6.5 Stationary Random Processes

- A discrete-time or continuous-time random process $X(t)$ is stationary if

$$F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k)$$

for all time shifts τ, all k, and all choices of sample times $t_1,\ldots,t_k$.

- Jointly stationary

For two processes $X(t)$ and $Y(t)$,
the joint cdf's of $X(t_1),\dots,X(t_k)$ and $Y(t_1'),\dots,Y(t_j')$
do not depend on the placement of the time origin
for all k and j and all choices of sampling times
$t_1,\dots,t_k$ and $t_1',\dots,t_j'$

- The first-order cdf of a stationary random process must be independent of time.

  $\rightarrow F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(\mathrm{x})$ all $t,\ \tau$.

  $\rightarrow m_X(t) = E[X(t)] = m$ for all t.

  $\mathrm{VAR}[X(t)] = E[(X(t)-m)^2] = \sigma^2$ for all t.

- The second-order cdf of a stationary random process can depend only on the time difference between the samples.

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \quad \text{for all } t_1, t_2$$

- **Function of (t_2-t_1)**

$$
\begin{aligned}
C_X(t_1,t_2) &= E[\{X(t_1)-m_X(t_1)\}\{X(t_2)-m_X(t_2)\}] \\
&= R_X(t_1,t_2)-m_X(t_1)m_X(t_2) \\
&= R_X(t_2-t_1)-m^2 \\
&= C_X(t_2-t_1) \qquad \text{for all } t_1,t_2.
\end{aligned}
$$

❖ Ex. 6.27

Is the sum process a discrete-time stationary process?

- sol) $S_n = X_1 + X_2 + \ldots + X_n$, where X_i are an iid sequence and n is time index.

cf) S_n : independent increment

stationary increment

$m_S(n) = nm$, $\quad \mathrm{VAR}[S_n] = n\sigma^2$

<u>Note</u> : Stationary process → Constant mean and variance.

∴ Cannot be a stationary process.

- Ex. 6.28
  - Random process (telegraph signal) $X(t)$ that assumes the values ± 1.
  - $X(0) = \pm 1$ with probability of ½.
  - $X(t)$ changes polarity with each occurrence of an event in a Poisson process of rate α.
  - Show that $X(t)$ is a stationary random process.

    Show that $X(t)$ settles into a stationary behavior as $t \to \infty$ even if $P[X(0) = \pm 1] \neq$ ½.

- sol) Need to show

$$P[X(t_1)=a_1,\ldots,X(t_k)=a_k]$$
$$=P[X(t_1+\tau)=a_1,\ldots,X(t_k+\tau)=a_k]$$

for any k, any $t_1<\cdots<t_k$ and any $a_j=\pm 1$

- The independent increments property of the Poisson process.

$$P[X(t_1)=a_1,\ldots,X(t_k)=a_k]=P\left[X(t_1)=a_1\right]$$
$$\times P[X(t_2)=a_2 \,|\, X(t_1)=a_1]\cdots P[X(t_k)=a_k \,|\, X(t_{k-1})=a_{k-1}]$$

- cf) the sum process

$$P[S_{n_1} = y_1, S_{n_2} = y_2, \ldots, S_{n_k} = y_k] = P[S_{n_1} = y_1]$$
$$\times P[S_{n_2} - S_{n_1} = y_2 - y_1] \cdots P[S_{n_k} - S_{n_{k-1}} = y_k - y_{k-1}]$$
$$= P[S_{n_1} = y_1] P[S_{n_2 - n_1} = y_2 - y_1] \cdots P[S_{n_k - n_{k-1}} = y_k - y_{k-1}]$$

$\because$ The values of the random telegraph at the times $t_1, \ldots, t_k$ is determined by the number of occurrences of the Poisson process in the time intervals (t_j, t_{j+1}).

- Similarly

$$P[X(t_1+\tau)=a_1,\ldots,X(t_k+\tau)=a_k]$$
$$=P[X(t_1+\tau)=a_1]P[X(t_2+\tau)=a_2 \,|\, X(t_1+\tau)=a_1]\cdots$$
$$\times P[X(t_k+\tau)=a_k \,|\, X(t_{k-1}+\tau)=a_{k-1}]$$

- Conditional probability (ex 6.22)

$$P[X(t_{j+1})=a_{j+1} \,|\, X(t_j)=a_j]=\begin{cases}\dfrac{1}{2}\left\{1+e^{-2\alpha(t_{j+1}-t_j)}\right\} & \text{if } \; a_j=a_{j+1}\\[2ex] \dfrac{1}{2}\left\{1-e^{-2\alpha(t_{j+1}-t_j)}\right\} & \text{if } \; a_j\neq a_{j+1}\end{cases}$$

- cf) $P\left[X(t)=\pm 1 \middle| X(0)=\pm 1\right]=P[N(t)=\text{even integer}]$

$$=\sum_{j=0}^{\infty}\frac{(\alpha t)^{2j}}{(2j)!}e^{-\alpha t}=e^{-\alpha t}\sum_{j=0}^{\infty}\frac{(\alpha t)^{2j}}{(2j)!}$$

$$=e^{-\alpha t}\frac{1}{2}(e^{-\alpha t}+e^{-\alpha t})=\frac{1}{2}(1+e^{-2\alpha t})$$

where $\quad e^{\alpha}=1+\alpha+\frac{1}{2!}\alpha^{2}\cdots,\quad e^{-\alpha}=1-\alpha+\frac{1}{2!}\alpha^{2}-\cdots$

- cf) Poisson process

$$P[N(t)=k]=\frac{(\lambda t)^{k}}{k!}e^{-\lambda t}$$

- $$P[X(t_{j+1}+\tau)=a_{j+1} \mid X(t_j+\tau)=a_j]$$
$$=\begin{cases} \frac{1}{2}\left\{1+e^{-2\alpha(t_{j+1}+\tau-t_j-\tau)}\right\} & \text{if } a_j = a_{j+1} \\ \frac{1}{2}\left\{1-e^{-2\alpha(t_{j+1}+\tau-t_j-\tau)}\right\} & \text{if } a_j \neq a_{j+1} \end{cases}$$

- The joint probabilities differ only in the first term.
  $\rightarrow P[X(t_1)=a_1]$ and $P[X(t_1+\tau)=a_1]$

- cf)

$$P[X(t)=1]=P[X(t)=1 \mid X(0)=1]P[X(0)=1]$$
$$+P[X(t)=1 \mid X(0)=-1]P[X(0)=-1]$$
$$=\frac{1}{2}\cdot\frac{1}{2}\left\{1+e^{-2\alpha t}\right\}+\frac{1}{2}\cdot\frac{1}{2}\left\{1-e^{-2\alpha t}\right\}$$
$$=\frac{1}{2}$$

$$P[X(t)=-1]=\frac{1}{2}$$

$$\therefore P[X(t_1)=a_1]=P[X(t_1+\tau)=a_1]=\frac{1}{2} \quad \text{with} \quad P[X(0)=\pm 1]=\frac{1}{2}$$

- If $P[X(0)=\pm 1] \neq \frac{1}{2} \Rightarrow P[X(t_1)=a_1] \neq P[X(t_1+\tau)=a_1]$

If $P[X(0)=1]=1,$

$$P\big[X(t)=a\big] = P[X(t)=a \,|\, X(0)=1]\cdot 1$$

$$= \begin{cases} \frac{1}{2}\left\{1+e^{-2\alpha t}\right\} & \text{if } a=1 \\ \frac{1}{2}\left\{1-e^{-2\alpha t}\right\} & \text{if } a=-1 \end{cases}$$

$$= P[X(t_1)=a_1] \rightarrow \frac{1}{2} \quad \text{as } t_1 \text{ becomes large}$$

The process forgets the initial condition
and settles down into steady state
→ stationary behavior.

Wide-Sense Stationary Random Processes

- Cannot determine whether a random process is stationary

- Can determine whether

  $m_X(t) = \mathrm{m}$ for all t.

  $C_X(t_1,\ t_2) = C_X(t_1 - t_2)$ for all $t_1,\ t_2$

  Function of t_1-t_2 only

  X(t) is wide-sense stationary (WSS).

- Jointly Wide-Sense Stationary

  ① $X(t)$ and $Y(t)$ are both wide-sense stationary.

  ② Cross-Covariance depends only on $t_1 - t_2$

- Note
  - $X(t)$ is Wide-Sense Stationary

    $\rightarrow$ auto covariance $C_X(t_1,\ t_2) = C_X(\tau)$ and

    auto correlation $R_X(t_1,\ t_2) = R_X(\tau)$

    where $\tau = t_1 - t_2$

- Note
  - All stationary random processes are wide-sense stationary.
  - Some wide-sense stationary processes are not stationary.

Ex. 6.29

- X_n : Consist of two interleaved sequences of independent r.v.'s.

  For n even, $P[X_n = \pm 1] = \frac{1}{2}$

  For n odd, $P\left[X_n = \frac{1}{3}\right] = \frac{9}{10}$, $P[X_n = -3] = \frac{1}{10}$

- X_n is not stationary since its pmf varies with n.
- $m_X(n) = 0$

$$C_X(i,j) = \begin{cases} E[X_i]E[X_j] = 0 & \text{for } i \neq j \\ E[X_i^2] = 1 & \text{for } i = j \end{cases}$$

$\rightarrow X_n$: Wide-Sense Stationary.

Properties of Autocorrelation Function of WSS Process

- Average power of the process.
  - $R_X(0) = E[X^2(t)]$ for all t.

- Even function of τ
  - $R_X(\tau) = E[X(t+\tau)\, X(t)] = E[X(t)\, X(t+\tau)] = R_X(-\tau)$

- Measure of the rate of change of a random process.
  - The change in the process from time t to $t+\tau$:

$$P\left[\left|X(t+\tau)-X(t)\right| > \varepsilon\right] = P[(X(t+\tau)-X(t))^2 > \varepsilon^2]$$

$$\leq \frac{E[(X(t+\tau)-X(t))^2]}{\varepsilon^2} = \frac{2\{R_X(0)-R_X(\tau)\}}{\varepsilon^2}$$

- cf) Markov inequality

$$P[X \geq a] \leq \frac{E[X]}{a}$$

- Observation:

  If $R_X(0) - R_X(\tau)$ is small, the probability of a large change in $X(t)$ in τ seconds is small.

cf) $R_X(0) - R_X(\tau)$ is small

$\rightarrow R_X(\tau)$ drops off slowly.

- $R_X(\tau)$ is maximum at $\tau = 0$
  - Proof)

① $E[XY]^2 \le E[X^2]E[Y^2]$ for any two r.v.'s X and Y.

- Can be proved using the approach used to prove $|\rho| \le 1$. - HW

② $R_X(\tau)^2 = E[X(t+\tau)X(t)]^2 \le E[X^2(t+\tau)]E[X^2(t)] = R_X(0)^2$

  - Thus

$$\left| R_X(\tau) \right| \le R_X(0)$$

cf) $R_X(0)$ is positive $\because R_X(0) = E[X^2(t)]$

- If $R_X(0) = R_X(d)$, then $R_X(\tau)$ is periodic with period d and $X(t)$ is <u>mean square periodic</u>.

  $\rightarrow$ $\boldsymbol{E[(X(t+d) - X(t))^2] = 0}$

  pf) $E[(X(t+\tau+d) - X(t+\tau))X(t)]^2$

  $$\leq E[(X(t+\tau+d) - X(t+\tau))^2]E[X^2(t)]$$

  $$\rightarrow \{R_X(\tau+d) - R_X(\tau)\}^2 \leq 2\{R_X(0) - R_X(d)\}R_X(0)$$

  $\therefore\ R_X(0) = R_X(d) \ \rightarrow$ R.H.S. is zero

  $\therefore\ R_X(\tau) = R_X(\tau+d)$ for all $\tau \ \rightarrow \ R_X(\tau)$ is periodic with period d.

  - Mean square periodic:

    $$E[(X(t+d) - X(t))^2] = 2\{R_X(0) - R_X(d)\} = 0$$

- Let $X(t) = m + N(t)$, where $N(t)$ is a zero-mean process for which $R_N(\tau) \to 0$ as $\tau \to \infty$, then

$$\begin{aligned} R_X(\tau) &= E[(m+N(t+\tau))(m+N(t))] \\ &= m^2 + 2mE[N(t)] + R_N(\tau) \\ &= m^2 + R_N(\tau) \;\to\; m^2 \quad \text{as } \tau \to \infty \end{aligned}$$

  - Note

    $R_X(\tau)$ approaches the square of the mean of $X(t)$ as $\tau \to \infty$.

- Summary: Three type of components

① $R_{X1}(\tau) \to 0$ as $|\tau| \to \infty$

② $R_{X2}(\tau) = R_{X2}(\tau + d)$

③ $R_{X3}(\tau) \to m^2$ as $|\tau| \to \infty$

WSS Gaussian Random Processes

- If a Gaussian random process is wide-sense stationary, then it is also stationary.

Proof)

- The joint pdf of a Gaussian random process is completely determined by the mean $m_X(t)$ and autocovariance $C_X(t_1, t_2)$.

- $X(t)$ is wide sense stationary $\rightarrow$ its mean is constant its autocovariance is only the function of the difference of the sampling times $t_i - t_j$ $\rightarrow$ the joint pdf of $X(t)$ depends only on this set of differences $\rightarrow$ invariant with respect to time shifts
- Thus the process is also stationary

Cyclostationary Random Processes

- $$F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k) = F_{X(t_1+mT),X(t_2+mT),\ldots,X(t_k+mT)}(x_1, x_2, \ldots, x_k)$$

  For all k, m and all choices of sampling times $t_1,\ldots,t_k$

- Wide-Sense Cyclostationary.

  :If the mean and autocovariance functions are invariant with respect to shifts in the time origin by integer multiples of T

  $$m_X(t+mT) = m_X(t)$$

  $$C_X(t_1+mT, t_2+mT) = C_X(t_1, t_2)$$

- Note
  - If $X(t)$ is cyclostationary, then $X(t)$ is also wide-sense cyclostationary.

- $X(t)$ is a cyclostationary process with period T.

$\rightarrow$ $X(t)$ is stationarized by observing a randomly phase-shifted version of $X(t)$

- $X_S(t) = X(t+\Theta)$, Θ uniform in $[0, T]$,

  where Θ is independent of $X(t)$.

$\rightarrow$ If $X(t)$ is a cyclostationary, $X_S(t)$ is a stationary random process.

- If $X(t)$ is a wide-sense cyclostationary random process, then $X_S(t)$ is a wide-sense stationary random process

$$E[X_s(t)] = \frac{1}{T}\int_0^T m_X(t)dt$$

$$R_{X_s}(\tau) = \frac{1}{T}\int_0^T R_X(t+\tau, t)dt$$

6.6 Continuity, Derivatives and Integrals of Random Processes

- The system having dynamics: described by linear differential eqs.
- Each sample function of a random process: deterministic signal
- Input to the system: Sample function of continuous-time random process

  Output of the system: A sample function of another random process
- Probabilistic methods for addressing the continuity, differentiability and integrability of random processes

cf) A random process: the ensemble of sample functions

Mean Square Continuity

- $X(t, \zeta)$: A particular deterministic sample function for each point ζ in S of random process

- The continuity of the sample function at a point t_0 for each point ζ :

  If given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|t - t_0| < \delta$ implies that $|X(t, \zeta) - X(t_0, \zeta)| < \varepsilon$

  - $\lim_{t \to t_0} X(t, \zeta) = X(t_0, \zeta)$

- All sample functions of the random process are continuous at t_0, then the random process is continuous
- The continuity of random process in a probabilistic sense is considered.
- Mean square continuity: $\underset{t \to t_0}{\text{l.i.m.}} X(t) = X(t_0)$
- The random process $X(t)$ is continuous at the point t_0 in the mean square sense if

$$E[(X(t) - X(t_0))^2] \to 0 \quad \text{as} \quad t \to t_0$$

- <u>Note:</u> Mean square continuity does not imply that all the sample functions are continuous

- Considering the mean square difference:

$$E[(X(t)-X(t_0))^2]=R_X(t,t)-R_X(t_0,t)-R_X(t,t_0)+R_X(t_0,t_0)$$

Therefore, if $R_X(t_1, t_2)$ is continuous in both t_1 and t_2 at the point (t_0, t_0), then $X(t)$ is mean square continuous at the point t_0.

- If $X(t)$ is <u>mean square continuous</u> at t_0, then the mean function $m_X(t)$ must be continuous at t_0.

$$\lim_{t\to t_0} m_X(t)=m_X(t_0)$$

- Proof

$$0 \le \mathrm{VAR}[X(t) - X(t_0)] = E[(X(t) - X(t_0))^2] - E[X(t) - X(t_0)]^2$$

$$\therefore E[(X(t) - X(t_0))^2] \ge E[X(t) - X(t_0)]^2 = [m_X(t) - m_X(t_0)]^2$$

If $X(t)$ is mean square continuous, L.H.S. $\rightarrow$ 0 as $t \rightarrow t_0$, then R.H.S. $\rightarrow$ 0, i.e., $m_X(t) \rightarrow m_X(t_0)$

- <u>Note:</u> If $X(t)$ is mean square continuous at t_0, then we can interchange the order of the limit and the expectation

$$\lim_{t \to t_0} E[X(t)] = E\left[\underset{t \to t_0}{\mathrm{l.i.m.}}\, X(t) \right]$$

- For the WSS random process $X(t)$,

$$E[(X(t_0+\tau)-X(t_0))^2]=2(R_X(0)-R_X(\tau))$$

: If $R_X(\tau)$ is continuous at $\tau=0$, then the WSS random process $X(t)$ is mean square continuous at every point t_0.

Mean Square Derivatives

- The derivative of a deterministic function

$$\lim_{\varepsilon \to 0} \frac{X(t+\varepsilon,\zeta) - X(t,\zeta)}{\varepsilon}$$

: this limit may exist for some sample functions and it may fail to exist for other sample functions

- Mean Square Derivative

$$X'(t) \equiv \underset{\varepsilon \to 0}{\mathrm{l.i.m.}} \frac{X(t+\varepsilon,\zeta) - X(t,\zeta)}{\varepsilon} \equiv \frac{dX(t)}{dt}$$

Provided that the mean square limit exists, that is,

$$\lim_{\varepsilon \to 0} E\left[\left(\frac{X(t+\varepsilon,\zeta) - X(t,\zeta)}{\varepsilon} - X'(t)\right)^2\right] = 0$$

- Note: The existence of the mean square derivative does not imply the existence of the derivative for all sample functions.

- The mean square derivative of $X(t)$ at the point t exists if

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$$

exists at the point $(t_1, t_2) = (t, t)$

Proof) Use the Cauchy criterion

- If the random process $X(t)$ is WSS

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1 - t_2)$$

$$= \frac{\partial}{\partial t_1}\left(-\frac{d}{d\tau} R_X(t_1 - t_2)\right) = -\frac{\partial^2}{\partial \tau^2} R_X(\tau)$$

The mean square derivative of a WSS random process $X(t)$ exists if $R_X(\tau)$ has derivatives up to order two at $\tau=0$.

- For a Gaussian random process $X(t)$, if $X'(t)$ exists, then $X'(t)$ must be a Gaussian random process

- Mean of $X'(t)$

$$E[X'(t)] = E\left[\underset{\varepsilon\to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon)-X(t)}{\varepsilon}\right] = \lim_{\varepsilon\to 0} E\left[\frac{X(t+\varepsilon)-X(t)}{\varepsilon}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{m_X(t+\varepsilon)-m_X(t)}{\varepsilon} = \frac{d}{dt} m_X(t)$$

- The cross-correlation between $X(t)$ and $X'(t)$

$$R_{X,X'}(t_1,t_2) = E\left[X(t_1)\underset{\varepsilon\to 0}{\text{l.i.m.}} \frac{X(t_2+\varepsilon)-X(t_2)}{\varepsilon}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{R_X(t_1,t_2+\varepsilon)-R_X(t_1,t_2)}{\varepsilon} = \frac{\partial}{\partial t_2} R_X(t_1,t_2)$$

- The autocorrelation of $X'(t)$

$$R_{X'}(t_1,t_2) = E\left[\underset{\varepsilon\to 0}{\text{l.i.m.}}\left\{\frac{X(t_1+\varepsilon)-X(t_1)}{\varepsilon}\right\}X'(t_2)\right]$$

$$= \lim_{\varepsilon\to 0}\frac{R_{X,X'}(t_1+\varepsilon,t_2)-R_{X,X'}(t_1,t_2)}{\varepsilon}$$

$$= \frac{\partial}{\partial t_1}R_{X,X'}(t_1,t_2)$$

$$= \frac{\partial^2}{\partial t_1 \partial t_2}R_X(t_1,t_2)$$

- For the WSS random process $X(t)$

$$R_{X,X'}(\tau) = \frac{\partial}{\partial t_2} R_X(t_1 - t_2) = -\frac{\partial}{\partial \tau} R_X(\tau)$$

$$R_{X'}(\tau) = \frac{\partial}{\partial t_1}\left\{\frac{\partial}{\partial t_2} R_X(t_1 - t_2)\right\} = -\frac{\partial^2}{\partial \tau^2} R_X(\tau)$$

Mean Square Integrals

- Imply the integral of a random process in the sense of mean square convergence

- The integral of the random process $X(t)$
  - The mean square limit of the sequence I_n as the width of the subintervals approaches zero:

$$I_n = \sum_{k=1}^{n} X(t_k)\Delta_k$$

$$Y(t) = \int_{t_0}^{t} X(t')dt' = \underset{\Delta_k \to 0}{\text{l.i.m.}} \sum_{k} X(t_k)\Delta_k$$

- Conditions that ensure the existence of the mean square integral

$$E\left[\left\{\sum_j X(t_j)\Delta_j - \sum_k X(t_k)\Delta_k\right\}^2\right] \to 0 \quad \text{as } \Delta_j, \Delta_k \to 0$$

:The Cauchy criterion

- Expanding the square inside the expected value

$$E\left[\sum_j \sum_k X(t_j) X(t_k) \Delta_j \Delta_k\right] = \sum_j \sum_k R_X(t_j, t_k) \Delta_j \Delta_k$$

- The limit of the right hand side approaches a double integral

$$\lim_{\Delta_j,\Delta_k \to 0} \sum_j \sum_k R_X(t_j, t_k)\Delta_j \Delta_k = \int_{t_0}^{t} \int_{t_0}^{t} R_X(u,v)\,du\,dv$$

- The mean square integral of $X(t)$ exists if the double integral of the autocorrelation function exists
- If $X(t)$ is a mean square continuous random process, then its integral exists.

❖ The mean and autocorrelation function of $Y(t)$

$$m_Y(t) = E\left[\int_{t_0}^{t} X(t')dt'\right] = \int_{t_0}^{t} E\left[X(t')\right]dt'$$

$$= \int_{t_0}^{t} m_X(t')dt'$$

$$R_Y(t_1, t_2) = E\left[\int_{t_0}^{t_1} X(u)du \int_{t_0}^{t_2} X(v)dv\right]$$

$$= \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u, v)dudv$$

6.7 Time Averages of Random Processes and Ergodic Theorems

- To estimate the mean $m_X(t)$ of a random process $X(t, \zeta)$,

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t, \xi_i)$$

where N is the number of repetitions of the experiment

- In estimating the mean or autocorrelation functions from the time average of a single realization

$$\left\langle X(t) \right\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t, \xi)\,dt$$

- Ergodic theorems : when time averages converge to the ensemble average (expected value)

  cf) Strong law of large numbers :

  : if X_n is an iid discrete-time random process with finite mean $E[X_n] = m$, then

$$P\left[\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i = m\right] = 1$$

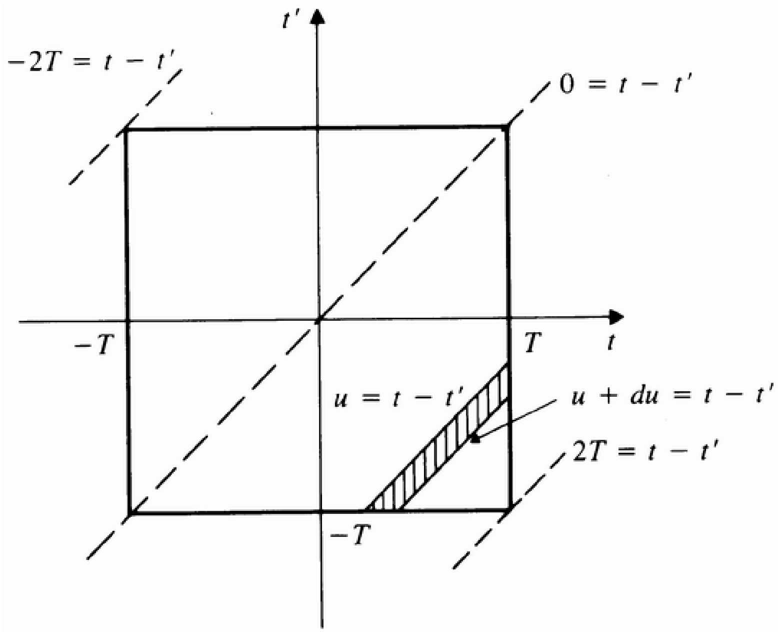
- $\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\xi)dt$ yields a single number

  $\Rightarrow$ consider process for which $m_X(t) = m$

- An ergodic theorem for the time average of wide-sense stationary processes
  - $X(t)$: WSS process

$$E[\langle X(t)\rangle_T] = E\left[\frac{1}{2T}\int_{-T}^{T} X(t)dt\right]$$
$$= \frac{1}{2T}\int_{-T}^{T} E[X(t)]dt = m$$

$$\mathrm{VAR}[\langle X(t)\rangle_T] = E[(\langle X(t)\rangle_T - m)^2]$$
$$= E\left[\left\{\frac{1}{2T}\int_{-T}^{T}(X(t)-m)dt\right\}\left\{\frac{1}{2T}\int_{-T}^{T}(X(t')-m)dt'\right\}\right]$$



$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t,t')\,dt\,dt'$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t-t')\,dt\,dt'$$

$$= \frac{1}{4T^2}\int_{-2T}^{2T} (2T-|u|)C_X(u)\,du$$

$$= \frac{1}{2T}\int_{-2T}^{2T} (1-\frac{|u|}{2T})C_X(u)\,du$$

$\Rightarrow$ $\langle X(t)\rangle_T$ will approach m in the mean square sense,

i.e., $E[(\langle X(t)\rangle_T - m)^2] \to 0$ if and only if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)C_X(u)\,du = 0$$

- Discrete-time random process

$$\langle X_n \rangle_T = \frac{1}{2T+1} \sum_{n=-T}^{T} X_n$$

$$\langle X_{n+k} X_n \rangle_T = \frac{1}{2T+1} \sum_{n=-T}^{T} X_{n+k} X_n$$

- If X_n is WSS

$$E[\langle X_n \rangle_T] = m$$

$$\mathrm{VAR}[\langle X_n \rangle_T] = \frac{1}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) C_X(k)$$

  - ($\mathrm{VAR}[<X_n>_T] \to 0$: mean square sense)

Home work

- Ch. 6 Problems
- 3,5,7,10,15,18,21,24,29,34,
- 37,40,44,48,52,55,59,63,67,69,
- 71,74,79,83,87,89