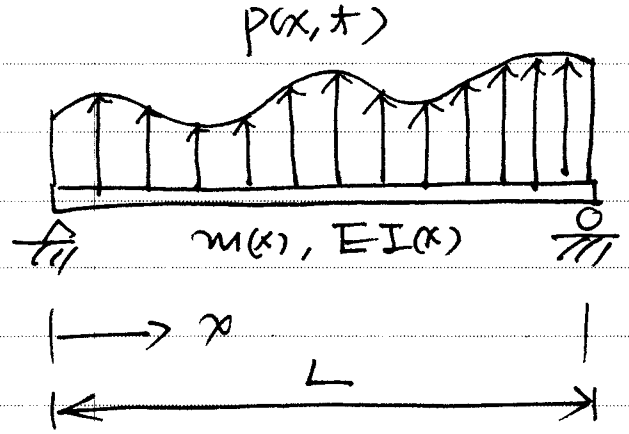
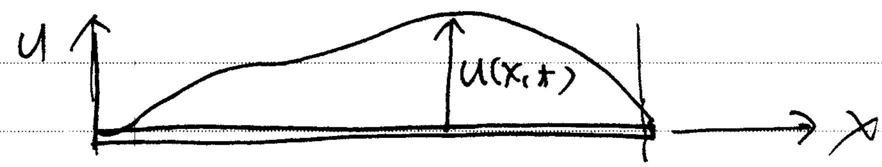
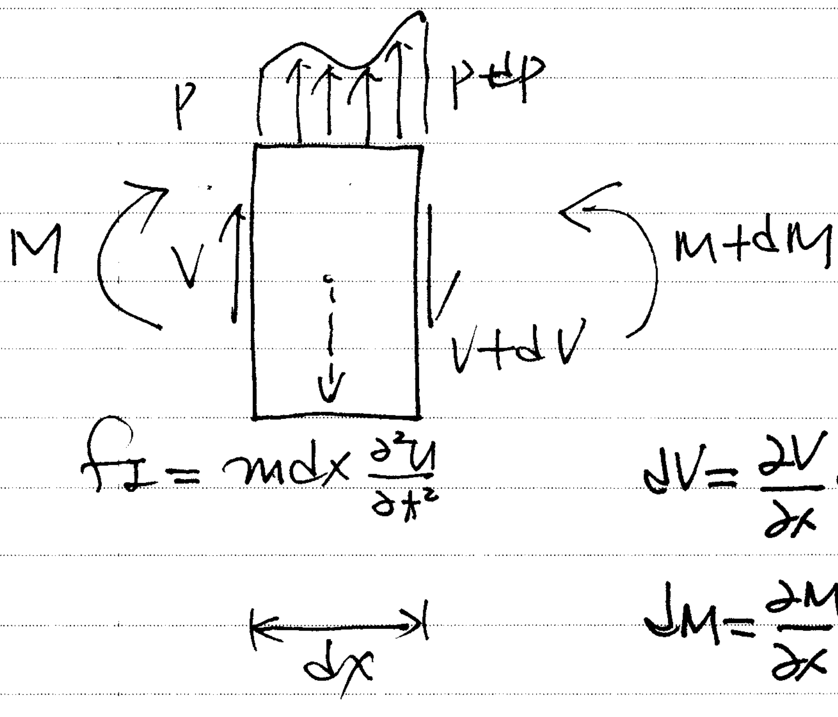
제 16 장 분포질량을 가진 탄성시스템

16.1. 보의 부감쇠 운동 방정식







$$f_I = m\,dx\,\frac{\partial^2 u}{\partial t^2}$$

$$dV = \frac{\partial V}{\partial x}dx$$

$$dM = \frac{\partial M}{\partial x}dx$$

y축(연직)방향에 대한 평형방정식

$$V-(V+dV)+p\,dx+\frac{dp}{2}dx-m\,dx\frac{\partial^2 u}{\partial t^2}=0$$

$$-dV+p\,dx-m\frac{\partial^2 u}{\partial t^2}dx=0$$

$$\frac{\partial V}{\partial x}=p-m\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

모멘트 평형방정식

$$-M+(M+dM)-V\,dx-\frac{dV}{2}dx=0$$

$$dM-V\,dx=0$$

$$\frac{\partial M}{\partial x}dx-V\,dx=0$$

$$V=\frac{\partial M}{\partial x} \qquad (2)$$

$$\frac{\partial V}{\partial x}=\frac{\partial^2 M}{\partial x^2} \qquad (3)$$

(3) $\Rightarrow$ (1)

$$\frac{\partial^2 M}{\partial x^2}+m\frac{\partial^2 u}{\partial t^2}=p \qquad (4)$$

Moment-Curvature relation

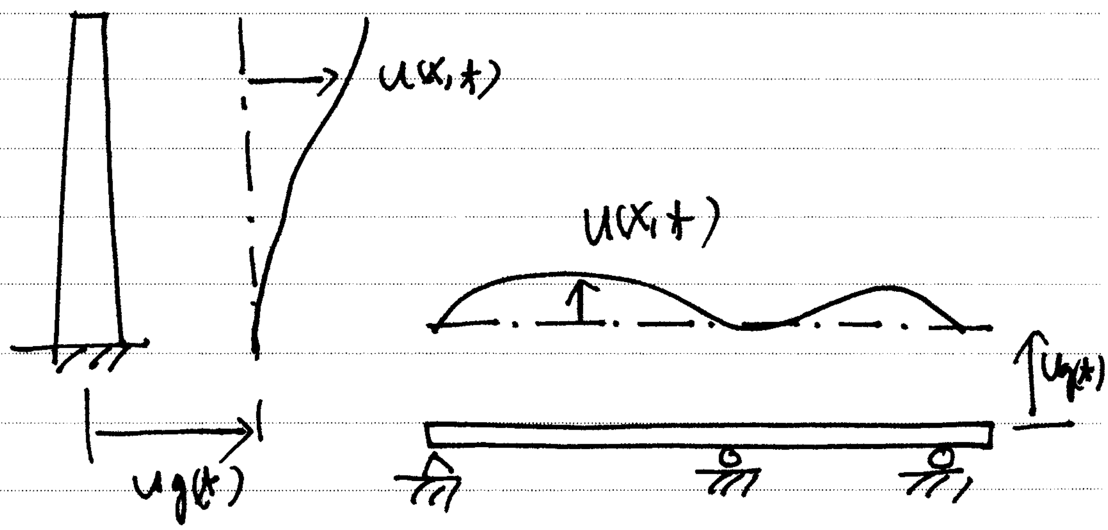
$$M = EI(x)\frac{\partial^2 y}{\partial x^2} \quad (5)$$

(5) → (4)

$$m\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = p(x,t) \quad (6)$$

16.2. 지점가진

$$u^t(x,t) = u(x,t) + u_g(t)$$



$$\frac{\partial V}{\partial x} = -m\frac{\partial^2 u^t}{\partial t^2} = -m\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u_g}{\partial t^2}\right)$$

$$= -m\frac{\partial^2 u}{\partial t^2} - m\frac{d^2 u_g}{dt^2} \qquad (17)$$

(17) → (1) → (3) → (4) → (6)

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = -m(x)\ddot{u}_g(t) \qquad (8)$$

$$P_{eff} = -m(x)\ddot{u}_g(t) \qquad (9)$$

16.3 고유진동수와 진동모드

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = 0$$

→ 공간의존

$$u(x,t) = \phi(x)\, q(t)$$

→ 시간의존

↑ separation of variables

$$\dot{u}(x,t) = \phi(x)\dot{q}(t), \quad \ddot{u}(x,t) = \phi(x)\ddot{q}(t)$$

$$\frac{\partial^2 y}{\partial t^2} = \phi(x)\,\ddot{q}(t)$$

$$\frac{\partial^2 y}{\partial x^2} = \phi''(x)\,q(t)$$

$$m(x)\,\phi(x)\,\ddot{q}(t) + \frac{\partial^2}{\partial x^2}\left[EI(x)\,\phi''(x)\,q(t)\right] = 0$$

$$m(x)\,\phi(x)\,\ddot{q}(t) + q(t)\left[EI(x)\,\phi''(x)\right]'' = 0$$

$$-\frac{\ddot{q}(t)}{q(t)} = \frac{\left[EI(x)\,\phi''(x)\right]''}{m(x)\,\phi(x)} = \lambda\ (=\omega^2)$$

$$\ddot{q}(t) + \lambda q(t) = 0$$

$$\left[EI(x)\,\phi''(x)\right]'' - \lambda\, m(x)\,\phi(x) = 0$$

Prismatic Beam

$$\ddot{q}(t) + \omega^2 q(t) = 0$$

$$EI\,\phi^{IV} - \omega^2 m\phi = 0$$

$$\phi^{IV} - \beta^4\phi = 0, \quad \beta^4 = \frac{\omega^2 m}{EI}$$

일반해

$$\phi^{IV} - \beta^4 \phi = 0$$

$$\phi(x) = Ae^{rx}, \quad \phi^{IV} = Ar^4 e^{rx}$$

$$Ae^{rx}(r^4 - \beta^4) = 0$$

항상 성립하기 위해서는

$$r^4 - \beta^4 = 0$$

$$(r^2+\beta^2)(r^2-\beta^2) = 0$$

$$(r+i\beta)(r-i\beta)(r+\beta)(r-\beta) = 0$$

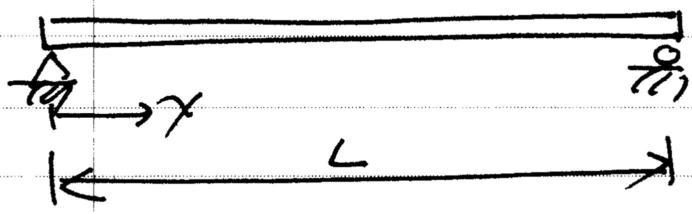
$$r = \beta, -\beta, i\beta, -i\beta$$

$$\underbrace{e^{\beta x}, e^{-\beta x}}_{\sinh\beta x,\ \cosh\beta x}, \quad \underbrace{e^{i\beta}, e^{-i\beta}}_{\sin\beta x,\ \cos\beta x}$$

$$\phi(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

$$\phi''(x) = -\beta^2 C_1 \sin\beta x - \beta^2 C_2 \cos\beta x + \beta^2 C_3 \sinh\beta x + \beta^2 C_4 \cosh\beta x$$

단순보



$u(0,t)=0$ $\qquad$ $u(L,t)=0$

$M(0,t)=0$ $\qquad$ $M(L,t)=0$

$\phi(0)=0$ $\qquad$ $\phi(L)=0$

$\phi''(0)=0$ $\qquad$ $\phi''(L)=0$

$$\phi(0)=0 = C_2 + C_4 = 0$$

$$\phi''(0) = \beta^2(-C_2 + C_4) = 0$$

$$c \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$C_2 = C_4 = 0$$

$$\phi(x) = C_1 \sin \beta x + C_3 \sinh \beta x$$

$$\phi(L) = C_1 \sin \beta L + C_3 \sinh \beta L = 0$$

$$\phi''(L) = \beta^2(-C_1 \sin \beta L + C_3 \sinh \beta L) = 0$$

$$C_3 \sinh \beta L = 0$$

$$\phi(x) = C_1 \sin \beta x, \quad \phi(L) = C_1 \sin \beta L = 0$$

$$\beta L = n\pi, \quad n = 1, 2, 3, \cdots$$

$$\beta = \frac{n\pi}{L}, \quad \beta^2 = \frac{n^2\pi^2}{L^2}$$

$$\omega^2 = \beta^4 \frac{EI}{m}, \quad \omega = \beta^2 \sqrt{\frac{EI}{m}}$$

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

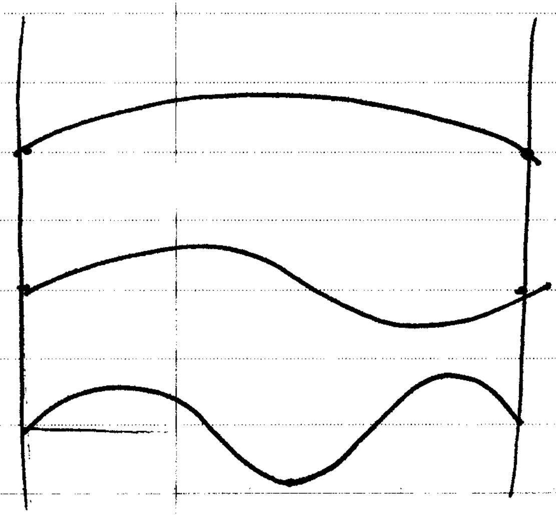
↑

nth mode의 고유진동수

$$\phi_n(x) = C_1 \sin \frac{n\pi}{L} x$$

↑

nth mode의 고유진동모드



$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

$$\omega_2 = \frac{4\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

$$\omega_3 = \frac{9\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

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16.4 모드의 직교성

변수분리후

$$\left[EI(x)\phi_r''(x)\right]'' = \omega_r^2 m(x)\phi_r(x)$$

$$\int_0^L \phi_n(x)\left[EI(x)\phi_r''(x)\right]'' dx = \omega_r^2 \int_0^L m(x)\phi_n(x)\phi_r(x)\,dx$$

$$\text{좌변} = \int_0^L \phi_n(x)\left[EI(x)\phi_r''(x)\right]'' dx$$

$$= \left\{\phi_n(x)\left[EI(x)\phi_r''(x)\right]'\right\}_0^L - \int_0^L \phi_n'(x)\left[EI(x)\phi_r''(x)\right]' dx$$

$$= \left\{\phi_n(x)\left[EI(x)\phi_r''(x)\right]'\right\}_0^L - \left\{\phi_n'(x)\left[EI(x)\phi_r''(x)\right]\right\}_0^L$$

$$+ \int_0^L \phi_n''(x)\left[EI(x)\phi_r''(x)\right] dx$$

$$EI(x)\phi''(x)$$

Boundary condition에 의해

$$\left\{\cdots\right\}_0^L = 0$$

$$\int_0^L \phi_n(x)\left[EI(x)\phi_r''(x)\right]''dx = \int_0^L \phi_n''(x)\left[EI(x)\phi_r''(x)\right]dx$$

마찬가지 절차에 따라서

$$\int_0^L \phi_r(x)\left[EI(x)\phi_n''(x)\right]''dx = \int_0^L \phi_r''(x)\left[EI(x)\phi_n''(x)\right]dx$$

그러므로

$$\omega_r^2\int_0^L m(x)\phi_n(x)\phi_r(x)\,dx = \omega_n^2\int_0^L m(x)\phi_r(x)\phi_n(x)\,dx$$

만약 $(\omega_r^2-\omega_n^2)\neq 0$ 이면

$$\int_0^L m(x)\phi_r(x)\phi_n(x)\,dx = 0 \quad \text{if } r\neq n$$

$$\int_0^L \phi_n(x)\left[EI(x)\phi_r''(x)\right]''dx = 0 \quad \text{if } r\neq n$$

다중근을 가질경우에도 직교조건을 만족하는 특성함수 $\phi_n(x)$가 존재한다.

16.5 강제 진동의 모드해석

구가 (0, L) 에서 임의의 함수는 특성함수 $\phi_n(x)$의 선형조합으로 표현할 수 있다.

$$u(x,t)=\sum_{r=1}^{\infty}\phi_r(x)\,q_r(t)$$

식(1)에 대입한다.

$$m\sum_{r=1}^{\infty}\phi_r(x)\,\ddot{q}_r(t)+\frac{\partial^2}{\partial x^2}\left[EI(x)\sum_{r=1}^{\infty}\phi_r''(x)\,q_r(t)\right]=p(x,t)$$

$$\sum_{r=1}^{\infty}m(x)\phi_r(x)\,\ddot{q}_r(t)+\sum_{r=1}^{\infty}\left[EI(x)\phi_r''(x)\right]''q_r(t)=p(x,t)$$

여기서 직교성을 이용한다.

$$\int_0^L\phi_n(x)\sum_{r=1}^{\infty}m(x)\phi_r(x)\,\ddot{q}_r(t)\,dx+\int_0^L\phi_n(x)\sum\left[EI(x)\phi_r''(x)\right]''q_r(t)\,dx$$

$$=\int_0^L\phi_n(x)\,p(x,t)\,dx$$

$$\int_0^L\phi_n(x)m(x)\phi_n(x)\,\ddot{q}_n(t)\,dx+\int_0^L\phi_n(x)\left[EI(x)\phi_n''(x)\right]''q_n(t)\,dx$$

$$= \int_0^L \phi_n(x)\, p(x,t)\, dx$$

$$\left(\int_0^L \phi_n^2(x)\, m(x)\, dx\right)\ddot{q}_n(t) + \left(\int_0^L \phi_n(x)\left[EI(x)\phi_n''(x)\right]'' dx\right) q_n(t)$$

$$= \int_0^L \phi_n(x)\, p(x,t)\, dx$$

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t)$$

여기서 $\int_0^L \phi_n(x)\left[EI(x)\phi_n''(x)\right]'' dx$

$$= \int_0^L \phi_n''(x)\left[EI\,\phi_n''(x)\right] dx$$

따라서

$$K_n = \int_0^L EI(x)\left(\phi_n''(x)\right)^2 dx$$

$q_n(t)$를 구하면

$$u_n(x,t) = \phi_n(x)\, q_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\, q_n(t)$$

$$M(x) = EI(x)u''(x)$$

$$V(x) = \frac{dM}{dx} = [EI(x)u''(x)]'$$

$$M_n(x,t) = EI(x)\phi_n''(x)q_n(t)$$

$$V_n(x,t) = [EI(x)\phi_n''(x)]'q_n(t)$$

$$M(x,t) = \sum_{n=1}^{\infty} M_n(x,t) = \sum_{n=1}^{\infty} EI(x)\phi_n''(x)q_n(t)$$

$$V(x,t) = \sum_{n=1}^{\infty} V_n(x,t) = \sum_{n=1}^{\infty} [EI(x)\phi_n''(x)]'q_n(t)$$