Ch.6 Random Processes

- Random Process = Stochastic Process
  - Numerical quantities that evolve randomly in time or space.
  - Indexed family of random variables.

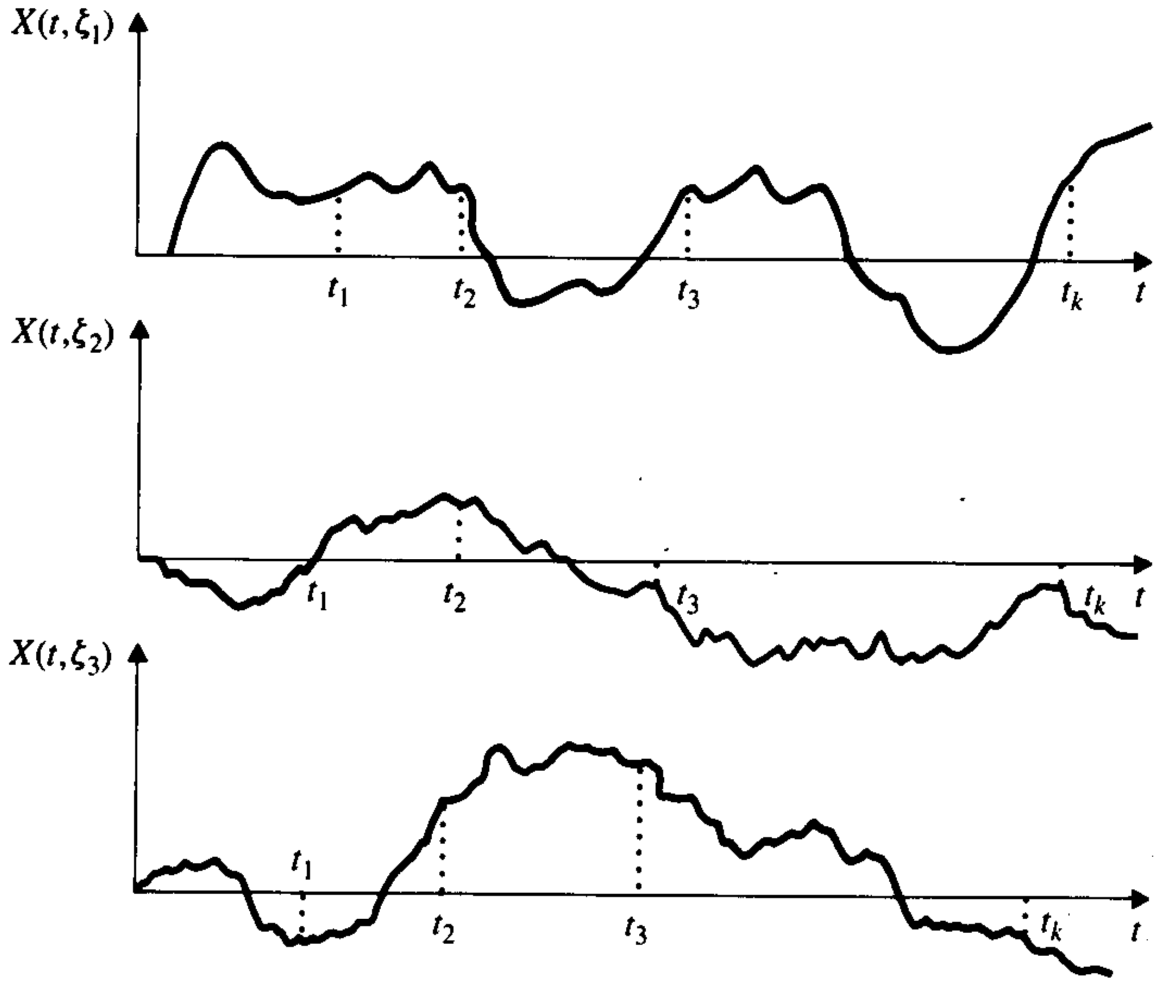
6.1 Definition of a Random Process

- For every outcome $\zeta \in S$, a function of time

$$X(t, \zeta), \quad t \in I$$

- Realization, Sample path or Sample function of the random process

  $\rightarrow$ the graph of $X(t, \zeta)$ versus t for ζ fixed.



$X(t,\zeta_1)$
t_1
t_2
t_3
t_k
t
$X(t,\zeta_2)$
t_1
t_2
t_3
t_k
t
$X(t,\zeta_3)$
t_1
t_2
t_3
t_k
t

- For each fixed t_k from the index set I,

  $X(t_k, \zeta)$ = a random variable.

- Indexed family of random variables.

  $\{X(t, \zeta),\ t \in I\} \rightarrow$ Random Process or Stochastic Process.

cf) discrete-time if the index set I is a countable set

continuous-time if I is continuous.

- The randomness in ζ induces randomness in the observed function $X(t, \zeta)$

- Ex.6.4

cf) $$f_Y(y) = \sum_k f_X(x) \left| \frac{dx}{dy} \right| \Bigg|_{x=x_k}$$

6.2 Specifying a Random Process

- Joint Distributions of Time Samples

  $$X_1 = X(t_1, \zeta),\ X_2 = X(t_2, \zeta), \ldots,\ X_k = X(t_k, \zeta)$$

  $\rightarrow$ stochastic process is specified by the collection of k th-order joint cdf.

$$F_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k)$$

$$= P[X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k]$$

for any k and any choice of sampling instants $t_1, \ldots, t_k$

- Discrete-valued stochastic process,
  pmf specifies the stochastic process.

$$p_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k)$$
$$= P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k]$$

- Continuous-valued stochastic process,
  pdf specifies the stochastic process.

$$f_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k)$$

Independent Increments

- Two fundamental processes : Poisson process and Wiener process $\rightarrow$ independent increments, Markov.

- A random process $X(t)$ is said to have "independent increments" if for any k and any choice of sampling instants $t_1 < t_2 < \ldots < t_k$, the random variables $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_k) - X(t_{k-1})$ are independent random variables

Markov Process

- For any k and any choice of sampling instants

  $t_1 < t_2 < \dots < t_k$ and for any $x_1, x_2, \dots, x_k$,

  $$f_{X(t_k)}(x_k \,|\, X(t_{k-1}) = x_{k-1}, \dots, X(t_1) = x_1)$$

  $$= f_{X(t_k)}(x_k \,|\, X(t_{k-1}) = x_{k-1})$$ if $X(t)$ is continuous-valued.

  $$P[X(t_k) = x_k \,|\, X(t_{k-1}) = x_{k-1}, \dots, X(t_1) = x_1]$$

  $$= P[X(t_k) = x_k \,|\, X(t_{k-1}) = x_{k-1}]$$ if $X(t)$ is discrete-valued.

  then $X(t)$ is said to be a Markov process

- Note

"A random process that has independent increments is also a Markov process."

"The converse is not true."

$$
\begin{aligned}
X(t_k) &= X(t_{k-1}) + \Delta X_{k-1} \\
&= X(t_{k-2}) + \Delta X_{k-2} + \Delta X_{k-1} \\
&= X(t_1) + \Delta X_1 + \Delta X_2 + \cdots + \Delta X_{k-1}
\end{aligned}
$$

$$
\Delta X_1 = X(t_2) - X(t_1), \ldots, \Delta X_{k-1} = X(t_k) - X(t_{k-1})
$$

: independent random variable

The Mean, Autocorrelation, Autocovariance

- The mean $m_X(t)$ of a random process $X(t)$

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$$

↓ **Funtion of time**

- Autocorrelation $R_X(t_1, t_2)$: a function of t_1 and t_2

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),X(t_2)}(x, y)dxdy$$

where $f_{X(t_1),X(t_2)}(x, y)$ is the second-order pdf of $X(t)$

- Autocovariance $C_X(t_1, t_2)$

$$C_X(t_1,t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]$$
$$= R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

- Variance of $X(t)$

$$\mathrm{VAR}[X(t)] = \mathrm{E}[(X(t) - m_X(t))^2] = C_X(t,t)$$

- Correlation coefficient

$$\rho_X(t_1,t_2)=\frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)}\sqrt{C_X(t_2,t_2)}}$$

- cf)
  - The mean, autocorrelation, and autocovariance functions are only partial descriptions of a random process.
  - It is possible for two quite different random processes to have the same mean, autocorrelation, and autocovariance function.

- Ex. 6.7
  - $X(t) = \cos(\omega t + \Theta)$, Θ is uniformly distributed in the interval $(-\pi, \pi)$

$$m_X(t) = E[\cos(\omega t + \Theta)]$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)d\theta = 0$$
$$C_X(t_1, t_2) = R_X(t_1, t_2) = E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)]$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\{\cos(\omega(t_1 - t_2)) + \cos(\omega(t_1 + t_2) + 2\theta)\}d\theta$$
$$= \frac{1}{2}\cos(\omega(t_1 - t_2))$$

$$\rho_X(t_1,t_2) = \frac{C_X(t_1,t_2)}{\sqrt{C_X(t_1,t_1)}\sqrt{C_X(t_2,t_2)}} = \frac{\frac{1}{2}\cos(\omega(t_1-t_2))}{\frac{1}{2}}$$

$$= \cos(\omega(t_1-t_2))$$

Gaussian Random Processes

- $X_1 = X(t_1), X_2 = X(t_2), \ldots, X_k = X(t_k)$ are jointly Gaussian r.v.'s for all k and all choices of $t_1, \ldots, t_k$.

- The joint pdf of jointly Gaussian r.v.'s

$$f_{X_1, X_2, \ldots, X_k}(x_1, \ldots, x_k) = \frac{e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{\mathrm{T}} K^{-1} (\mathbf{x}-\mathbf{m})}}{(2\pi)^{k/2} |K|^{1/2}}$$

where $\mathbf{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_k) \end{bmatrix}$ $\quad K = \begin{bmatrix} C_X(t_1, t_1) & C_X(t_1, t_2) & \ldots & C_X(t_1, t_k) \\ C_X(t_2, t_1) & C_X(t_2, t_2) & \ldots & C_X(t_2, t_k) \\ \vdots & \vdots & & \vdots \\ C_X(t_k, t_1) & & \ldots & C_X(t_k, t_k) \end{bmatrix}$

Ex. 6.8: iid Gaussian sequence

- Mean m, Variance σ^2
- $\{C_X(t_i, t_j)\} = \{\sigma^2 \delta_{ij}\} = \sigma^2 I$

  $\delta_{ij} = 1$ when $i = j$ and 0 otherwise.

  I = identity matrix.

$$f_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = \frac{1}{(2\pi\sigma^2)^{k/2}} \exp\left\{-\sum_{i=1}^{k} (x_i - m)^2 / 2\sigma^2\right\}$$

$$= f_X(x_1) f_X(x_2) \ldots f_X(x_k)$$

Multiple Random Processes

- For a pair of random processes $X(t)$ and $Y(t)$, we must specify all possible joint density functions of $X(t_1),\ldots,X(t_k)$ and $Y(t_1'),\ldots,Y(t_j')$ for all k, j, and all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$

- The processes $X(t)$ and $Y(t)$ are said to be independent if the vector r.v.'s $(X(t_1),\ldots,X(t_k))$ and $(Y(t_1'),\ldots,Y(t_j'))$ are independent for all k, j, and all choices of $t_1,\ldots,t_k$ and $t_1',\ldots,t_j'$

- The cross-correlation $R_{X,Y}(t_1, t_2)$ of $X(t)$ and $Y(t)$
  - $R_{X,Y}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$

    $\rightarrow$ $X(t)$ and $Y(t)$ are orthogonal if

    $R_{X,Y}(t_1, t_2) = 0$ for all t_1 and t_2

- The cross-covariance $C_{X,Y}(t_1, t_2)$ of $X(t)$ and $Y(t)$
  - $C_{X,Y}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}]$

    $= R_{X,Y}(t_1, t_2) - m_X(t_1) m_Y(t_2)$

    $\rightarrow$ $X(t)$ and $Y(t)$ are uncorrelated if

    $C_{X,Y}(t_1, t_2) = 0$ for all t_1 and t_2

6.3 Examples of Discrete-Time Random Processes

- iid Random Process X_n
  - Consisting of a sequence of independent, identically distributed random variables with common cdf $F_X(x)$, mean m, and variance σ^2
  - $$F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = P[X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k]$$
    $$= F_X(x_1)F_X(x_2)\ldots F_X(x_k)$$

    : the joint cdf for any time instants $n_1, n_2, \ldots, n_k$
  - Mean of an iid process

    $m_X(n) = E[X_n] = m$ for all n

    └→ Constant

- Autocovariance

  - If $n_1 \neq n_2$

$$C_X(n_1, n_2) = E[(X_{n_1} - m)(X_{n_2} - m)]$$
$$= E[X_{n_1} - m]E[X_{n_2} - m] = 0$$

  - If $n_1 = n_2 = n$

$$C_X(n_1, n_2) = E[(X_n - m)^2] = \sigma^2$$

$$\therefore C_X(n_1, n_2) = \sigma^2 \delta_{n_1, n_2}$$

where $\delta_{n_1, n_2} = 1$ if $n_1 = n_2$ and 0 otherwise

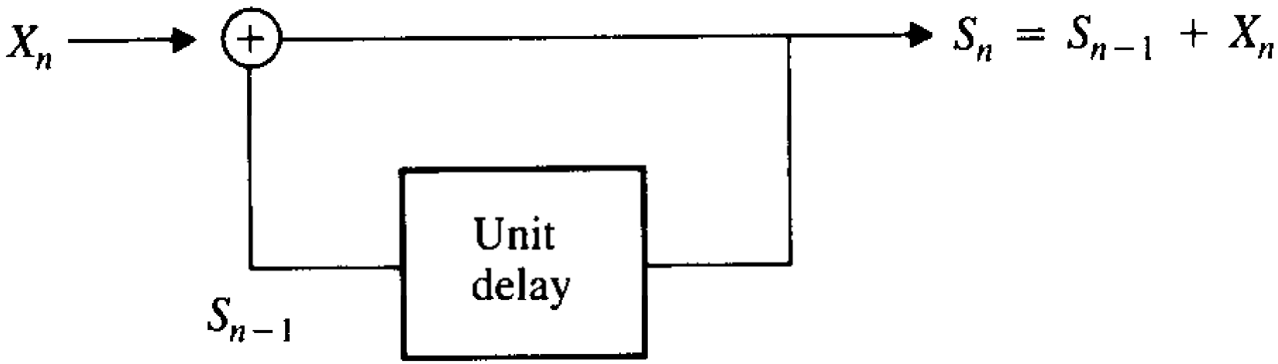
- Autocorrelation function

$$R_X(n_1, n_2) = C_X(n_1, n_2) + m^2$$

Sum Processes : The Binomial Counting and Random Walk Processes

- Sum of a sequence of iid random variables, $X_1, X_2 \ldots$

  $S_n = X_1 + X_2 + \ldots + X_n = S_{n-1} + X_n, \quad n = 1, 2, \ldots \text{(time)}$



$$f_{S_n}(s) = F^{-1}\{\Phi_{X_1}(\omega)\cdots\Phi_{X_n}(\omega)\}$$

- Note : S_n is independent of the past when S_{n-1} is known.
  $\rightarrow S_n$ is a Markov process.

Ex. 6.13

- I_i : Sequence of independent Bernoulli random variables.
- S_n : Sum Process $\rightarrow$ S_n is the counting process for successes
  $\rightarrow$ Gives the number of successes in the first n Bernoulli trials.
- Sample function for S_n corresponding to a particular sequence of I_i's is given in Fig. 6.4.

$$S_n = \sum_{i=1}^{n} I_i \rightarrow$$ S_n is a binomial random variable with parameters n and $p = P[I = 1]$

$$P[S_n = j] = \binom{n}{j} p^j (1-p)^{n-j}$$ for $0 \le j \le n$ and zero otherwise

- $E[S_n] = np$
- $\mathrm{VAR}[S_n] = np(1-p)$

$\longrightarrow$ Grows linearly with time

Independent increments:

two time intervals having no overlapping

$n_0 < n \le n_1$ and $n_2 < n \le n_3$ where $n_1 \le n_2$

$$S_{n_1} - S_{n_0} = X_{n_0+1} + \cdots + X_{n_1}$$

$$S_{n_3} - S_{n_2} = X_{n_2+1} + \cdots + X_{n_3}$$

$\rightarrow$ no common X_n's in the above

two random variables $S_{n_1} - S_{n_0}$ and $S_{n_3} - S_{n_2}$

If X_n is independent then

the increments $(S_{n_1} - S_{n_0})$ and $(S_{n_3} - S_{n_2})$

are independent r.v.'s.

- For $n' > n$, $S_{n'} - S_n =$ sum of $n' - n$ iid r.v.'s

  $\rightarrow$ **the same distribution as** $S_{n'-n}$**,**

  the sum of the first $(n'-n)X$'s

- $P[S_{n'} - S_n = y] = P[S_{n'-n} = y]$ $\rightarrow$ **stationary increments**

- Note : the increments in intervals of the same length have the same distribution regardless of when the interval begins.

- The joint pmf/pdf of S_n for any number of time instants
  - X_n's (iid): integer valued $\rightarrow S_n$: integer valued.

$$P[S_{n_1} = y_1, S_{n_2} = y_2, S_{n_3} = y_3]$$

$$= P[S_{n_1} = y_1, S_{n_2} - S_{n_1} = y_2 - y_1, S_{n_3} - S_{n_2} = y_3 - y_2]$$

$$\because \; S_{n_2} = S_{n_1} + X_{n_1+1} + \cdots + X_{n_2} = y_1 + X_{n_1+1} + \cdots + X_{n_2} = y_2$$

$$S_{n_2} - S_{n_1} = X_{n_1+1} + \cdots + X_{n_2} = y_2 - y_1$$

  - independent and stationary increments

$$P[S_{n_1} = y_1, S_{n_2} = y_2, S_{n_3} = y_3]$$

$$= P[S_{n_1} = y_1] P[S_{n_2} - S_{n_1} = y_2 - y_1] P[S_{n_3} - S_{n_2} = y_3 - y_2]$$

Stationary increments

$$= P[S_{n_1} = y_1] P[S_{n_2 - n_1} = y_2 - y_1] P[S_{n_3 - n_2} = y_3 - y_2]$$

Generalization

- For integer-valued $\underline{X}_n$

$$P[S_{n_1} = y_1, S_{n_2} = y_2, \ldots, S_{n_k} = y_k]$$
$$= P[S_{n_1} = y_1] P[S_{n_2 - n_1} = y_2 - y_1] \cdots P[S_{n_k - n_{k-1}} = y_k - y_{k-1}]$$

- For continuous-valued X_n

$$f_{S_{n_1}, S_{n_2}, \ldots, S_{n_k}}(y_1, y_2, \ldots, y_k)$$
$$= f_{S_{n_1}}(y_1) f_{S_{n_2 - n_1}}(y_2 - y_1) \cdots f_{S_{n_k - n_{k-1}}}(y_k - y_{k-1})$$

- Ex. 6.16 the joint pmf for the binomial counting process at times n_1 and n_2

$$P\left[S_{n_1} = y_1, S_{n_2} = y_2\right] = P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1]$$

$$= \binom{n_2 - n_1}{y_2 - y_1} p^{y_2-y_1}(1-p)^{n_2-n_1-y_2+y_1} \binom{n_1}{y_1} p^{y_1}(1-p)^{n_1-y_1}$$

$$= \binom{n_2 - n_1}{y_2 - y_1}\binom{n_1}{y_1} p^{y_2}(1-p)^{n_2-y_2}$$

❖ Mean, variance and autocovariance of sum process

S_n = Sum of n iid r.v.'s

$$m_S(n) = E[S_n] = nE[X] = nm$$

$$\text{VAR}[S_n] = n\,\text{VAR}[X] = n\sigma^2$$

cf) $\text{VAR}[X_1 + X_2 + \cdots + X_n]$

$$= \text{VAR}[X_1] + \cdots + \text{VAR}[X_n] + \sum\sum \text{COV}(X_i, X_j)$$

iid $\rightarrow \text{COV}(X_i, X_j) = 0$ for $i \neq j$

autocovariance

$$C_S(n,k) = E[(S_n - E[S_n])(S_k - E[S_k])]$$

$$= E\left[(S_n - nm)(S_k - km)\right]$$

$$= E\left[\left\{\sum_{i=1}^{n}(X_i - m)\right\}\left\{\sum_{j=1}^{k}(X_j - m)\right\}\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} E[(X_i - m)(X_j - m)]$$

$$C_S(n,k) = \sum_{i=1}^{\min(n,k)} C_X(i,i) = \min(n,k)\sigma^2$$

cf) $C_X(i,j) = \sigma^2 \delta_{i,j}$: autocovariance of the iid process X_n

- Or $n \le k \rightarrow n = \min(n, k)$

$$
\begin{aligned}
C_S(n,k) &= E[(S_n - nm)(S_k - km)] \\
&= E[(S_n - nm)\{(S_n - nm) + (S_k - km) - (S_n - nm)]] \\
&= E[(S_n - nm)^2] + E[(S_n - nm)(S_k - S_n - (k-n)m)] \\
&= E[(S_n - nm)^2] + E[(S_n - nm)]E[(S_k - S_n - (k-n)m)]
\end{aligned}
$$

S_n and the increment $S_k - S_n$ are independent for $k>=n$

$$
\begin{aligned}
\therefore\ C_S(n,k) &= E[(S_n - nm)^2] \\
&= \mathrm{VAR}[S_n] = n\sigma^2
\end{aligned}
$$

6.4 Examples of Continuous-Time Random Processes

❖ Poisson Process

- $N(t)$: the number of event occurrences in the time interval $[0, t]$
  $\rightarrow$ Nondecreasing, integer valued, continuous-time random process.

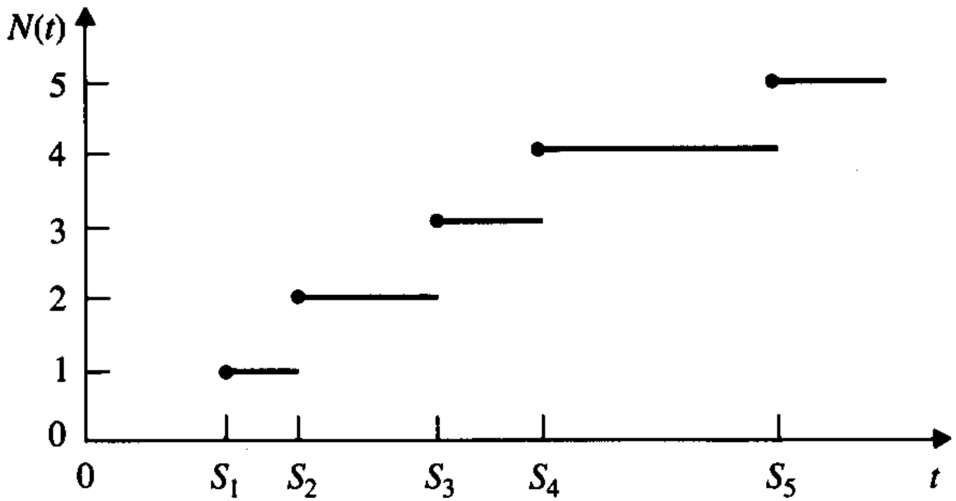
- The interval $[0, t]$ is divided into n subintervals of very short duration $\delta = t/n$

① The probability of more than one event occurrence in a subinterval $\ll$ the probability of observing one or zero events
$\rightarrow$ Bernoulli trial.

② An event occurrence in a subinterval is independent of each other. → independence of Bernoulli trial.

∴ $N(t)$: approximated by the binomial counting process.

- A sample path of the Poisson counting process.



- The probability of an event occurrence in each subinterval = p

$\rightarrow$ the expected number of event occurrences in the interval $[0, t] = np$ (n sub-intervals)

- λ : The rate of event occurrence $\rightarrow$ the average number of events in the interval $[0, t] = \lambda t$

$$\therefore \lambda t = np$$

cf) For a large n and a very small p

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \cong \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{for } k = 0,1,\ldots \qquad \text{:Eq. (3.31)}$$

with $\alpha = np = \lambda t$

- The number of event occurrence $N(t)$ in the interval $[0, t]$: the Poisson process

$$P[N(t)=k]=\frac{(\lambda t)^k}{k!}e^{-\lambda t} \quad \text{for} \quad k=0,1,\ldots$$

$\rightarrow$ a Poisson distribution with mean λt

- Note

The Poisson process $N(t)$'s properties (from the underlying binomial process)

① Independent increment

② Stationary increment

- The properties of independent and stationary increments

$\rightarrow$ the distribution for the number of occurrences in any interval of length t.

$$P[N(t)=k]=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

- The joint pmf for $N(t)$ at any number of points.

For $t_1 < t_2$

$$P\left[N(t_1)=i, N(t_2)=j\right]=P[N(t_1)=i]P[N(t_2)-N(t_1)=j-i]$$

$$=P[N(t_1)=i]P[N(t_2-t_1)=j-i]$$

$$=\frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!}\cdot\frac{[\lambda(t_2-t_1)]^{j-i}e^{-\lambda(t_2-t_1)}}{(j-i)!}$$

- The autocovariance of $N(t)$

  For $t_1 \le t_2$

$$
\begin{aligned}
C_N(t_1,t_2) &= E[(N(t_1)-\lambda t_1)(N(t_2)-\lambda t_2)] \\
&= E[(N(t_1)-\lambda t_1)\{N(t_2)-N(t_1)-\lambda t_2+\lambda t_1+(N(t_1)-\lambda t_1)\}] \\
&= E[N(t_1)-\lambda t_1]E[N(t_2)-N(t_1)-\lambda(t_2-t_1)]+\mathrm{VAR}[N(t_1)] \\
&= \mathrm{VAR}[N(t_1)]=\lambda t_1 \\
&= \lambda \min(t_1,t_2)
\end{aligned}
$$

cf) Poisson r.v. $p_k = \dfrac{\alpha^k e^{-\alpha}}{k!} \Rightarrow E[X]=\alpha,\ \mathrm{VAR}[X]=\alpha$

- The inter-event time T : time between event occurrences in a Poisson process.

  - The time interval $[0, t]$
  - $\delta = t/n$
  - n Bernoulli trials

$$
\begin{aligned}
P[T > t] &= P[\text{no events in } t \text{ seconds}] \\
&= (1-p)^n \\
&= \left(1 - \frac{\lambda t}{n}\right)^n \Rightarrow e^{-\lambda t} \quad \text{as } n \to \infty
\end{aligned}
$$

- Note : T is an exponential r.v. with parameter λ

cf) - $N(t)$: approximated by binomial counting process.

- T : independent geometric random variables

(memoryless r.v.)

$\rightarrow$ exponential r.v. as n goes infinite

$$P[T = t] = (1 - p)^n p$$

- The sequence of inter-event times in a Poisson process is composed of independent r.v.'s.
- Note : “The inter-event times in a Poisson process form an iid sequence of exponential random variables with mean $1/\lambda$.

- S_n : the time at which the nth event occurs in a Poisson process

  $S_n = T_1 + T_2 + \ldots + T_n, \quad T_j$: iid exponential interarrival times.

- cf) $\Phi_X(\omega) = \dfrac{\lambda}{\lambda - j\omega}$: The Characteristic function of a single exponential r.v.

  $$\Phi_{S_n}(\omega) = \left\{ \frac{\lambda}{\lambda - j\omega} \right\}^n$$ → m-Erlang r.v.

  $$\therefore f_{S_n}(y) = \frac{(\lambda y)^{n-1}}{(n-1)!} \lambda e^{-\lambda y}, \quad y \geq 0$$

- cf) $f_X(x) = \lambda e^{-\lambda x}$: exponential r.v.

- Arrivals occurs “at random”.

  Assumption : only one arrival in $[0, t]$.

  X : the arrival time of the single customer.

  For $0 < x < t$, $N(x)$: the number of events upto time x.

  $N(t) - N(x)$: the increment in the interval $(x, t]$.

- $$P[X \le x] = P[N(x) = 1 \mid N(t) = 1]$$

$$= \frac{P[N(x) = 1 \text{ and } N(t) = 1]}{P[N(t) = 1]}$$

$$= \frac{P[N(x) = 1 \text{ and } N(t) - N(x) = 0]}{P[N(t) = 1]}$$

$$= \frac{P[N(x)=1]P[N(t)-N(x)=0]}{P[N(t)=1]}$$

$$= \frac{\lambda x e^{-\lambda x} \cdot e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}}$$

Poisson distribution with $k=1$

$$= \frac{x}{t}$$

$\rightarrow$ the arrival time is uniformly distributed in the interval **[0, *t*]**.

- Note

If the number of arrivals in the interval $[0, t]$ is k, then the individual arrival times are distributed independently and uniformly in the interval

Wiener Process and Brownian Motion

- The symmetric random walk process (i.e., $p = 1/2$)

  cf) D_n : the iid process of ± 1 random variables and

  $$S_n = D_1 + D_2 + \dots + D_n$$

  $\rightarrow$ one-dimensional random walk (steps with magnitude h).

- Magnitude of each step = h at every δ seconds.

  $\rightarrow$ at time t,

  $$X_\delta(t) = h(D_1 + D_2 + \dots + D_{[t/\delta]}) = hS_n$$

  where $n = t/\delta$ (intervals for t sec)

- $E[X_\delta(t)] = hE[S_n] = 0$

  $\mathrm{VAR}[X_\delta(t)] = h^2 n \mathrm{VAR}[D_n] = h^2 n$

  where $\mathrm{VAR}[D_n] = \mathrm{VAR}[2I_n - 1] = 2^2\ \mathrm{VAR}[I_n] = 4p(1-p)$

  cf) $D_n = +1 \text{ or } -1$ while $I_n = 1 \text{ or } 0$

  $\therefore \mathrm{VAR}[D_n] = 4 \cdot 1/2 \cdot (1 - 1/2) = 1$

  Variance of Bernoulli trial

  i.e., $n \to \infty$

- Let $\delta \to 0$, $h \to 0$ with $h = \sqrt{\alpha\delta}$ (step)

  and let $X(t)$ be the resulting process.

  $E[X(t)] = 0$

  $\mathrm{VAR}[X(t)] = (\sqrt{\alpha\delta})^2 (t/\delta) = \alpha t$

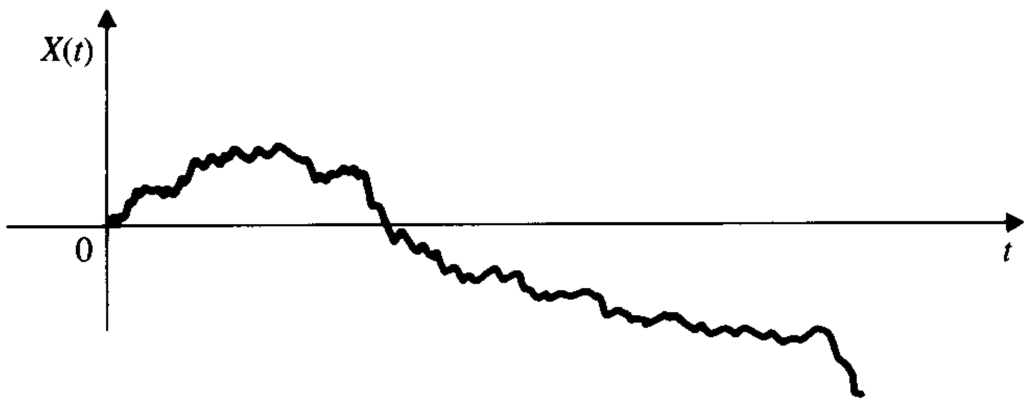
→ Continuous time process $X(t)$: Wiener random process.

① Begins at the origin

② Zero mean for all time

③ Variance increases linearly with time.

④ Used to model Brownian motion.



- As $\delta \to 0$, $X(t)$ approaches the sum of an infinite number of random variable.
- cf) $n = \dfrac{t}{\delta} \to \infty, X_\delta(t) \to X(t)$

$$X(t) = \lim_{\delta \to 0} hS_n = \lim_{n \to 0} \sqrt{\alpha t} \frac{S_n}{\sqrt{n}}$$

$$(\because h = \sqrt{\alpha \delta})$$

- The pdf of $X(t) \to$ pdf of Gaussian r.v. with mean zero and variance αt

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t}$$ by CLT

- The property of independent and stationary increments from the random walk process. ($\because$ sum of iid r.v.'s)

$$f_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = f_{X(t_1)}(x_1)\, f_{X(t_2-t_1)}(x_2 - x_1)\cdots f_{X(t_k-t_{k-1})}(x_k - x_{k-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1{}^2}{\alpha t_1} + \frac{(x_2 - x_1)^2}{\alpha(t_2 - t_1)} + \cdots + \frac{(x_k - x_{k-1})^2}{\alpha(t_k - t_{k-1})}\right]\right\}}{\sqrt{(2\pi\alpha)^k t_1 (t_2 - t_1)\cdots(t_k - t_{k-1})}}$$

- H.W.

  - Autocovariance of $X(t)$

$$C_X(t_1, t_2) = \alpha \min(t_1, t_2)$$

- Note : The Wiener Process and the Poisson Process have the same covariance function despite the fact that the two processes have very different sample functions.
  - ✓ Mean and autocovariance represent only partial information