

$$(\underline{f_s})_1 = \begin{Bmatrix} 48 \\ 48 \end{Bmatrix}, \quad (\underline{f_s})_2 = \begin{Bmatrix} 96 \\ -48 \end{Bmatrix}$$

$$(\underline{f_s}) = (\underline{f_s})_1 + (\underline{f_s})_2 = \begin{Bmatrix} 144 \\ 0 \end{Bmatrix}, \ \text{ton-f}$$

Check)

$2k = 2\times 720$ ton-f/m

$(0.1)\times(2\times 720 \text{ ton-f}) = 144$ ton-f

10.9 감쇠를 가진 시스템의 자유진동

$$\underset{\sim}{m}\ddot{\underset{\sim}{u}} + \underset{\sim}{c}\dot{\underset{\sim}{u}} + \underset{\sim}{k}\underset{\sim}{u} = 0$$

At $t=0$, $u = u(0)$, $\dot{u} = \dot{u}(0)$

$$\underset{\sim}{u} = \underset{\sim}{\Phi}\underset{\sim}{q} = \sum_{n=1}^{N} \phi_n q_n$$

$\underset{\sim}{\Phi}$: 모드행렬, 무감쇠시스템

$$\underset{\sim}{q} = \begin{Bmatrix} q_1 \\ \vdots \\ q_N \end{Bmatrix}$$

$$\underset{\sim}{m}\underset{\sim}{\Phi}\ddot{\underset{\sim}{q}} + \underset{\sim}{c}\underset{\sim}{\Phi}\dot{\underset{\sim}{q}} + \underset{\sim}{k}\underset{\sim}{\Phi}\underset{\sim}{q} = \underset{\sim}{0}$$

$$\underbrace{\underset{\sim}{\Phi}^T\underset{\sim}{m}\underset{\sim}{\Phi}}_{\underset{\sim}{M}}\ddot{\underset{\sim}{q}} + \underbrace{\underset{\sim}{\Phi}^T\underset{\sim}{c}\underset{\sim}{\Phi}}_{\underset{\sim}{C}}\dot{\underset{\sim}{q}} + \underbrace{\underset{\sim}{\Phi}^T\underset{\sim}{k}\underset{\sim}{\Phi}}_{\underset{\sim}{K}}\underset{\sim}{q} = 0$$

$$\underset{\sim}{M}\ddot{\underset{\sim}{q}} + \underset{\sim}{C}\dot{\underset{\sim}{q}} + \underset{\sim}{K}\underset{\sim}{q} = 0$$

$\underset{\sim}{M}$, $\underset{\sim}{K}$: 대각행렬.

$\underset{\sim}{C}$ 이 대각행렬 : 고전적 감쇠, q_n 에 대한 N 개의 비연계 미분방정식

$\underset{\sim}{C}$ 가 비대각행렬 : 비고전적 감쇠, q_n 에 대해서 연계된 미분방정식

23/10

고전적 감쇠시스템

NDOF system의 경우 N개의 비연계 미분 방정식

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = 0$$

$$M_n = \underline{\phi}_n^T \underline{m}\, \underline{\phi}_n, \quad K_n = \underline{\phi}_n^T \underline{k}\, \underline{\phi}_n$$

$$C_n = \underline{\phi}_n^T \underline{c}\, \underline{\phi}_n$$

$$\ddot{q}_n + \frac{C_n}{M_n}\dot{q}_n + \frac{K_n}{M_n} q_n = 0$$

$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = 0$$

$$\zeta_n = \frac{C_n}{2M_n\omega_n}$$

$\uparrow$

nth mode 감쇠비

10.10 고전적 감쇠시스템의 자유진동

$$\omega_{nD} = \omega_n\sqrt{1-\zeta_n^2}$$

$$q_n(t) = e^{-\zeta_n\omega_n t}\left[q_n(0)\cos\omega_{nD} t + \frac{\dot{q}_n(0) + \zeta_n\omega_n q_n(0)}{\omega_{nD}} \times \sin\omega_{nD} t\right]$$

24/10

$$\underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n q_n(t) = \sum_{n=1}^{N} \underline{u}_n$$

$$\underline{u}_n = \underline{\phi}_n q_n(t)$$

10.1 고유값 문제의 해법

$$\underline{k}\,\underline{\phi} = \lambda\,\underline{m}\,\underline{\phi}$$

$$p(\lambda) = \det(\underline{k} - \lambda \underline{m}) = 0$$

(1) Vector Iteration (벡터 반복법)
: $\underline{k}\underline{\phi} = \lambda \underline{m}\underline{\phi}$ 이용

(2) 변환법 : 모드의 직교성 이용

(3) 다항식 반복법 : $p(\lambda) = 0$ 이용

✓ 모든 고유값 풀이법은 반복적

✓ 고유쌍 $(\lambda_n, \underline{\phi}_n)$

∘ 반복법에 의한 λ_n 계산시

$$(\underline{k} - \lambda_n \underline{m})\underline{\phi}_n = 0$$

$\underline{\phi}_n$은 해석적으로 계산

○ 반복법에 의한 ϕ_n 계산

$$\lambda_n = \frac{\phi_n^T K \phi_n}{\phi_n^T M \phi_n} \quad \text{(Rayleigh quotient)}$$

λ_n은 Rayleigh 지수를 이용해서 계산

✓ 구조공학분야: 수개의 저차모드가 지배적

○ 역벡터 반복법이 효과적

○ 부분공간 반복법과 Lanczos 방법은

이 책에서는 취급하지 않는다.

○ 변환법과 다항식 반복기법으로 해석

10.12. Rayleigh quotient (레일리 지수)

$$\phi^T K \phi = \lambda \phi^T M \phi$$

$$\lambda = \frac{\phi^T K \phi}{\phi^T M \phi}$$

: Rayleigh quotient.

1. $\phi = \phi_n$, $\lambda = \lambda_n$

26/10

2. 모드는 ω_n에 대한 1계 ~~오~~무한소 오차근사

→ λ는 λ_n에 대한 2계 무한소 오차근사

→ 레일리 지수는 고유벡터 근처에서 stationary

3. $$\underset{\omega_1^2}{\lambda_1} \le \underset{\text{Rayleigh 지수}}{\lambda} \le \underset{\omega_N^2}{\lambda_N}$$

10.13. Vector Iteration method

the fundamental frequency & mode

$\boxed{1st}$ $\underline{X}_1$ = Assumed mode shape

$\underline{R}_1 = \underline{m}\,\underline{X}_1$

$\underline{k}\,\overline{\underline{X}}_2 = \underline{R}_1$ solve for $\overline{X}_2$

~~normalize $\overline{X}_2$~~

$$\cancel{\underline{X}_2 = \frac{\overline{X}_2}{(\overline{X}_2^T \underline{m}\,\overline{X}_2)^{1/2}}}$$

$$\lambda^{(2)} = \frac{\overline{X}_2\,\underline{k}\,\overline{X}_2}{\overline{X}_2^T\,\underline{m}\,\overline{X}_2} = \frac{\overline{X}_2\,\underline{m}\,\underline{X}_1}{\overline{X}_2^T\,\underline{m}\,\overline{X}_2}$$

$$\frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}} \le \text{허용오차}(?)$$

수렴기준이 만족되지 않으면 정규화

$$\underline{X}_2 = \frac{\overline{X}_2}{(\overline{X}_2^T \underset{\sim}{m} \overline{X}_2)^{1/2}}$$

$\boxed{2nd}$

$$R_2 = \underset{\sim}{m} \underline{X}_2$$
$$\underset{\sim}{k} \overline{X}_3 = R_2 \longrightarrow \overline{X}_3$$

$$\lambda^{(3)} = \frac{\overline{X}_3^T k \overline{X}_3}{\overline{X}_3^T \underset{\sim}{m} \overline{X}_3} = \frac{\overline{X}_3^T \cancel{m} \underline{X}_3}{\overline{X}_3^T m \overline{X}_3}$$

$$\frac{\lambda^{(3)} - \lambda^{(2)}}{\lambda^{(3)}} \le \text{허용오차}(?)$$

수렴기준 불만족시 정규화

$$\underline{X}_3 = \frac{\overline{X}_3}{(\overline{X}_3^T \underset{\sim}{m} \overline{X}_3)^{1/2}}$$

$\boxed{3rd}$

⋮

$\boxed{jth}$

$$R_j = \underset{\sim}{m} \underline{X}_j$$

$$\underset{\sim}{k} X_{j+1} = R_j \longrightarrow X_{j+1}$$