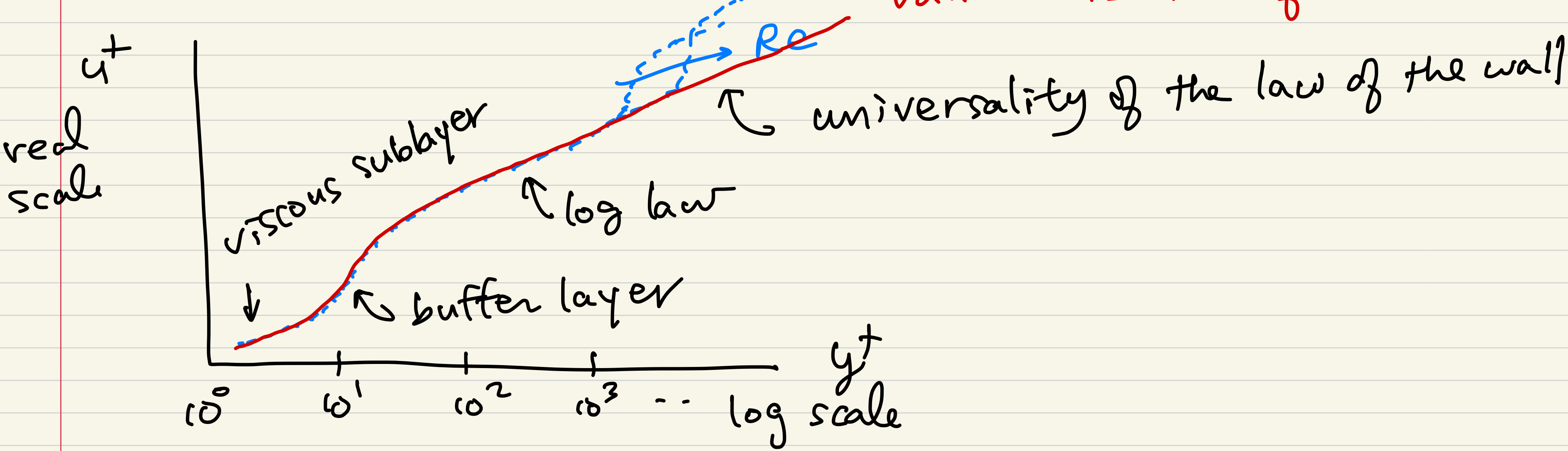
- Law of the wall

Van Driest's law of the wall



From mixing-length hypothesis,

$$\frac{1}{\rho}\tau(y) = \nu\frac{\partial\langle U\rangle}{\partial y} - \langle uv\rangle = \nu\frac{\partial\langle U\rangle}{\partial y} + \nu_T\frac{\partial\langle U\rangle}{\partial y}$$

$$\left(l_m^+ = \frac{l_m}{\delta_\nu} = \frac{l_m u_\tau}{\nu}\right) \qquad = \nu\frac{\partial\langle U\rangle}{\partial y} + l_m^2\left(\frac{\partial\langle U\rangle}{\partial y}\right)^2$$

normalize this eq. in wall units

$$\rightarrow \quad \frac{\tau}{\tau_w} = \frac{\partial u^+}{\partial y^+} + \left(l_m^+\frac{\partial u^+}{\partial y^+}\right)^2 \qquad \tau_w = \rho u_\tau^2$$

Solve for $\frac{\partial u^+}{\partial y^+}$

$$\rightarrow \frac{\partial u^+}{\partial y^+} = \frac{2\tau/\tau_w}{1+\left[1+(4\tau/\tau_w)(l_m^+)^2\right]^{\frac{1}{2}}}$$

In the inner layer ($\tau/\tau_w \simeq 1$)

$$\rightarrow \boxed{u^+ = f_w(y^+) = \int_0^{y^+} \frac{2}{1+\left[1+4\,\underbrace{l_m^{+2}}_{l_m^+(y^+)}\right]^{1/2}}\, dy^+} \quad \text{①}$$

In the log-law region, $l_m = \kappa y \rightarrow l_m^+ = \kappa y^+$

In the viscous sublayer, $u^+ = y^+$

$$\nu_T \frac{\partial \langle U \rangle}{\partial y} = l_m^2 \left(\frac{\partial \langle U \rangle}{\partial y}\right)^2 \sim y^2 \quad (\text{if } l_m = \kappa y)$$

(modeled one)

$$\uparrow$$

$$-\langle uv \rangle \sim y \cdot y^2 \sim y^3$$

(real one)

contradiction

$\downarrow$

correction is required.

or l_m should be damped near the wall

→ Van Driest (1956) : $\boxed{l_m^+ = \kappa y^+ [1 - \exp(-y^+/A^+)]}$ — ②

van Driest damping ft.

① + ② : Van Driest's law of the wall.

matching condition between buffer layer & log law

→ $A^+ = 26$ for $B = 5.3$ and $\kappa = 0.41$.

Boundary layers w/ various dP_0/dx

duct flow

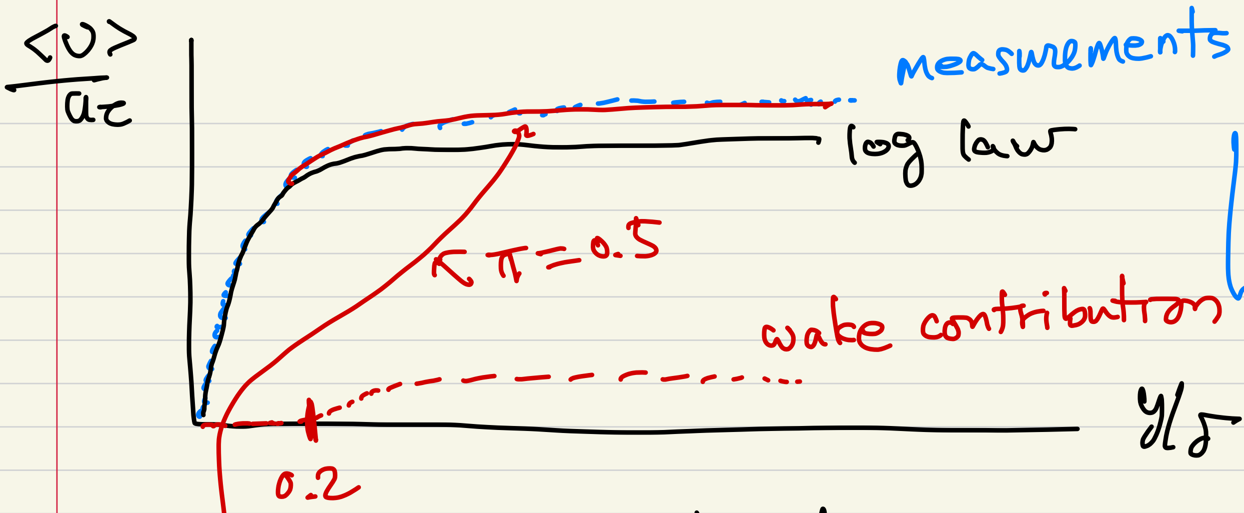
→ log law is observed. : $\kappa = 0.41$

A^+ increases significantly

when $-\frac{\partial \tau}{\partial y}\Big|_{y=0} > 2\times10^{-3} \frac{\tau_w}{\delta_\nu}$

- velocity-defect law

In the defect layer ($y/\delta > 0.2$), the mean velocity deviates from the log law.



Coles (1956)

wake ft.

$$\frac{\langle U\rangle}{u_\tau} = \underbrace{f_w\left(\frac{y}{\delta_\nu}\right)}_{\text{law of the wall}} + \underbrace{\frac{\Pi}{\kappa}\, w\left(\frac{y}{\delta}\right)}_{\text{law of the wake}}$$

$\Pi \neq \pi\ (=3.14\ldots)$

Π : wake strength parameter depending on the flow type

$$w\left(\frac{y}{\delta}\right) = 2\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$: an approximation.

the wake ft, w, is assumed to be universal and is defined to satisfy $w(0)=0$ & $w(1)=2$.

$$\rightarrow \frac{\langle U\rangle}{u_\tau} = \frac{1}{\kappa}\ln\frac{y}{\delta_\nu} + B + \frac{\Pi}{\kappa}\, w\left(\frac{y}{\delta}\right)$$

$$- \quad \frac{U_0}{u_\tau} = \frac{1}{\kappa}\ln\frac{\delta}{\delta_\nu} + B + \frac{\Pi}{\kappa}\,\overset{2}{\cancel{w(1)}} \quad @\ y=\delta$$

$$\frac{U_0 - \langle U\rangle}{u_\tau} = \frac{1}{\kappa}\left\{ -\ln\frac{y}{\delta} + \Pi\left[2 - w\left(\frac{y}{\delta}\right)\right]\right\}$$

$$\frac{U_0}{u_\tau} = \frac{1}{\kappa}\ln\left(\frac{\delta u_\tau}{\nu}\right) + B + \frac{2\Pi}{\kappa}$$

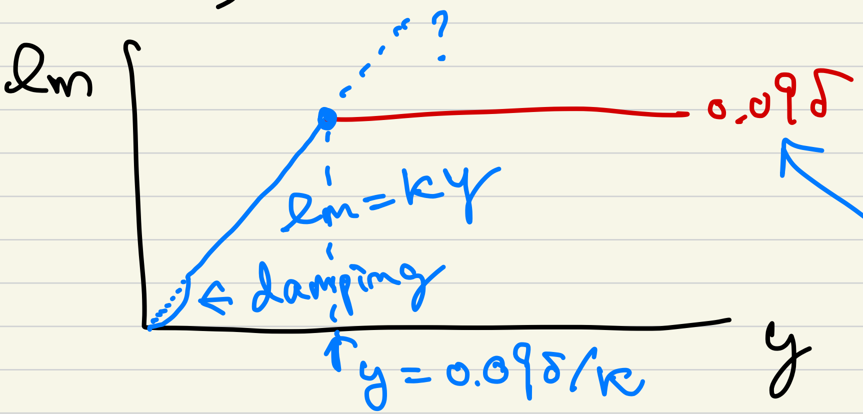
$$= \frac{1}{\kappa}\ln\left(Re_\delta \frac{u_\tau}{U_0}\right) + B + \frac{2\Pi}{\kappa}$$

$$\frac{c_f}{2} = \left(\frac{u_\tau}{U_0}\right)^2$$

$$\rightarrow \boxed{\sqrt{\frac{2}{c_f}} = \frac{1}{\kappa}\ln\left(Re_\delta \sqrt{\frac{c_f}{2}}\right) + B + \frac{2\Pi}{\kappa}}$$ friction law

$$c_f = 0.370\left[\log_{10} Re_x\right]^{-2.584}$$: Schultz-Grunow formula.

- mixing length l_m



$$\left.\frac{\partial\langle U\rangle}{\partial y}\right|_{real} > \left.\frac{\partial\langle U\rangle}{\partial y}\right|_{log\ law}$$

& $\tau > \tau_w$

$$\nu_T = l_m^2\left(\frac{\partial\langle U\rangle}{\partial y}\right)^2$$

$$= \kappa^2 y^2 \frac{u_\tau}{\kappa y}$$

$$= u_\tau \kappa y$$

$\nu_T = \tau \Big/ \frac{\partial\langle U\rangle}{\partial y}$ is less than $u_\tau \kappa y$.

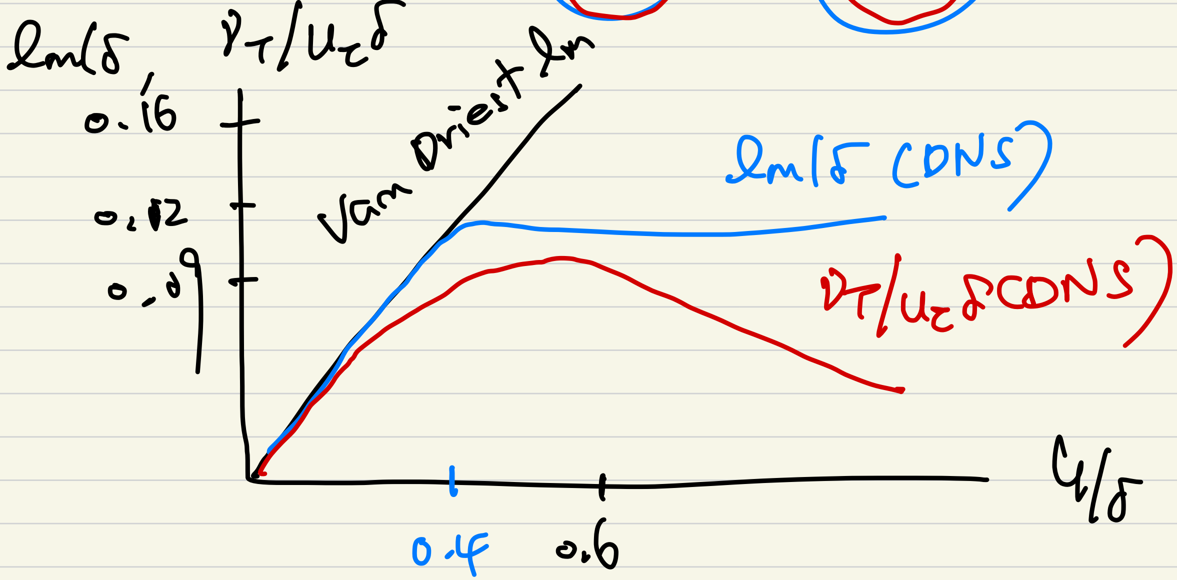
$\rightarrow l_m < \kappa y. \rightarrow l_m = \min(\kappa y, 0.09\delta)$ by Escudier (1966)

$$\frac{1}{\rho}\tau(y) = \nu\frac{\partial\langle U\rangle}{\partial y} + l_m^2\left(\frac{\partial\langle U\rangle}{\partial y}\right)^2$$

$$\text{or} = \nu\frac{\partial\langle U\rangle}{\partial y} + \nu_T\frac{\partial\langle U\rangle}{\partial y}$$

O's are obtained by DNS.

So, obtain l_m & ν_T from DNS.



- overlap region reconsidered.

log law $\frac{\partial\langle U\rangle}{\partial y}\Big/\frac{u_\tau}{y} \neq f\left(\frac{y}{\delta_\nu}\right)$: universal

$= \text{const}$ indep. of Re.

But the Reynolds stresses depend on Re # in the overlap region.

Alternative assump.

inner layer $\quad u^+ = f_I(y^+) \qquad y^+ = y/\delta_\nu$

outer " $\quad \dfrac{U_0 - \langle U \rangle}{u_0} = F_0(\eta) \qquad \eta = y/\delta$

may depend on Re.

u_0 : velocity scale for outer layer

$u_0 = U_0$ boundary layer

$u_0 = U_0 - \bar{U}$ pipe flow

in the overlap layer ($\delta_\nu \ll y \ll \delta$),

matching allows two ft. forms,

① log law, $\quad u^+ = \frac{1}{\kappa} \ln y^+ + B$

② power law, $\quad u^+ = c\, y^{+\alpha}$

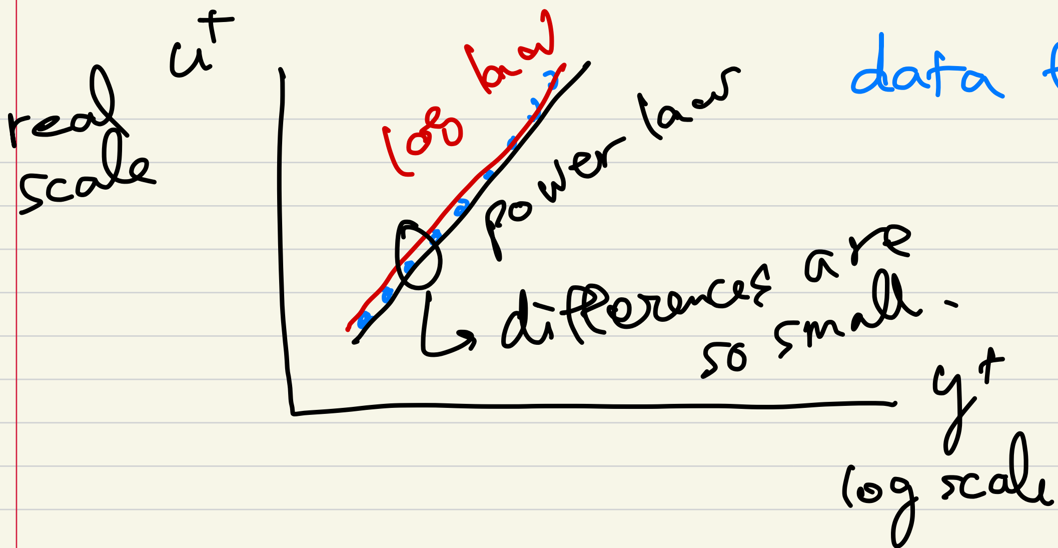
κ, B, c, α are allowed to dep. on Re.

If these are indep. of Re, laws are universal.

HW

Solve Exercise 7.20

due by May 27.



data from Zagarola & Smits (1997)
turb. pipe flow
$Re = 32 \times 10^3 \sim 30 \times 10^6$

$\kappa = 0.436$ and $B = 6.13$

$$\alpha = \frac{1.085}{\ln Re} + \frac{6.535}{(\ln Re)^2}.$$

§ 7.3.5 Reynolds-stress balances — skip

Additional effects

- Mean press. gradients

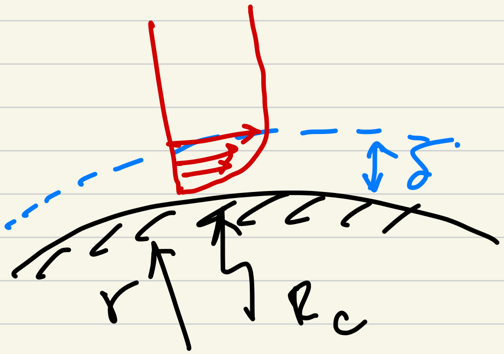
favorable press. grad. $\frac{dp_0}{dx} < 0$, $\frac{dU_0}{dx} > 0$ → steeper mean vel.
H : decreases
c_f : increases.

adverse " " $\frac{dp_0}{dx} > 0$, $\frac{dU_0}{dx} < 0$ → less steep mean vel.
H : increases
c_f : decreases.

for strong adverse " ", bdry layers seperate.

- <u>surface curvature</u>

<u>convex surface</u>



$\delta / R_c = 0.01 \sim 0.1$

angular mtm increases with radius

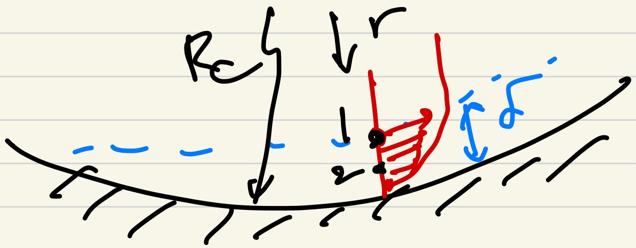
$\Downarrow$

stabilizing according to the Rayleigh's criterion

$\boxed{\begin{array}{l}\text{* Rayleigh's circulation criterion} \\ \text{for } r_2 > r_1, \quad (r_2 v_{\theta_2})^2 > (r_1 v_{\theta_1})^2 \rightarrow \text{stable}\end{array}}$

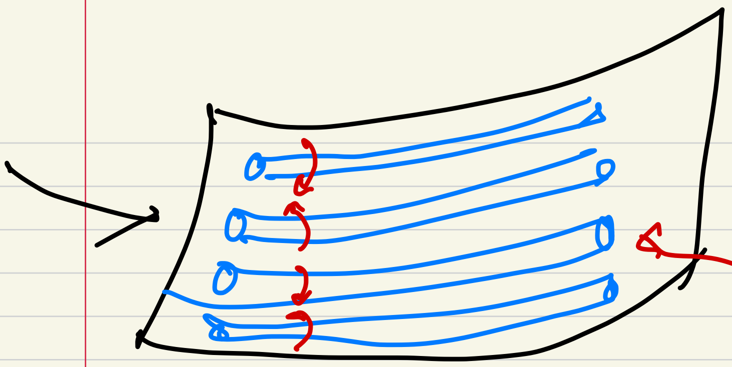
reduction in $\langle u_i u_j \rangle$ & C_f
(compared to those in flat-plate bdry layer)

<u>Concave surface</u>



$(r_2 v_{\theta_2})^2 < (r_1 v_{\theta_1})^2$

angular mtm decreases with radius



→ destabilizing

→ longitudinal Taylor - Görtler vortices

→ $\langle u_i u_j \rangle$ and c_f increase.

7.4 Turbulent structures