4.6 Functions of Several r.v.'s

- One function of several r.v.'s
  - $Z = g(X_1, X_2, \ldots, X_n)$

    To find cdf of Z $\rightarrow$ the equivalent event of $\{Z \le z\}$

    $\rightarrow R_Z = \{\mathbf{x} = (x_1, \ldots x_n)$ such that $g(\mathbf{x}) \le z\}$

$$F_Z(z) = P[\mathbf{X} \text{ in } R_Z]$$

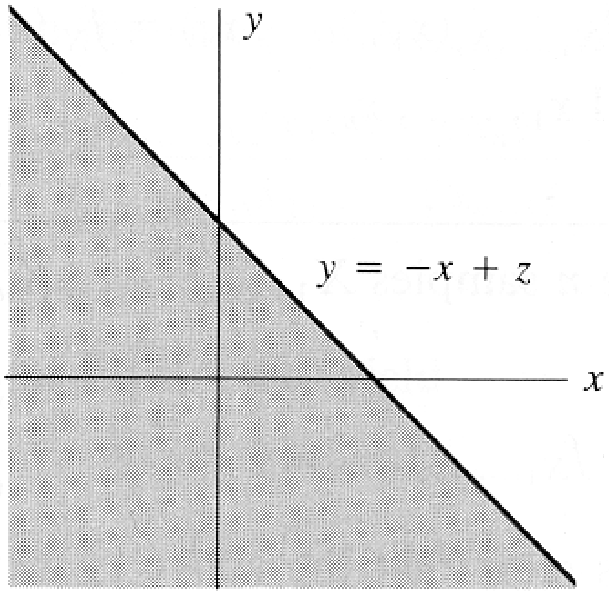
$$= \int \underset{\mathbf{x} \text{ in } R_Z}{\cdots} \int f_{X_1,\ldots,X_n}(x_1', \ldots, x_n') dx_1' \ldots dx_n'$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

- Ex. 4.31

$Z = X + Y$, find $F_Z(z)$ and $f_Z(z)$

$\rightarrow$ the equivalent event of $\{Z \le z\}$



i.e., $P[Z \le z] = P[X + Y \le z]$

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x'} f_{X,Y}(x', y') dy' dx'$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x', z - x') dx'}_{\text{superposition integral}}$$

- If X and Y are independent r.v.'s

$$\rightarrow \quad f_Z(z) = \underbrace{\int_{-\infty}^{\infty} f_X(x') f_Y(z - x') dx'}_{\text{convolution integral}}$$

- Ex.4.32

The sum of two non-independent Gaussian r.v.'s is also a Gaussian r.v.

$Z = X + Y$, where X and Y are zero-mean, unit variance Gaussian r.v.'s with correlation coefficient $\rho = -1/2$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x', z - x')dx'$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} e^{-[x'^2 - 2\rho x'(z-x') + (z-x')^2]/2(1-\rho^2)} dx'$$

$$= \frac{1}{2\pi(3/4)^{1/2}} \int_{-\infty}^{\infty} e^{-(x'^2 - x'z + z^2)/2(3/4)} dx'$$

$$= \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

- The conditional pdf and a function of several r.v.'s

Let $Z = g(X, Y)$

Suppose that $Y = y$ then $Z = g(X, y)$ is a function of one r.v.

$\therefore$ the pdf of Z given $Y = y$

$f_Z(z \mid Y = \mathrm{y})$ $\rightarrow$ conditional pdf of a function of one r.v.

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_Z(z \mid y') f_Y(y') dy'}$$

- cf) $$f_Z(z\,|\,y) = \frac{f_{Y,Z}(y,z)}{f_Y(y)}$$

$$\therefore\ f_{Y,Z}(y,z) = f_Z(z\,|\,y)\,f_Y(y)$$

- Marginal pdf of r.v. Z

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)dy$$

$$= \int_{-\infty}^{\infty} f_Z(z\,|\,y)\,f_Y(y)dy$$

- Ex. 4.34

$$Z = \frac{X}{Y} \rightarrow$$ Find the pdf of Z if X and Y are independent and both exponentially distributed with mean one.

- sol) Assume $Y = y$, then $Z = \dfrac{X}{y}$: scaled version of X

- cf) $Y = aX + b$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a > 0$$

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a < 0$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$\therefore\ f_Z(z \mid y) = |y|\, f_X(yz \mid y)$$

$$\therefore \quad f_Z(z) = \int_{-\infty}^{\infty} |y'| f_X(y'z \mid y') f_Y(y') dy'$$

$$= \int_{-\infty}^{\infty} |y'| f_{X,Y}(y'z, y') dy'$$

$$= \int_{-\infty}^{\infty} |y'| f_X(y'z) f_Y(y') dy' \qquad z > 0$$

$$= \int_{0}^{\infty} y' e^{-y'z} e^{-y'} dy'$$

$$= \frac{1}{(1+z)^2}, \qquad z > 0$$

Transformation of Random Vectors

$\mathbf{X} = (X_1, \ldots, X_n)$ → vector r.v

$\mathbf{Z} = (g_1(\mathbf{X}), g_2(\mathbf{X}), \ldots, g_n(\mathbf{X}))$ → vector r.v.

- The joint cdf of $Z_1, Z_2, \ldots, Z_n$ at the point $\mathbf{z} = (z_1, \ldots, z_n)$
  → the probability of the region of $\mathbf{x}$
  where $g_k(\mathbf{x}) \le z_k$ for $k = 1, \ldots, n$

$$F_{Z_1,\ldots,Z_n}(z_1, \ldots, z_n) = P[g_1(\mathbf{X}) \le z_1, \ldots, g_n(\mathbf{X}) \le z_n]$$

If $X_1, \ldots, X_n$ have a joint pdf

$$F_{Z_1,\ldots,Z_n}(z_1, \ldots, z_n) = \int \cdots \int_{\mathbf{x}' : g_k(\mathbf{x}') \le z_k} f_{X_1,\ldots,X_n}(x_1', \ldots, x_n') dx_1' \ldots dx_n'$$

pdf of Linear Transformation

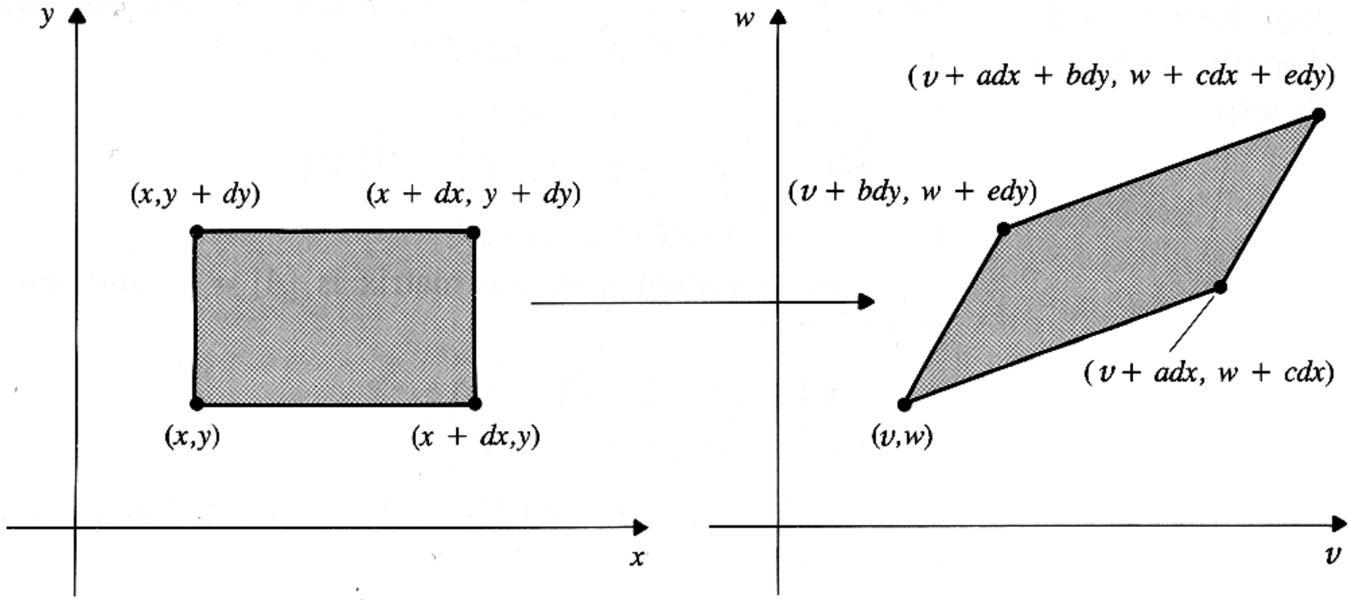
- The linear transformation of two r.v.'s

$$V = aX + bY \qquad \begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$
$$W = cX + eY$$

$$A = \begin{bmatrix} a & b \\ c & e \end{bmatrix}, \quad \text{determinant } |ae - bc| \neq 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} v \\ w \end{bmatrix}$$

- The joint pdf of $\mathbf{Z} \rightarrow$ the joint pdf $\mathbf{X}$ by finding equivalent events of infinitesimal rectangles.



$$v = ax + by$$
$$w = cx + ey$$

- $f_{X,Y}(x, y)dxdy \cong f_{V,W}(v, w)dP$

  dP : the area of the parallelogram

$$f_{V,W}(v,w) = \frac{f_{X,Y}(x,y)}{\left|\frac{dP}{dxdy}\right|}$$

cf) Area of parallelogram

$= AB\sin\theta$

$$\left|\frac{dP}{dxdy}\right| = \frac{|ae-bc|dxdy}{dxdy} = |ae-bc| = |A|$$

H.W. : Prove this

→ General Linear Transformation

- The n-dimensional vector $\mathbf{Z} = A\mathbf{X}$

$$f_{\mathbf{Z}}(\mathbf{z}) = f_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n) = \left.\frac{f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)}{|A|}\right|_{\mathbf{x}=A^{-1}\mathbf{z}}$$

$$= \frac{f_{\mathbf{X}}(A^{-1}\mathbf{z})}{|A|}$$

*pdf of General Transformations

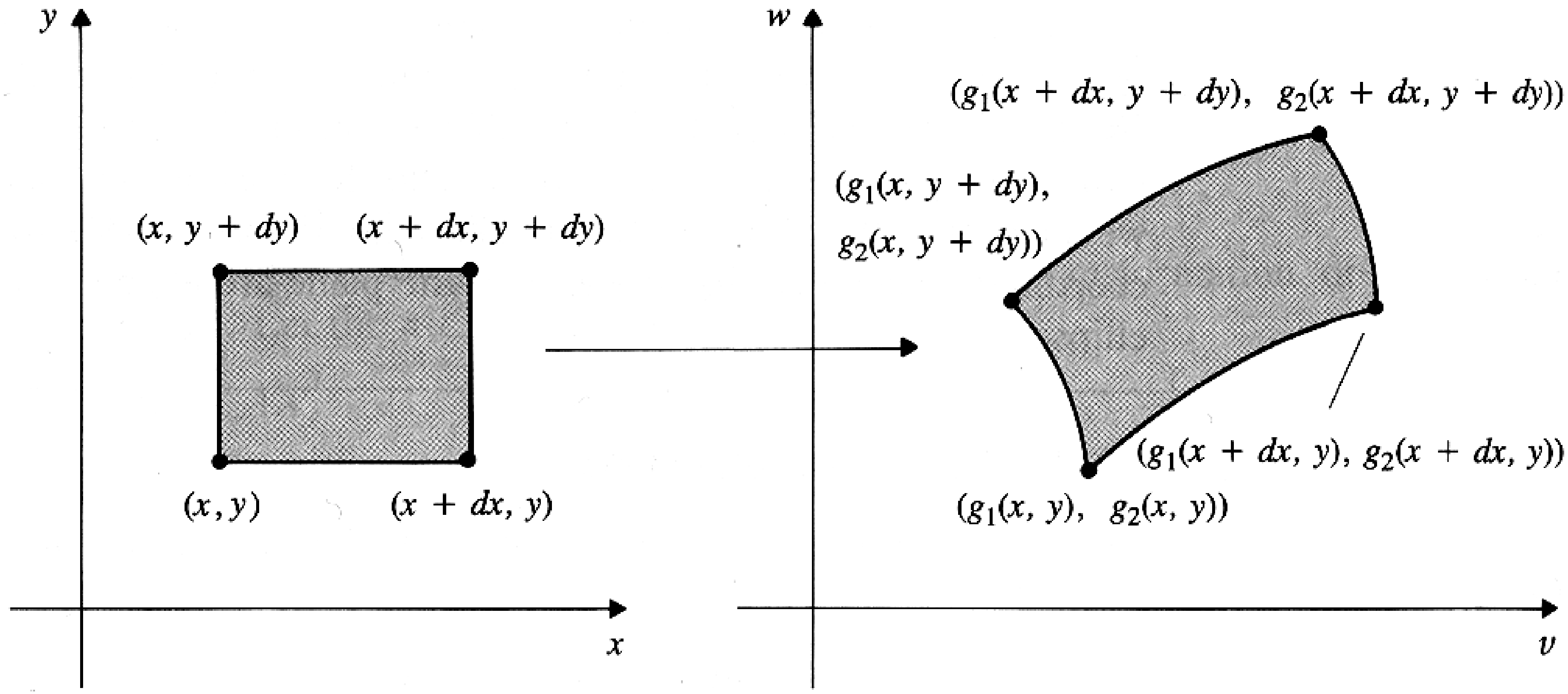
- The r.v.'s V and W : two nonlinear functions of X and Y

$$V = g_1(X, Y)\ , \qquad W = g_2(X, Y)$$

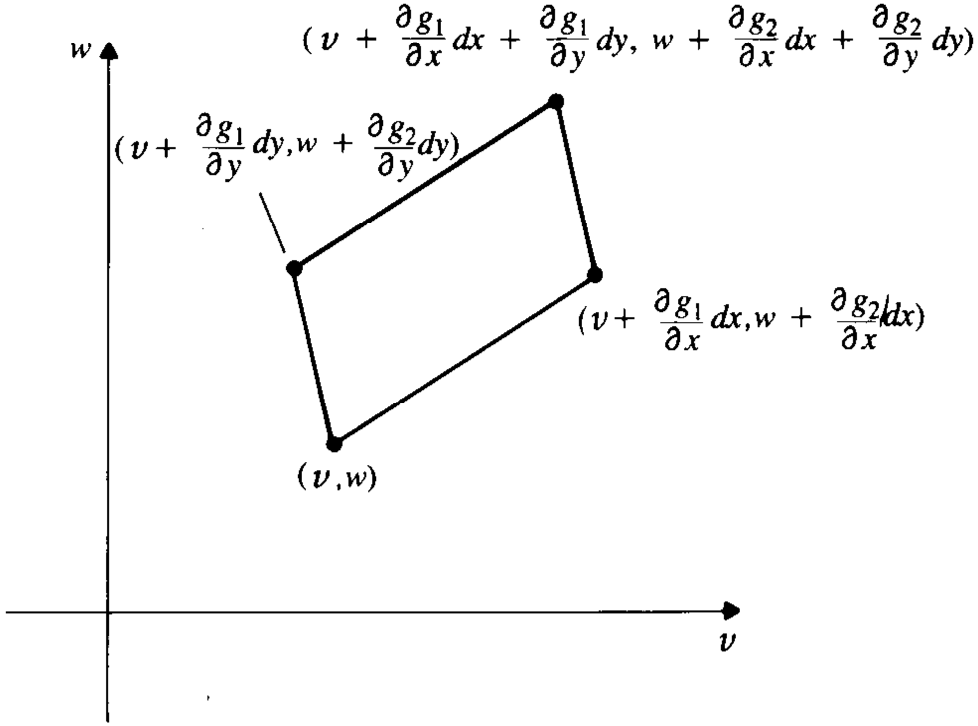
- Assume that the functions $g_1(x, y)$ and $g_2(x, y)$ are invertible

$$\rightarrow x = h_1(v, w)\ , \qquad y = h_2(v, w)$$

- The joint pdf of V and W : finding the equivalent event of infinitesimal rectangles.


y
x
(x, y + dy)
(x + dx, y + dy)
(x, y)
(x + dx, y)
w
v
(g1(x + dx, y + dy), g2(x + dx, y + dy))
(g1(x, y + dy),
g2(x, y + dy))
(g1(x + dx, y), g2(x + dx, y))
(g1(x, y), g2(x, y))

$\Downarrow$ Approximation



$$v = g_1(x,y)$$
$$w = g_2(x,y)$$

- cf) Compare with linear transformation

$$a \leftrightarrow \frac{\partial g_1}{\partial x}, b \leftrightarrow \frac{\partial g_1}{\partial y}$$

$$c \leftrightarrow \frac{\partial g_2}{\partial x}, e \leftrightarrow \frac{\partial g_2}{\partial y}$$

$$f_{X,Y}(x,y)dxdy = f_{V,W}(v,w)dP$$

$$\therefore f_{V,W}(v,w) = \frac{f_{X,Y}(h_1(v,w), h_2(v,w))}{\left|\frac{dP}{dxdy}\right|}$$

where dP is the area of the parallelogram.

$\rightarrow$ stretch factor at the point (v, w)

: given by the determinant of a matrix of partial derivatives.

$$\mathcal{J}(x, y) = \det \begin{bmatrix} \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix}$$ $\Rightarrow$ Jacobian of the transformation.

cf) $|A| = |ae - bc|$

$$\mathcal{J}(v,w) = \det \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{bmatrix} \Rightarrow$$ Jacobian of the inverse transformation

- note $\left|\mathcal{J}(v,w)\right| = \dfrac{1}{\left|\mathcal{J}(x,y)\right|}$

- pf) $x = h_1(v, w)$, $\qquad y = h_2(v, w)$
  $v = g_2(x, y)$, $\qquad w = g_2(x, y)$

$$\frac{dx}{dx} = 1 = \frac{\partial h_1}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial h_1}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial x}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial x}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{dy}{dy} = 1 = \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial y}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{dx}{dx} \cdot \frac{dy}{dy} = 1 = \frac{\partial x}{\partial v} \cdot \frac{\partial v}{\partial x} \cdot \frac{\partial y}{\partial w} \cdot \frac{\partial w}{\partial y} + \frac{\partial x}{\partial v} \cdot \frac{\partial v}{\partial x} \cdot \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$+ \frac{\partial x}{\partial w} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial x}{\partial w} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial y}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{dy}{dx} = 0 = \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial y}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{dx}{dy} = 0 = \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial x}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$\begin{aligned}\frac{dy}{dx}\cdot\frac{dx}{dy} = 0 = &\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial w}\cdot\frac{\partial w}{\partial y}\\ &+ \frac{\partial y}{\partial w}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial y}{\partial w}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial w}\cdot\frac{\partial w}{\partial y}\end{aligned}$$

$$\begin{aligned}\frac{dx}{dx}\cdot\frac{dy}{dy} - \frac{dy}{dx}\cdot\frac{dx}{dy} = 1 = &\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial y}{\partial w}\cdot\frac{\partial w}{\partial y} + \frac{\partial x}{\partial w}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial y}\\ &- \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial w}\cdot\frac{\partial w}{\partial y} - \frac{\partial y}{\partial w}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y}\end{aligned}$$

$$\therefore \quad \left|\mathcal{J}(x,y)\right|\left|\mathcal{J}(v,w)\right| = \left|\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial w} - \frac{\partial x}{\partial w}\frac{\partial y}{\partial v}\right)\right|$$

$$= \left| \begin{array}{l} \frac{\partial v}{\partial x}\frac{\partial w}{\partial y}\frac{\partial x}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}\frac{\partial x}{\partial w}\frac{\partial y}{\partial v} \\ - \frac{\partial v}{\partial x}\frac{\partial w}{\partial y}\frac{\partial x}{\partial w}\frac{\partial y}{\partial v} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}\frac{\partial y}{\partial w} \end{array} \right| = 1$$

$$\therefore \quad f_{V,W}(v,w) = \frac{f_{X,Y}(h_1(v,w), h_2(v,w))}{\left|\mathcal{J}(x,y)\right|}$$

$$= f_{X,Y}(h_1(v,w), h_2(v,w))\left|\mathcal{J}(v,w)\right|$$

- Ex. 4.37

X, Y : zero-mean, unit-variance independent Gaussian r.v.'s

Find the joint pdf of V and W defined by

$$V = (X^2 + Y^2)^{1/2}$$

$$W = \angle(X, Y)$$

sol) $x = v\cos w, \quad y = v\sin w$

$$\mathcal{J}(v,w) = \begin{vmatrix} \cos w & -v\sin w \\ \sin w & v\cos w \end{vmatrix} = v$$

$$\therefore \quad f_{V,W}(v,w) = f_{X,Y}(x(v,w), y(v,w)) \left| \mathcal{J}(v,w) \right|$$

$$= \frac{v}{2\pi} e^{-(v^2 \cos^2 w + v^2 \sin^2 w)/2}$$

$$= \frac{1}{2\pi} v e^{-v^2/2} \qquad v \geq 0, \quad 0 \leq w \leq 2\pi$$

- V : Rayleigh random variable

  $f_V(v) = v e^{-v^2/2} \qquad v \geq 0 \quad \Rightarrow$ Rayleigh random variable

- W : uniformly distributed in the interval $(0, 2\pi)$

4.7 Expected Value of Functions of r.v.'s

$$Z = g(X, Y)$$

$$E[Z] = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy & \text{for jointly continuous } X, Y \\ \sum_i \sum_n g(x_i, y_n) p_{X,Y}(x_i, y_n) & \text{for discrete } X, Y \end{cases}$$

- Note

  Let $Z = X + Y$, then

$$E[Z] = E[X + Y]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x' + y') f_{X,Y}(x', y') dx' dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x' f_{X,Y}(x', y') dy' dx' + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y' f_{X,Y}(x', y') dx' dy'$$

$$= \int_{-\infty}^{\infty} x' f_X(x') dx' + \int_{-\infty}^{\infty} y' f_Y(y') dy'$$

$$= E[X] + E[Y]$$

→ "The expected value of a sum of n r.v.'s is equal to the sum of the expected values"

- $E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n]$
  Regardless of independence of r.v.'s

- Note
  If $X_1, \ldots, X_n$ are independent r.v.'s, then
  $E[g_1(X_1)g_2(X_2)\ldots g_n(X_n)] = E[g_1(X_1)]E[g_2(X_2)]\ldots E[g_n(X_n)]$

Correlation and Covariance of Two r.v.'s

- The jk th joint moment of X and Y

$$E[X^jY^k] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)\,dxdy & \text{for continuous} \\ \sum_i \sum_n x_i^j y_n^k p_{X,Y}(x_i, y_n) & \text{for discrete} \end{cases}$$

- The $j = 1$ and $k = 1$ moment, $E[XY]$ is called as "the correlation of X and Y".
- Note if $E[XY] = 0$, then X and Y are orthogonal.

- The jk th central moment of X and Y is defined as
  $E[(X-E[X])^j(Y-E[Y])^k]$

- The $j=1$ and $k=1$ central moment is called as "the covariance of X and Y".
- $$\begin{aligned}\mathrm{COV}(X, Y) &= E[(X-E[X])(Y-E[Y])] \\ &= E[XY] - XE[Y] - YE[X] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$
- Note $\mathrm{COV}(X, Y) = E[XY]$ if either of the r.v.'s has mean zero
  $\mathrm{COV}(X, Y) = 0$ if X and Y are independent.

- The correlation coefficient of X and Y

$$\rho_{X,Y} = \frac{COV(x,y)}{\sigma_X \sigma_Y} = \frac{E[XY]-E[X]E[Y]}{\sigma_X \sigma_Y}, \qquad -1 \le \rho_{X,Y} \le 1$$

$$\sigma_X = \sqrt{VAR(X)} \qquad \sigma_Y = \sqrt{VAR(Y)}$$

$$\begin{aligned} 0 &\le E\left\{\left(\frac{X-E[X]}{\sigma_X} \pm \frac{Y-E[Y]}{\sigma_Y}\right)^2\right\} \\ &= 1 \pm 2\rho_{X,Y} + 1 \\ &= 2(1 \pm \rho_{X,Y}) \quad \Rightarrow \quad -1 \le \rho_{X,Y} \le 1 \end{aligned}$$

cf) $Y = aX + b$, (X and Y are related linearly)

$\rho_{X,Y} = 1$ if $a > 0$

$\rho_{X,Y} = -1$ if $a < 0$

- Proof:

$$\rho_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}$$

$$Y = aX + b$$

$$\begin{aligned} E[XY] &= E[aX^2 + bX] \\ &= aE[X^2] + bE[X] \\ E[X]E[Y] &= E[X] \cdot E[aX + b] \\ &= E[X]\{aE[X] + b\} \\ &= a\{E[X]\}^2 + bE[X] \end{aligned}$$

$$\therefore E[XY] - E[X]E[Y] = a\{E[X^2] - E[X]^2\}$$

$$= a\sigma_X{}^2$$

$$\sigma_Y{}^2 = \mathrm{VAR}[aX + b] = a^2\,\mathrm{VAR}[X] = a^2\sigma_X{}^2$$

$$\therefore \sigma_Y = |a|\sigma_X$$

$$\therefore \rho_{X,Y} = \frac{a\sigma_X{}^2}{|a|\sigma_X{}^2} = \begin{cases} 1 & \text{for } a > 0 \\ -1 & \text{for } a < 0 \end{cases}$$

- If $\rho_{X,Y} = 0$ then X and Y are said to be uncorrelated.

  cf) If X and Y are independent, then $\mathrm{COV}(X, Y) = 0$

  $\rightarrow \rho_{X,Y} = 0$

- $\rho_{X,Y}$: Statistical measure of the extent to which Y can be predicted by a linear function of X.

- Note

X, Y are independent $\rightarrow$ $\rho_{X,Y} = 0$: Always true.

$\rho_{X,Y} = 0$ does not necessarily mean that X, Y are independent.

$\rightarrow$ It is possible for X and Y to be uncorrelated ($\rho_{X,Y} = 0$) but not independent.

- Ex. 4.42
- Θ : Uniformly distributed in the interval (0, 2π)
- $X = \cos\Theta, \quad Y = \sin\Theta$
- sol) from Ex. 3.28,

$$f_X(x) = \frac{1}{2\pi\sqrt{1-x^2}} + \frac{1}{2\pi\sqrt{1-x^2}}$$

$$= \frac{1}{\pi\sqrt{1-x^2}} \qquad \text{for} \quad -1 < x < 1$$

cf) $$f_Y(y) = \sum_k \frac{f_X(x)}{\left|dy/dx\right|}\Bigg|_{X=x_k}$$

- Where $Y = g(X)$ and x_k is the solution of $y = g(x)$

Here $f_\Theta(\theta) = \dfrac{1}{2\pi}, \quad x = \cos\theta, \quad \theta_0 = \cos^{-1} x$

$$\frac{dx}{d\theta} = -\sin\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta_0} = -\sqrt{1-x^2}, \quad \left.\frac{dx}{d\theta}\right|_{\theta_1 = 2\pi - \theta_0} = \sqrt{1-x^2}$$

$$0 \le \theta_0 \le \pi$$

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}} \quad \text{for} \quad -1 < x < 1$$

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}} \quad \text{for} \quad -1 < y < 1$$

The point (X, Y) : the point on the unit circle specified by the angle Θ

- The product of the marginal pdf's

  $f_X(x)\, f_Y(y) \neq 0$ in the square defined by $-1 < x < 1, \quad -1 < y < 1$

  $\rightarrow$ If X and Y are independent, the point (X, Y) would assume all values in this square

  $\rightarrow$ Discrepancy with the preceding definition of the point (X, Y)

  $\rightarrow$ Dependent

$$E[XY] = E[\sin\Theta\cos\Theta]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\cos\theta d\theta$$

$$= \frac{1}{4\pi}\sin 2\theta \,|_0^{2\pi} = 0$$

$$E[X] = 0,\ E[Y] = 0$$

$$\therefore \mathrm{COV}(X, Y) = E[XY] - E[X]E[Y] = 0$$

$$\therefore \rho_{X,Y} = \frac{\mathrm{COV}(X,Y)}{\sigma_X\sigma_Y} = 0 \quad \Rightarrow$$ uncorrelated

Joint Characteristic Function

- The joint characteristic function of n r.v.'s

$$\Rightarrow \Phi_{X_1, X_2, \ldots, X_n}(\omega_1, \omega_2, \ldots, \omega_n) = E[e^{j(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n)}]$$

- For two r.v.'s

$$\Phi_{X,Y}(\omega_1, \omega_2) = E[e^{j(\omega_1 X + \omega_2 Y)}]$$

- Assume X, Y are jointly continuous r.v.'s, then

$$\Phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j(\omega_1 x + \omega_2 y)} dxdy$$

- Marginal characteristic function

$$\Phi_X(\omega) = \Phi_{X,Y}(\omega, 0)$$

$$\Phi_Y(\omega) = \Phi_{X,Y}(0, \omega)$$

- X, Y are independent

$$\begin{aligned}\Phi_{X,Y}(\omega_1, \omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] \\ &= E[e^{j\omega_1 X} \cdot e^{j\omega_2 Y}] \\ &= E[e^{j\omega_1 X}]E[e^{j\omega_2 Y}] \\ &= \Phi_X(\omega_1)\Phi_X(\omega_2)\end{aligned}$$

- $Z = aX + bY$

$$\Phi_Z(\omega) = E[e^{j\omega(aX+bY)}] = E[e^{j\omega aX + j\omega bY}]$$
$$= \Phi_{X,Y}(a\omega, b\omega)$$

- X, Y are independent.

$Z = aX + bY$

$$\Phi_Z(\omega) = \Phi_{X,Y}(a\omega, b\omega) = \Phi_X(a\omega)\Phi_Y(b\omega)$$

- The joint moments of X and Y

$$\Phi_{X,Y}(\omega_1,\omega_2) = E[e^{j\omega_1 X} e^{j\omega_2 Y}]$$

$$= E\left[\sum_{i=0}^{\infty} \frac{(j\omega_1 X)^i}{i!} \sum_{k=0}^{\infty} \frac{(j\omega_2 Y)^k}{k!}\right]$$

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} E[X^i Y^k] \frac{(j\omega_1)^i}{i!} \frac{(j\omega_2)^k}{k!}$$

$$E[X^i Y^k] = \frac{1}{j^{i+k}} \frac{\partial^i \partial^k}{\partial \omega_1{}^i \partial \omega_2{}^k} \Phi_{X,Y}(\omega_1,\omega_2)\Bigg|_{\omega_1=0,\omega_2=0}$$

4.8 Jointly Gaussian Random Variables

- Jointly Gaussian r.v.'s X, Y

  If their joint pdf has the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{X,Y}^2}} \times$$

$$\exp\left\{\frac{-1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho_{X,Y}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]\right\}$$

for $-\infty < x < \infty, \quad -\infty < y < \infty,$

- Comment

①The pdf is centered at the point (m_1, m_2)

②Bell shape. That depends on the value of $\sigma_1, \sigma_2,$ and $\rho_{X,Y}$

③The pdf is constant for values x and y for which the argument of the exponent is constant.

$$\left[\left(\frac{x-m}{\sigma_1}\right)^2 - 2\rho_{X,Y}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2\right] = \text{constant}$$

- cf) Contours of constant pdf for various values of σ_1, σ_2 and $\rho_{X,Y}$

When

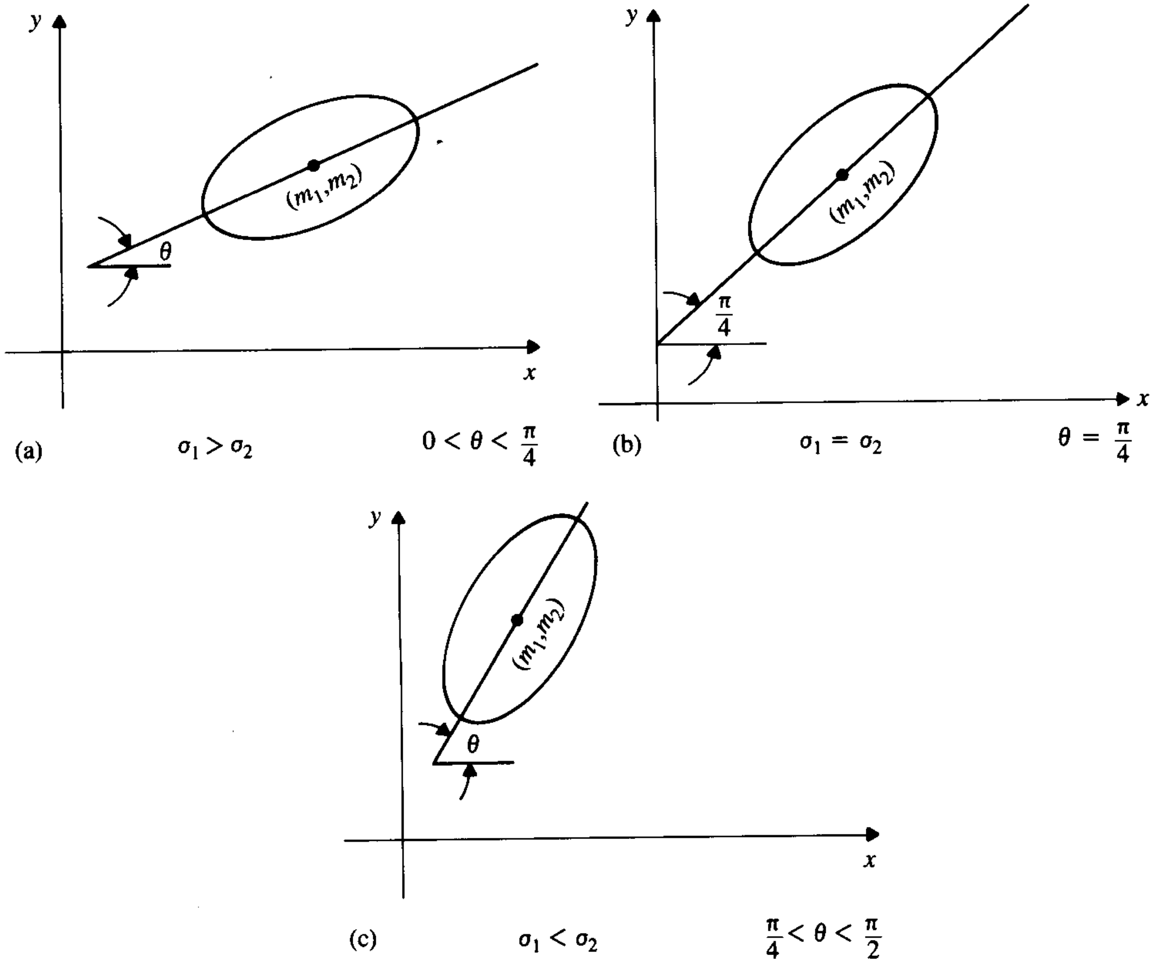
① $\rho_{X,Y} = 0$ (X, Y are independent)

$\rightarrow$ Contour is an ellipse with principal axes aligned with the x-and y-axes

② when $\rho_{X,Y} \neq 0$, the major axes is oriented along the angle.

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho_{X,Y}\sigma_1\sigma_2}{\sigma_1^{\,2} - \sigma_2^{\,2}}\right)$$

HW: Derive this formula

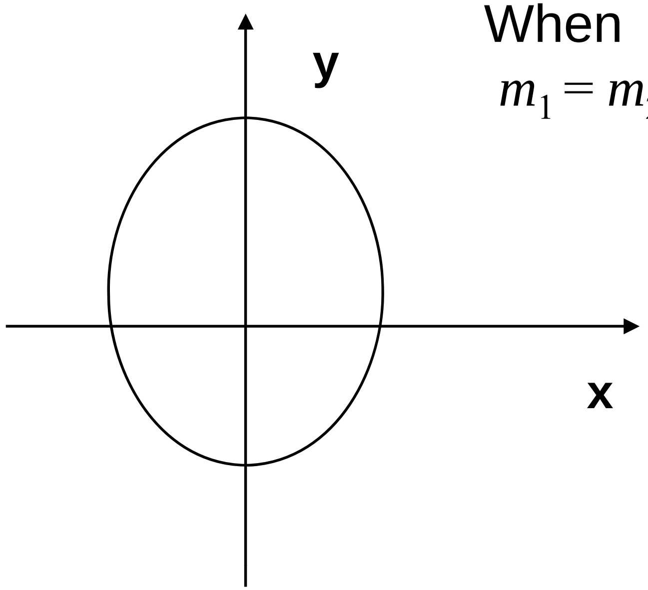


y
x
(m_1, m_2)
θ
(a)
$\sigma_1 > \sigma_2$
$0 < \theta < \frac{\pi}{4}$
y
x
(m_1, m_2)
$\frac{\pi}{4}$
(b)
$\sigma_1 = \sigma_2$
$\theta = \frac{\pi}{4}$
y
x
(m_1, m_2)
θ
(c)
$\sigma_1 < \sigma_2$
$\frac{\pi}{4} < \theta < \frac{\pi}{2}$

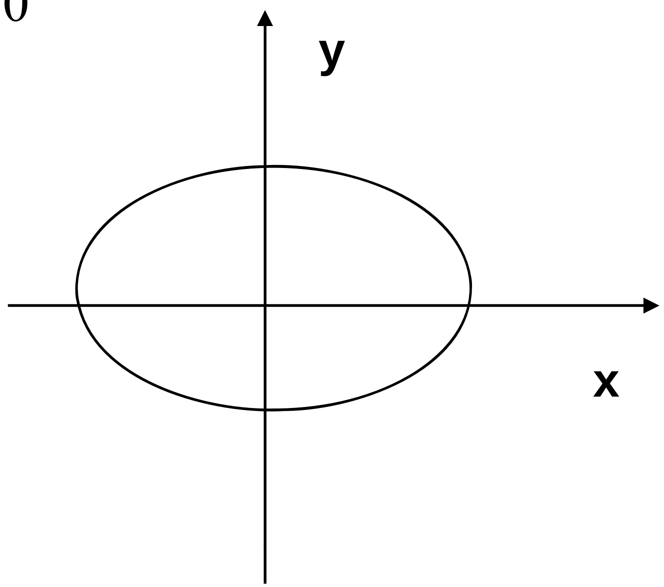
- H.W.) Derive the expression for θ with $m_1 = m_2 = 0$, $\rho_{X,\,Y} \neq 0$ and plot the contour with $\sigma_1 > \sigma_2$, $\sigma_1 = \sigma_2$ and $\sigma_1 < \sigma_2$

Ex) Contour for $\rho_{X,Y} = 0$

When $\rho_{X,Y} = 0$
$m_1 = m_2 = 0$



or



- The marginal pdf

$$f_X(x) = \frac{e^{-(x-m_1)^2 / 2\sigma_1^2}}{\sqrt{2\pi}\sigma_1}$$

H.W.: Prove this

- The conditional pdf's $f_X(x \mid y)$ and $f_Y(y \mid x)$

$$f_X\left(x \mid y\right) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\exp\left\{\dfrac{-1}{2(1-\rho_{X,Y}^2)\sigma_1^2}\left[x - \rho_{X,Y}\dfrac{\sigma_1}{\sigma_2}(y - m_2) - m_1\right]^2\right\}}{\sqrt{2\pi\sigma_1^2(1-\rho_{X,Y}^2)}}$$

- ✓ Gaussian with conditional mean: $m_1 + \rho_{X,Y}\dfrac{\sigma_1}{\sigma_2}(y - m_2)$
- ✓ Conditional variance: $\sigma_1^{\ 2}(1-\rho_{X,Y}^2)$

- Note: when $\rho_{X,Y} = 0$,
  "The conditional pdf of X given $Y = y$ equals the marginal pdf of X" $\rightarrow$ Independent

- cf) When $\rho_{X,Y} = 0$, then Gaussian r.v.'s X and Y are independent
  For other r.v.'s, $\rho_{X,Y} = 0$ does not guarantee that X and Y are independent.

- When $|\rho_{X,Y}| \to 1$, the variance of X about the conditional mean $\to 0$

  So the conditional pdf approaches a delta function at the conditional mean.

- cf) $|\rho_{X,Y}| \to 1$ (i.e., $\rho_{X,Y} = 1$) means that X and Y are linearly related. i.e., $Y = aX + b$
- $\therefore$ the conditional variance is zero

  $\to X$ is equal to the conditional mean with probability one.

❖ Pf) $\rho_{X,Y}$ for jointly Gaussian r.v.'s

$$\mathrm{COV}(X,Y) = E[(X-m_1)(Y-m_2)]$$
$$= E[E[(X-m_1)(Y-m_2)\,|\,Y]]$$

cf) $f_X(x\,|\,y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$$\therefore\ f_{X,Y}(x,y) = f_X(x\,|\,y)\,f_Y(y)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-m_1)(y-m_2)\,f_{X,Y}(x,y)\,dxdy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-m_1)(y-m_2)\,f_X(x\,|\,y)\,f_Y(y)\,dxdy$$

$$= \int_{-\infty}^{\infty} E[(X - m_1)(Y - m_2) \mid Y = y] f_Y(y) dy$$

$$= E[E[(X - m_1)(Y - m_2) \mid Y]]$$

Now

$$E\big[(X - m_1)(Y - m_2) \mid Y = y\big] = (y - m_2) E[(X - m_1) \mid Y = y]$$

$$= (y - m_2)(E[X \mid Y = y] - m_1)$$

$$= (y - m_2)\left(m_1 + \rho_{X,Y} \frac{\sigma_1}{\sigma_2}(y - m_2) - m_1 \right)$$

$$= (y - m_2)\left(\rho_{X,Y} \frac{\sigma_1}{\sigma_2}(y - m_2) \right)$$

$$\Rightarrow E[(X-m_1)(Y-m_2)\,|\,Y=y]=\rho_{X,Y}\frac{\sigma_1}{\sigma_2}(y-m_2)^2$$

$$\therefore \mathrm{COV}(X,Y)=\rho_{X,Y}\frac{\sigma_1}{\sigma_2}E[(Y-m_2)^2]=\rho_{X,Y}\frac{\sigma_1}{\sigma_2}\cdot{\sigma_2}^2=\rho_{X,Y}\sigma_1\sigma_2$$

$$\therefore \rho_{X,Y}=\frac{\mathrm{COV}(X,Y)}{\sigma_1\sigma_2}$$

n jointly Gaussian Random Variables

- pdf of n jointly Gaussian r.v.'s

$$f_{\mathbf{X}}(\mathbf{x}) \triangleq f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$$

$$= \frac{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{\mathrm{T}} K^{-1}(\mathbf{x}-\mathbf{m})\right\}}{(2\pi)^{n/2}|K|^{1/2}}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} E[x_1] \\ E[x_2] \\ \vdots \\ E[x_n] \end{bmatrix}$

$$K = \begin{bmatrix} \mathrm{VAR}(X_1) & \mathrm{COV}(X_1, X_2) & \dots & \mathrm{COV}(X_1, X_n) \\ \mathrm{COV}(X_2, X_1) & \mathrm{VAR}(X_2) & \dots & \mathrm{COV}(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ \mathrm{COV}(X_n, X_1) & \dots & & \mathrm{VAR}(X_n) \end{bmatrix}$$

: covariance matrix

cf) Covariance matrix is a symmetric matrix

$\because\ \mathrm{COV}(X_i, X_j) = \mathrm{COV}(X_j, X_i)$

- Note

  “the pdf of jointly Gaussian random variables is completely specified by the individual means and variances and the pairwise covariances”

- HW: 5, 11, 15, 29, 35, 40, 45, 54, 58, 60, 70,72, 78, 84, 86

Linear Transformation of Gaussian r.v.'s

- "The linear transformation of any n jointly Gaussian r.v.'s results in n random variables that are also jointly Gaussian."

- Let $\mathbf{X} = (X_1, \ldots, X_n)$: jointly Gaussian.

$$\mathbf{Y} = A\mathbf{X}, \quad \mathbf{Y} = (Y_1, \ldots, Y_n)$$

$A = n \times n$ invertible matrix.

$(|A| \neq 0)$

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{f_{\mathbf{X}}(A^{-1}\mathbf{y})}{|A|}$$

$$= \frac{\exp\left\{-\frac{1}{2}(A^{-1}\mathbf{y}-\mathbf{m})^{\mathrm{T}} K^{-1}(A^{-1}\mathbf{y}-\mathbf{m})\right\}}{(2\pi)^{n/2}|A||K|^{1/2}}$$

cf) $A^{-1}\mathbf{y}-\mathbf{m} = A^{-1}(\mathbf{y}-A\mathbf{m})$

$$(A^{-1}\mathbf{y}-\mathbf{m})^{\mathrm{T}} = (\mathbf{y}-A\mathbf{m})^{\mathrm{T}} A^{-1^{\mathrm{T}}}$$

$$\therefore (\mathbf{y}-A\mathbf{m})^{\mathrm{T}} A^{-1^{\mathrm{T}}} \cdot K^{-1} A^{-1} (\mathbf{y}-A\mathbf{m})$$

$$= (\mathbf{y}-A\mathbf{m})^{\mathrm{T}} (AKA^{\mathrm{T}})^{-1} (\mathbf{y}-A\mathbf{m})$$

let $\quad C = AKA^{\mathrm{T}}, \quad \mathbf{n} = A\mathbf{m}$

$$\det(C) = \det(AKA^{\mathrm{T}}) = \det(A)\det(K)\det(A^{\mathrm{T}})$$

$$= \det(A)^2 \det(K)$$

$$\therefore\ f_{\mathbf{Y}}(\mathbf{y}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{y}-\mathbf{n})^{\mathrm{T}} C^{-1} (\mathbf{y}-\mathbf{n})\right]}{(2\pi)^{n/2} |C|^{1/2}} \quad \Rightarrow$$

Jointly Gaussian with mean $\mathbf{n} = A\mathbf{m}$

Covariance matrix
***C* = *AKA*$^{\mathrm{T}}$**

- Transforming $\mathbf{X}$ to <u>a vector $\mathbf{Y}$ of independent Gaussian r.v.'s</u>

K : Symmetric covariance matrix.

Then $K = P\Lambda P^{\mathrm{T}}$

where Λ : diagonal matrix that consists of eigenvalues of K.

P : matrix whose columns consist of an orthonormal set of eigenvectors of K.

- It is always possible to find a matrix A such that
  - $\rightarrow AKA^{\mathrm{T}} = \Lambda$, where Λ is a diagonal matrix.
  - $\rightarrow$ then

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{y}-\mathbf{n})^{\mathrm{T}}\Lambda^{-1}(\mathbf{y}-\mathbf{n})\right]}{(2\pi)^{n/2}\left|\Lambda\right|^{1/2}}$$

$$= \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i - n_i)^2 / \lambda_i\right\}}{[(2\pi\lambda_1)(2\pi\lambda_2)\ldots(2\pi\lambda_n)]^{1/2}}$$

  - $\rightarrow$ independent r.v.'s with mean n_i and variance λ_i

- It is always possible to select the matrix A that diagonalizes K so that $\det(A) = 1$.

$\rightarrow$ The transformation $A\mathbf{X}$ corresponds to a rotation of the coordinate system so that the principal axes at the ellipsoid corresponding to the pdf are aligned to the axes of the system.

Joint Characteristic Function of Gaussian R.V.'s

- Joint Characteristic function of n jointly Gaussian r.v.'s $X_1, X_2, \ldots, X_n$

$$\Phi_{X_1, X_2 \ldots, X_n}(\omega_1, \omega_2, \ldots, \omega_n) = e^{j\sum_{i=1}^{n}\omega_i m_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\omega_i\omega_k \mathrm{COV}(X_i, X_k)}$$

$$\Rightarrow \Phi_{\mathbf{X}}(\boldsymbol{\omega}) \triangleq \Phi_{X_1, X_2 \ldots, X_n}(\omega_1, \omega_2, \ldots, \omega_n) = e^{j\boldsymbol{\omega}^{\mathrm{T}}\mathbf{m} - \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}} K \boldsymbol{\omega}}$$

  - Proof) Consider $Z = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$

$$\Phi_Z(\omega) = E[e^{j\omega Z}] = E[e^{j(\omega a_1 X_1 + \omega a_2 X_2 + \cdots + \omega a_n X_n)}]$$

$$= \Phi_{X_1, \ldots, X_n}(a_1\omega, a_2\omega, \ldots, a_n\omega)$$

cf) Definition of joint characteristic function

- cf) Z : Gaussian r.v. with mean $E[Z] = \sum_{i=1}^{n} a_i E[X_i]$

  And variance $\mathrm{VAR}[Z] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \,\mathrm{COV}(X_i, X_j)$

- Pf) Using auxiliary random variables

  $\mathbf{Z} = (Z, Z_2, \ldots, Z_n)$

  where $Z_2 = X_2, Z_3 = X_3, \ldots, Z_n = X_n$

  then $\mathbf{Z} = A\mathbf{X}$

Transformation matrix:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & \cdot & a_n \\ 0 & 1 & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdot & \cdots & 0 & 1 \end{bmatrix} \qquad A^{\mathrm{T}} = \begin{bmatrix} a_1 & 0 & \cdot & 0 \\ a_2 & 1 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & 0 \\ a_n & 0 & \cdot & 1 \end{bmatrix}$$

Covariance matrix:

$$K = \begin{bmatrix} \mathrm{VAR}(X_1) & \mathrm{COV}(X_1, X_2) & \ldots & \mathrm{COV}(X_1, X_n) \\ \mathrm{COV}(X_2, X_1) & \mathrm{VAR}(X_2) & \ldots & \mathrm{COV}(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ \mathrm{COV}(X_n, X_1) & \ldots & & \mathrm{VAR}(X_n) \end{bmatrix}$$

- Gaussian random variable Z has the following mean and variance

$$E[Z] = (A\mathbf{m})_1 = \sum_{i=1}^{n} a_i E[X_i]$$

$$\mathrm{VAR}[Z] = (AKA^{\mathrm{T}})_{1,1} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{COV}(X_i, X_j)$$

❖ Characteristic function of Gaussian random variable Z

$$\Phi_Z(w) = e^{j\omega E[Z] - \frac{1}{2}\mathrm{VAR}[Z]\omega^2}$$

$$= e^{j\omega \sum a_i m_i - \frac{1}{2}\omega^2 \sum_i \sum_k a_i a_k \,\mathrm{COV}(x_i, x_j)}$$

➢ For $\omega = 1$

$$\Rightarrow \quad \Phi_{X_1, X_2, \ldots, X_n}(a_1, a_2, \ldots, a_n) = e^{j\sum_i a_i m_i - \frac{1}{2}\sum_i \sum_k a_i a_k \,\mathrm{COV}(X_i, X_k)}$$

$$= \frac{1}{2} e^{j\mathbf{a}^{\mathrm{T}}\mathbf{m} - \frac{1}{2}\mathbf{a}^{\mathrm{T}} K \mathbf{a}}$$

then replacing $(a_1, a_2, \ldots, a_n)$ with $(\omega_1, \omega_2, \ldots, \omega_n)$