Functions of a Random Variable

- $Y = g(X)$,

  where $g(x)$ is a real-valued function defined on the real line.

  X and Y are random variables

- $P[Y \text{ in } C] = P[g(X) \text{ in } C] = P[X \text{ in } B]$

  where C and B are equivalent events.

- Three useful types of equivalent events
  - The event $\{g(X) = y_k\}$: the magnitude of the jump at a discontinuous point y_k of cdf
  - The event $\{g(X) \le y\}$: the cdf of Y
  - The event $\{y < g(X) \le y+h\}$: the pdf of Y

- Ex. 3.21
  - X: the number of active speakers in a group of N independent speakers
  - p: the probability that a speaker is active
  - M: # of transmittable voice signals at a time
  - If $X > M$, $X - M$ randomly selected signals are discarded.
    The number of signals discarded = r.v. Y

$$\therefore Y = (X - M)^+$$

$$\text{where } (x)^+ = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$$

- sol)

$$S_Y = \{0,1,\ldots,N-M\}$$

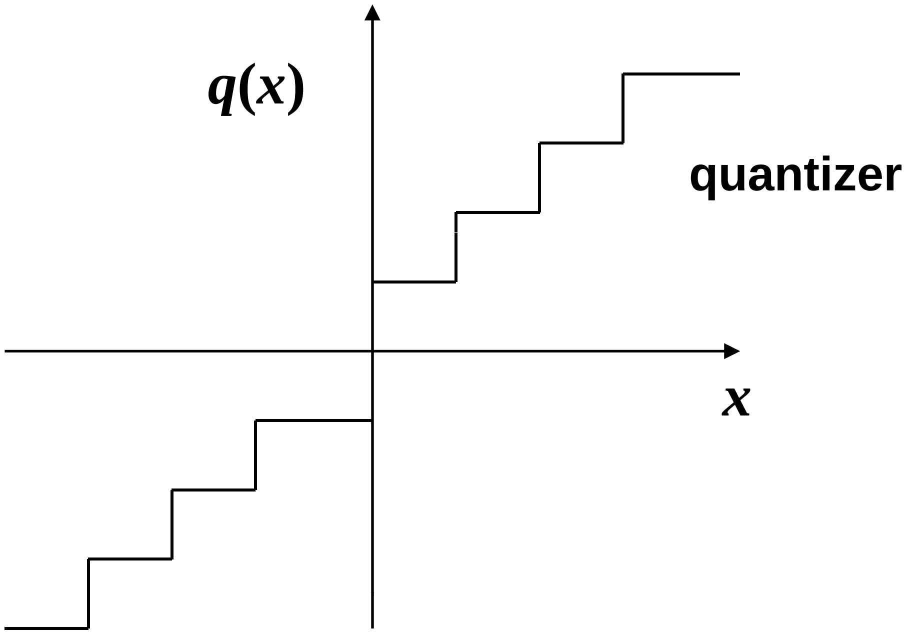
$$\rightarrow P[Y=0] = P[X \text{ in } \{0,1,\ldots,M\}] = \sum_{j=0}^{M} p_j$$

$$P[Y=k] = P[X = M+k] = p_{m+k} \qquad 0 < k \le N-M$$

cf) p_j : the pmf of a binomial random variable X.

- Ex. 3.22

A continuous random variable X can be mapped into a discrete random variable Y via a quantizer



- $Y = g(X)$ is constant during certain intervals and the pdf of X is nonzero in these interval

$\rightarrow$ Jump in the cdf of Y

$\rightarrow$ The pdf of Y contains delta functions

$\rightarrow$ Y : either discrete or mixed type

- Ex. 3.23
  - Let $Y = aX + b, \quad a \neq 0$
  - $F_X(x)$: the cdf of X
  - Find the cdf of Y : $F_Y(y)$

sol)

- The event $\{Y \leq y\}$ occurs when $A = \{aX + b \leq y\}$ occurs
- if $a > 0 \rightarrow A = \left\{X \leq \dfrac{(y-b)}{a}\right\}$

$$a < 0 \rightarrow A = \left\{X \geq \frac{(y-b)}{a}\right\}$$

$$\therefore F_Y(y) = P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right), \quad a > 0$$

$$F_Y(y) = P\left[X \ge \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right), \quad a < 0$$

- pdf $$f_Y(y) = \frac{dF}{dy} = \frac{dF}{du} \cdot \frac{du}{dy}$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right), \quad a > 0$$

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right), \quad a < 0$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

❖ Ex. 3.24

- X : a Gaussian random variable with mean m and standard deviation σ
- $Y = aX + b$: a linear function of X

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-(y-b-am)^2/2(a\sigma)^2}$$

- Mean $b + am$, standard deviation $|a|\sigma$
- A linear function of a Gaussian random variable is also a Gaussian random variable

- Ex. 3.25
  - $Y = X^2$, X : a continuous random variable
  - Find the cdf and pdf of Y

sol) $\{Y \le y\} \Rightarrow \{X^2 \le y\}$

$\Rightarrow \{-\sqrt{y} \le X \le \sqrt{y}\}$ for nonnegative y

cf) the event is null for $y < 0$

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \text{for } y > 0 \end{cases}$$

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

- Ex. 3.26

  X : a Gaussian random variable with $m = 0$ and $\sigma = 1$

  $\rightarrow$ A standard normal random variable

  $Y = X^2$

  Find the pdf of Y

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-(\sqrt{y})^2/2}}{2\sqrt{y}} + \frac{e^{-(-\sqrt{y})^2/2}}{2\sqrt{y}}\right)$$

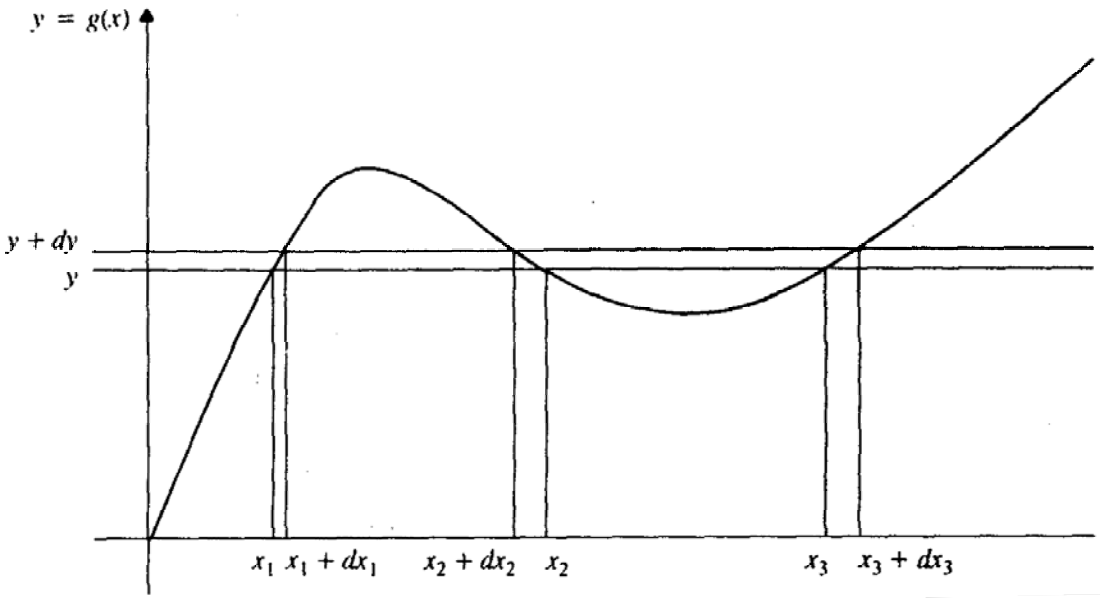
$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{y}} \cdot 2 \cdot e^{-y/2}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad \text{for } y \ge 0$$

- The pdf of a Chi-Square random variable with one degree of freedom:

$$f_X(x) = \frac{x^{(k-2)/2} e^{-x/2}}{2^{k/2}\Gamma(k/2)} \qquad \text{for } x > 0$$

❖ A nonlinear function $Y = g(X)$



① The event $C_y = \{y < Y < y + dy\}$

② Its equivalent event B_y

③ $g(x) = y$ has three solutions x_1, x_2, x_3

④ $B_y = \{x_1 < X < x_1 + dx_1\} \cup \{x_2 + dx_2 < X < x_2\} \cup \{x_3 < X < x_3 + dx_3\}$

⑤ $P[C_y] = f_Y(y)|dy|$

where $|dy|$ is the length of the interval $y < Y \le y + dy$

⑥ $P[B_y] = f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3|$

⑦ $P[C_y] = P[B_y]$

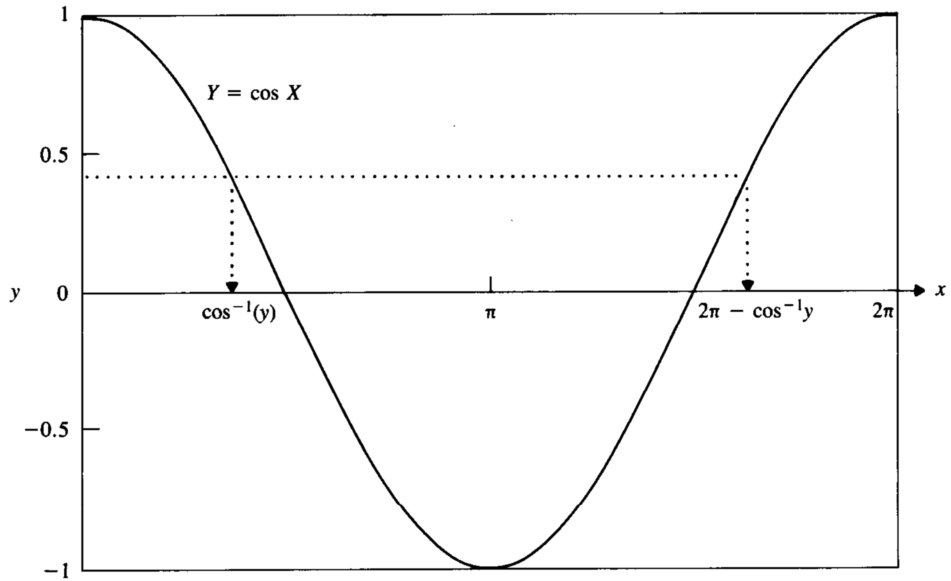
$$f_Y(y) = \sum_k \frac{f_X(x)}{|dy/dx|}\bigg|_{x=x_k}$$

$$= \sum_k f_X(x)\left|\frac{dx}{dy}\right|\bigg|_{x=x_k}$$

Ex. 3.28

- $Y = \cos(X)$
- X is uniformly distributed in the interval $(0,2\pi]$
- $Y = \cos(x)$ has two solutions for $-1 < y < 1$

$\rightarrow x_0 = \cos^{-1}(y), \qquad x_1 = 2\pi - x_0$



$$\left.\frac{dy}{dx}\right|_{x_0} = -\sin(x_0) = -\sin(\cos^{-1}(y)) = -\sqrt{1-y^2}$$

$$\left.\frac{dy}{dx}\right|_{x_1} = -\sin(2\pi - x_0) = +\sin(x_0) = +\sqrt{1-y^2}$$

$$f_X(x) = \frac{1}{2\pi}$$

$$\therefore\ f_Y(y) = \frac{1}{2\pi}\frac{1}{\sqrt{1-y^2}} + \frac{1}{2\pi}\frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\pi\sqrt{1-y^2}} \qquad \text{for } -1 < y < 1$$

$$F_Y(y) = \begin{cases} 0 & \text{for } \ y < -1 \\ \dfrac{1}{2} + \dfrac{\sin^{-1} y}{\pi} & \text{for } \ -1 \le \text{y} \le 1 \\ 1 & \text{for } \ y > 1 \end{cases}$$

cf) $F_Y(y) = \int_{-\infty}^{y} f_Y(y) dy$

But $f_Y(y) \neq 0$ for $-1 \le y \le 1$

$$
\begin{aligned}
F_Y(y) &= \int_{-\infty}^{y} f_Y(y)dy \\
&= \int_{-\infty}^{-1} f_Y(y)dy + \int_{-1}^{y} f_Y(y)dy \\
&= 0 + \int_{-1}^{y} f_Y(y)dy \\
&= \int_{-1}^{y} \frac{1}{\pi\sqrt{1-y^2}}dy \\
&= \left[\frac{1}{\pi}\sin^{-1} y\right]_{-1}^{y} = \frac{1}{2} + \frac{1}{\pi}\sin^{-1} y
\end{aligned}
$$

- H.W. 32, 34, 35, 37, 40, 42, 45, 50, 51, 54, 58, 59

3.6 Expected Value of Random Variables

- The expected value of X
  - Mean of X or the center of mass
  - Average of X

$$E[X] = \int_{-\infty}^{\infty} t f_X(t) dt \quad \text{for a continuous random variable } X$$

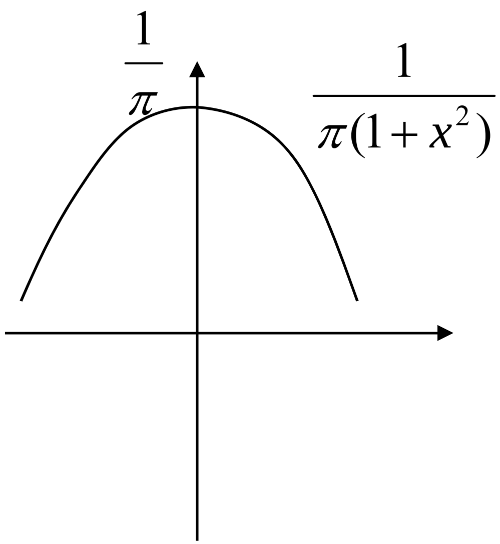
$$E[X] = \sum_k x_k p_X(x_k) \quad \text{for a discrete random variable } X$$

  - cf) Condition for the existence of the expected value (See Prob. 71 & 72 for examples of non-existence)

$$E[|X|] = \int_{-\infty}^{\infty} |t| f_X(t) dt < \infty$$

$$E[|X|] = \sum_k |x_k| p_X(x_k) < \infty$$

- cf) Cauchy Random Variable



$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

$$\int_{-\infty}^{\infty} \frac{|t|}{\pi(1+t^2)}dt = 2\int_0^{\infty} \frac{t}{\pi(1+t^2)}dt$$

$$= 2 \cdot \frac{1}{2\pi} \ln(1+t^2)\Big|_0^{\infty}$$

$$= \infty$$

- ✓ Note : Consider if the upper half plane converges or not.

- Then, why doesn't the expected value exist???

$$\int_{-\infty}^{\infty} \frac{t}{\pi(1+t^2)} dt = \int_{0}^{\infty} \frac{t}{\pi(1+t^2)} dt + \int_{-\infty}^{0} \frac{t}{\pi(1+t^2)} dt$$

$$= \int_{0}^{\infty} \frac{|t|}{\pi(1+t^2)} dt - \int_{0}^{\infty} \frac{|t|}{\pi(1+t^2)} dt$$

$$= \infty - \infty = ?$$

- cf) Gaussian

$$\int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_0^{\infty} x \cdot e^{-x^2/2\sigma^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} 2\left[-\sigma^2 e^{-x^2/2\sigma^2}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2\sigma < \infty$$

- When the pdf is symmetric about a point m, $E[X] = \mathrm{m}$

$$f_X(m-x) = f_X(m+x) \quad \text{for all} \quad x$$

$(m-t)$: odd symmetric about $t = m$

$f_X(t)$: symmetric about $t = m$

$\therefore\ (m-t)\, f_X(t)$: odd symmetric about $t = m$

$$0 = \int_{-\infty}^{\infty} (m-t) f_X(t)dt = m - \int_{-\infty}^{\infty} t f_X(t)dt$$

$$\therefore \int_{-\infty}^{\infty} t f_X(t)dt = m$$

- When X is a nonnegative random variable

$$E[X] = \int_0^\infty (1 - F_X(t))dt$$

$$E[X] = \sum_{k=0}^{\infty} P[X > k]$$

Proof)

① $E[X] = \int_0^{\infty} t f_X(t)dt$

$$= \lim_{x\to\infty} \int_0^{x} t f_X(t)dt$$

$$= \lim_{x\to\infty}\left[tF_X(t)\Big|_0^x - \int_0^x F_X(t)dt \right]$$

$$= \lim_{x\to\infty}\left[xF_X(x) - \int_0^x F_X(t)dt \right]$$

$$= \lim_{x\to\infty}\left[x(F_X(x) - 1) + \int_0^x (1 - F_X(t))dt \right]$$

$$= 0 + \int_0^{\infty} (1 - F_X(t))dt$$

② $$\sum_{k=0}^{\infty} P[X > k] = \sum_{k=0}^{\infty}\left(\sum_{j=k+1}^{\infty} P[X = j]\right)$$

$$= (P[X = 1] + P[X = 2] + \ldots) + (P[X = 2] + \ldots)$$

$$+ \big(P[X = 3] + \ldots\big) + \ldots$$

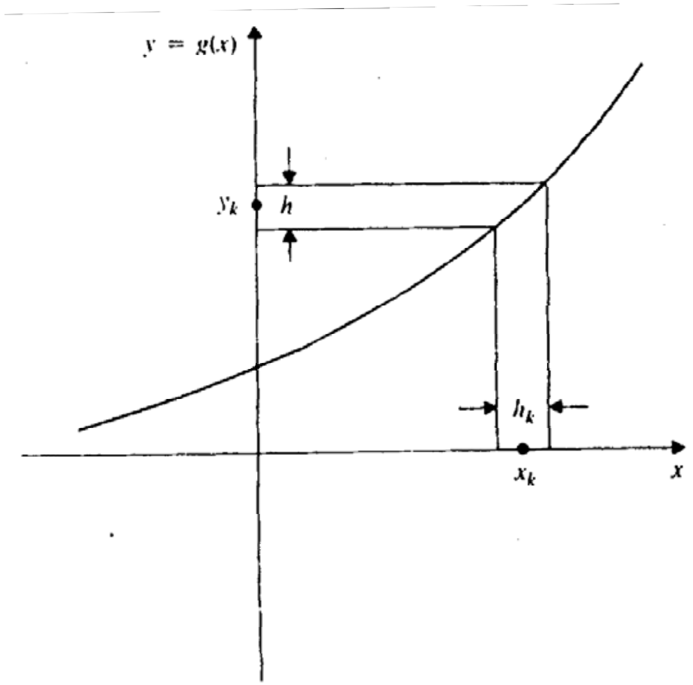
$$= P[X = 1] + 2P[X = 2] + 3P[X = 3] + \ldots$$

$$= \sum_{k=0}^{\infty} kP[X = k]$$

$$= E\big[X\big]$$

Expected Value of $Y = g(X)$

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$



$$E[Y] \cong \sum_k y_k f_Y(y_k) h$$

$\because y_k = g(x_k)$ and $f_Y(y_k) h = f_X(x_k) h_k$

$$\cong \sum_k g(x_k) f_X(x_k) h_k$$

Let $h \rightarrow 0$

then $E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

cf) $f_Y(y) = \sum_k f_X(x) \left| \frac{dx}{dy} \right| \Bigg|_{x=x_k}$

$$\int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} y \sum_k f_X(x) \left| \frac{dx}{dy} \right| \Bigg|_{x=x_k} dy = \int_{-\infty}^{\infty} g(x) \sum_k f_X(x) \left| \frac{dx}{dy} \right| \Bigg|_{x=x_k} dy$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

where $dy \rightarrow h,\ dx \rightarrow h_k,\ \left| \frac{dx}{dy} \right| dy \rightarrow \sum_k h_k$

Ex. 3.33

- $Y = a\cos(\omega t + \Theta)$, a, ω, : constants, t : variable
  Θ : r.v. in (0,2π)

$$E[Y] = E[a\cos(\omega t + \Theta)]$$

$$= \int_0^{2\pi} a\cos(\omega t + \theta) \cdot \frac{1}{2\pi} d\theta \qquad \Leftarrow \text{where} \quad f_\Theta(\theta) = \frac{1}{2\pi}$$

$$= -a\sin(\omega t + \theta)\Big|_0^{2\pi}$$

$$= 0$$

$$
\begin{aligned}
E[Y^2] &= E[a^2 \cos^2(wt+\Theta)] \\
&= E\left[\frac{a^2}{2} + \frac{a^2}{2}\cos(2wt+2\Theta)\right] \\
&= \frac{a^2}{2} + \frac{a^2}{2}\int_0^{2\pi} \cos(2wt+2\theta)\frac{1}{2\pi}d\theta \\
&= \frac{a^2}{2}
\end{aligned}
$$

❖ Note

$$E[c] = c$$

$$E[cX] = cE[X]$$

$$E[Y] = E\left[\sum_{k=1}^{n} g_k(X)\right]$$

$$= \sum_{k=1}^{n} E[g_k(X)]$$

$$E[X + c] = E[X] + c$$

Variance of X

- The extent of the rv's variation about its mean

  - $D = X - E[X]$: the deviation of X about its mean.
    $\rightarrow$ negative or positive

  - D^2 : always positive $\rightarrow$ information of variation of amplitude.

  - $E[D^2]$: the variance of the random variable

  - $\mathrm{VAR}[X] = \mathrm{E}[(X - E[X])^2]$: the variance of the r.v. X

- $\mathrm{STD}[X] = (\mathrm{VAR}[X])^{½}$: the standard deviation of the r.v. X

  → a measure of the "width" or "spread" of a distribution.

$$\begin{aligned}\mathrm{VAR}[X] &= E[X^2 - 2E[X]X + (E[X])^2] \\ &= E[X^2] - 2E[X]E[X] + (E[X])^2 \\ &= E[X^2] - (E[X])^2\end{aligned}$$

❖ Ex. 3.37

- Variance of Geometric Random Variable

$$\mathrm{VAR}[V] = E[N^2] - (E[N])^2$$

$$\sum_{k=1}^{\infty} k^2 \cdot pq^{k-1} - \sum_{k=1}^{\infty} k \cdot pq^{k-1} = p\sum_{k=1}^{\infty}(k^2 - k)q^{k-1}$$

$$= p\sum_{k=1}^{\infty} k(k-1)q^{k-1}$$

$$= p\left[\frac{d^2}{dq^2}\left(\sum_{k=1}^{\infty} q^k\right)\right] \cdot q$$

$$= pq\frac{d^2}{dq^2}\left(\sum_{k=1}^{\infty} q^k\right)$$

$$= pq\frac{d^2}{dq^2}\left(\frac{q}{1-q}\right) = pq\frac{d}{dq}\left(\frac{1}{(1-q)^2}\right)$$

$$= pq\frac{2}{(1-q)^3} = \frac{2q}{(1-q)^2}$$

$$E[N^2] - E[N] = \frac{2q}{(1-q)^2}, \qquad E[N] = \frac{1}{p}$$

$$E[N^2] = \frac{1+q}{p^2}$$

$$\mathrm{VAR}[N] = E[N^2] - (E[N])^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

- Note

$$\mathrm{VAR}[c] = 0$$
$$\mathrm{VAR}[X + c] = \mathrm{VAR}[X]$$
$$\mathrm{VAR}[cX] = c^2\mathrm{VAR}[X]$$

- Note The n<u>th</u> moment of the random variable X.

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(n)dx$$

- Note

  mean : The first moment

  variance : The second moment – the square of the first moment

3.7 The Markov and Chebyshev Inequalities

❖ Markov inequality

$$P[X \ge a] \le \frac{E[X]}{a}$$ → Bound → worst case analysis

for X nonnegative

➢ Proof)

$$E[X] = \int_0^{\infty} t f_X(t)dt = \int_0^{a} t f_X(t)dt + \int_a^{\infty} t f_X(t)dt$$

$$\ge \int_a^{\infty} t f_X(t)dt$$

$$\ge \int_a^{\infty} a f_X(t)dt = aP[X \ge a]$$

Chebyshev inequality

- $E[X] = m, \quad \mathrm{VAR}[X] = \sigma^2$

$$P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2}$$

- Proof) $\quad D^2 = (X - m)^2$

$$P[D^2 \geq a^2] \leq \frac{E[(X-m)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

- Note $\{D^2 \geq a^2\}$ and $\{|X - m| \geq a\}$ are equivalent events.

- cf) $\mathrm{VAR}[X] = 0$

$$\mathrm{P}[|X - m| \geq a] \leq 0$$
$$\longrightarrow \mathrm{P}[X = m] = 1$$

$\Rightarrow$ the random variable is equal its mean with probability one.

- Note

  Chebyshev inequality can give rather loose bounds.

3.9 Transform method

- The characteristic function

$$\Phi_X(\omega) = E[e^{j\omega X}] \leftarrow$$ Expected value of a function of X, e^{jwX}

$$= \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \leftarrow$$ Fourier transform of the pdf $f_X(x)$

- cf) $S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt$ "a reversal in the sign of the exponent"

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

- Every pdf and its characteristic function form a unique Fourier transform pair

- For a discrete random variable X

$$\Phi_X(\omega) = \sum_k p_X(x_k) e^{j\omega x_k}$$

discrete random variable

$$\Rightarrow \Phi_X(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

integer valued random variable

$\Rightarrow$ Fourier transform of the sequence $p_X(k)$

$$\Phi_X(\omega + 2\pi) = \Phi_X(\omega)$$

: periodic function of ω with period of 2π.

$$p_X(k) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_X(w)e^{-j\omega k}d\omega \qquad k = 0, \pm 1, \pm 2, \ldots$$

$\because$ $\Phi_X(\omega)$ is periodic $\rightarrow$ Fourier series expansion.

i.e., $p_X(k)$: the coefficients of the Fourier series of the periodic function $\Phi_X(\omega)$

❖ Moment theorem

$f_X(x)$ and $\Phi_X(\omega)$: a transform pair

$$E[X^n] = \frac{1}{j^n}\frac{d^n}{d\omega^n}\Phi_X(w)\bigg|_{\omega=0}$$

- Proof)

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\left\{1 + j\omega X + \frac{(j\omega X)^2}{2!} + \cdots\right\}dx$$

$$= 1 + j\omega E[X] + \frac{(j\omega)^2 E[X^2]}{2!} + \cdots + \frac{(j\omega)^n E[X^n]}{n!} + \cdots$$

$$\left.\frac{d}{d\omega}\Phi_X(\omega)\right|_{\omega=0} = jE[X]$$

$$\left.\frac{d^n}{d\omega^n}\Phi_X(\omega)\right|_{\omega=0} = j^n E[X^n]$$

The Probability Generating Function

- $G_N(z) = E[z^n]$ $\quad$ N = a nonnegative integer-valued r.v.

$$= \sum_{k=0}^{\infty} p_N(k) z^k$$

the z-transform of the pdf with a sign change in the exponent

- $\Phi_X(\omega) = G_N(e^{j\omega})$

$$p_N(k) = \frac{1}{k!}\frac{d^k}{dz^k}G_N(z)\bigg|_{z=0}$$

$\Rightarrow$ pmf by $G_N(z)$: Probability generating function

$$\frac{d}{dz}G_N(z)\bigg|_{z=1} = \sum_{k=0}^{\infty} p_N(k)kz^{k-1}\bigg|_{z=1}$$

$$= \sum_{k=0}^{\infty} kp_N(k)$$

$$= E[N]$$

- $$\left.\frac{d^2}{dz^2} G_N(z)\right|_{z=1} = \left.\sum_{k=0}^{\infty} p_N(k) k(k-1) z^{k-2}\right|_{z=1}$$
  $$= \sum_{k=0}^{\infty} p_N(k) k(k-1)$$
  $$= E[N(N-1)] = E[N^2] - E[N]$$

$$E[N] = G'_N(1)$$

$$\mathrm{VAR}[N] = G''_N(1) + G'_N(1) - (G'_N(1))^2$$

The Laplace Transform of the pdf

- Nonnegative continuous r.v.'s

$$X^*(s) = \int_0^\infty f_X(x) e^{-sx} dx = E[e^{-sX}]$$

Laplace transform of the pdf

$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s) \Big|_{s=0}$$

The moment theorem

- HW : 67, 69, 72, 74, 75, 76, 83, 86, 91, 93, 95, 99, 100