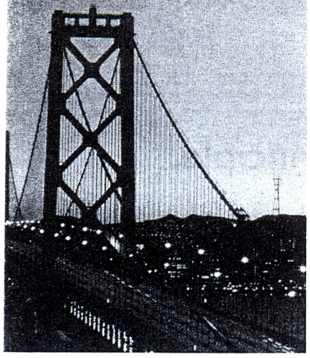

CHAPTER 3

Higher Order Linear ODEs

In this chapter we extend the concepts and methods of Chap. 2 for linear ODEs from order $n = 2$ to arbitrary order n. This will be straightforward and needs no new ideas. However, the formulas become more involved, the variety of roots of the characteristic equation (in Sec. 3.2) becomes much larger with increasing n, and the Wronskian plays a more prominent role.

Prerequisite: Secs. 2.1, 2.2, 2.6, 2.7, 2.10.
References and Answers to Problems: App. 1 Part A, and App. 2.

3.1 Homogeneous Linear ODEs

Recall from Sec. 1.1 that an ODE is of ***n*th order** if the nth derivative $y^{(n)} = d^n y/dx^n$ of the unknown function $y(x)$ is the highest occurring derivative. Thus the ODE is of the form

$$F(x, y, y', \cdots, y^{(n)}) = 0 \qquad \left(y^{(n)} = \frac{d^n y}{dx^n}\right)$$

where lower order derivatives and y itself may or may not occur. Such an ODE is called **linear** if it can be written

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x). \tag{1}$$

(For $n = 2$ this is (1) in Sec. 2.1 with $p_1 = p$ and $p_0 = q$). The **coefficients** $p_0, \cdots, p_{n-1}$ and the function r on the right are any given functions of x, and y is unknown. $y^{(n)}$ has coefficient 1. This is practical. We call this the **standard form.** (If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.) An nth-order ODE that cannot be written in the form (1) is called **nonlinear.**

If $r(x)$ is identically zero, $r(x) \equiv 0$ (zero for all x considered, usually in some open interval I), then (1) becomes

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{2}$$

and is called **homogeneous.** If $r(x)$ is not identically zero, then the ODE is called **nonhomogeneous.** This is as in Sec. 2.1.

A **solution** of an nth-order (linear or nonlinear) ODE on some open interval I is a function $y = h(x)$ that is defined and n times differentiable on I and is such that the ODE becomes an identity if we replace the unknown function y and its derivatives by h and its corresponding derivatives.

homogeneous = Autonomous

non — = non —

Homogeneous Linear ODE: Superposition Principle, General Solution

Sections 3.1–3.2 will be devoted to homogeneous linear ODEs and Sec. 3.3 to nonhomogeneous linear ODEs. The basic **superposition or linearity principle** in Sec. 2.1 extends to nth order homogeneous linear ODEs as follows.

THEOREM 1

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *sums and constant multiples of solutions on some open interval I are again solutions on I.* (This does ***not*** hold for a nonhomogeneous or nonlinear ODE!)

The proof is a simple generalization of that in Sec. 2.1 and we leave it to the student.

Our further discussion parallels and extends that for second-order ODEs in Sec. 2.1. So we define next a general solution of (2), which will require an extension of linear independence from 2 to n functions.

DEFINITION

General Solution, Basis, Particular Solution

A **general solution** of (2) on an open interval I is a solution of (2) on I of the form

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \text{ arbitrary}) \tag{3}$$

where $y_1, \cdots, y_n$ is a **basis** (or **fundamental system**) of solutions of (2) on I; that is, these solutions are linearly independent on I, as defined below.

A **particular solution** of (2) on I is obtained if we assign specific values to the n constants $c_1, \cdots, c_n$ in (3).

DEFINITION

Linear Independence and Dependence

n functions $y_1(x), \cdots, y_n(x)$ are called **linearly independent** *on some interval I* where they are defined if the equation

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \qquad \text{on } I \tag{4}$$

implies that all $k_1, \cdots, k_n$ are zero. These functions are called **linearly dependent** on I if this equation also holds on I for some $k_1, \cdots, k_n$ not all zero.

(As in Secs. 1.1 and 2.1, the arbitrary constants $c_1, \cdots, c_n$ must sometimes be restricted to some interval.)

If and only if $y_1, \cdots, y_n$ are linearly dependent on I, we can express (at least) one of these functions on I as a **"linear combination"** of the other $n - 1$ functions, that is, as a sum of those functions, each multiplied by a constant (zero or not). This motivates the term "linearly dependent." For instance, if (4) holds with $k_1 \neq 0$, we can divide by k_1 and express y_1 as the linear combination

$$y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n).$$

Note that when $n = 2$, these concepts reduce to those defined in Sec. 2.1.

EXAMPLE 1 Linear Dependence

Show that the functions $y_1 = x^2$, $y_2 = 5x$, $y_3 = 2x$ are linearly dependent on any interval.

Solution. $y_2 = 0y_1 + 2.5y_3$. This proves linear dependence on any interval. ■

EXAMPLE 2 Linear Independence

Show that $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ are linearly independent on any interval, for instance, on $-1 \leqq x \leqq 2$.

Solution. Equation (4) is $k_1 x + k_2 x^2 + k_3 x^3 = 0$. Taking (a) $x = -1$, (b) $x = 1$, (c) $x = 2$, we get

$$\text{(a) } -k_1 + k_2 - k_3 = 0, \qquad \text{(b) } k_1 + k_2 + k_3 = 0, \qquad \text{(c) } 2k_1 + 4k_2 + 8k_3 = 0.$$

$k_2 = 0$ from (a) + (b). Then $k_3 = 0$ from (c) -2(b). Then $k_1 = 0$ from (b). This proves linear independence.
A better method for testing linear independence of solutions of ODEs will soon be explained. ■

EXAMPLE 3 General Solution. Basis

Solve the fourth-order ODE

$$y^{\text{iv}} - 5y'' + 4y = 0 \qquad (\text{where } y^{\text{iv}} = d^4y/dx^4).$$

Solution. As in Sec. 2.2 we try and substitute $y = e^{\lambda x}$. Omitting the common factor $e^{\lambda x}$, we obtain the characteristic equation

$$\lambda^4 - 5\lambda^2 + 4 = 0.$$

This is a quadratic equation in $\mu = \lambda^2$, namely,

$$\mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0.$$

The roots are $\mu = 1$ and 4. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

provided those four solutions are linearly independent. This is true but will be shown later. ■

Initial Value Problem. Existence and Uniqueness

An **initial value problem** for the ODE (2) consists of (2) and n **initial conditions**

$$(5) \qquad y(x_0) = K_0, \qquad y'(x_0) = K_1, \qquad \cdots, \qquad y^{(n-1)}(x_0) = K_{n-1}$$

with given x_0 in the open interval I considered, and given $K_0, \cdots, K_{n-1}$.
In extension of the existence and uniqueness theorem in Sec. 2.6 we now have the following.

THEOREM 2 Existence and Uniqueness Theorem for Initial Value Problems

Piece

If the coefficients $p_0(x), \cdots, p_{n-1}(x)$ of (2) *are continuous on some open interval I and x_0 is in I, then the initial value problem* (2), (5) *has a unique solution $y(x)$ on I.*

Existence is proved in Ref. [A11] in App. 1. Uniqueness can be proved by a slight generalization of the uniqueness proof at the beginning of App. 4.

EXAMPLE 4 **Initial Value Problem for a Third-Order Euler–Cauchy Equation**

Solve the following initial value problem on any open interval I on the positive x-axis containing $x = 1$.

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0, \qquad y(1) = 2, \qquad y'(1) = 1, \qquad y''(1) = -4.$$

Solution. ***Step 1. General solution.*** As in Sec. 2.5 we try $y = x^m$. By differentiation and substitution,

$$m(m-1)(m-2)x^m - 3m(m-1)x^m + 6mx^m - 6x^m = 0.$$

$(m-1)(m-2)(m-3) = 0$

Dropping x^m and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$, we can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions x, x^2, x^3, which are linearly independent on I (see Example 2). [In general one shall need a root-finding method, such as Newton's (Sec. 19.2), also available in a CAS (Computer Algebra System).] Hence a general solution is

$$y = c_1x + c_2x^2 + c_3x^3$$

valid on any interval I, even when it includes $x = 0$ where the coefficients of the ODE divided by x^3 (to have the standard form) are not continuous.

Step 2. Particular solution. The derivatives are $y' = c_1 + 2c_2x + 3c_3x^2$ and $y'' = 2c_2 + 6c_3x$. From this and y and the initial conditions we get by setting $x = 1$

$$\begin{aligned}
&\text{(a)}\ \ y(1) &&= c_1 + c_2 + c_3 = 2\\
&\text{(b)}\ \ y'(1) &&= c_1 + 2c_2 + 3c_3 = 1\\
&\text{(c)}\ \ y''(1) &&= 2c_2 + 6c_3 = -4.
\end{aligned}$$

This is solved by Cramer's rule (Sec. 7.6), or by elimination, which is simple, as follows. (b) − (a) gives (d) $c_2 + 2c_3 = -1$. Then (c) − 2(d) gives $c_3 = -1$. Then (c) gives $c_2 = 1$. Finally $c_1 = 2$ from (a). *Answer:* $y = 2x + x^2 - x^3$. ■

Linear Independence of Solutions. Wronskian

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. Now Theorem 2 in Sec. 2.6 extends from order $n = 2$ to any n. This extended criterion uses the **Wronskian** W of n solutions $y_1, \cdots, y_n$ defined as the nth order determinant

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdots & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}. \tag{6}$$

Note that W depends on x since $y_1, \cdots, y_n$ does. The criterion states that these solutions form a basis if and only if W is not zero; more precisely:

THEOREM 3 **Linear Dependence and Independence of Solutions**

Let the ODE (2) have piecewise- *continuous coefficients $p_0(x), \cdots, p_{n-1}(x)$ on an open interval I. Then n solutions $y_1, \cdots, y_n$ of (2) on I are linearly dependent on I if and only if their Wronskian is zero for some $x = x_0$ in I. Furthermore, if W is zero for $x = x_0$, then W is identically zero on I. Hence if there is an x_1 in I at which W is not zero, then $y_1, \cdots, y_n$ are linearly independent on I, so that they form a basis of solutions of (2) on I.*

($\Rightarrow$) Necessity proof

PROOF **(a)** Let $y_1, \cdots, y_n$ be linearly dependent solutions of (2) on I. Then, by definition, there are constants $k_1, \cdots, k_n$ *not all zero*, such that for all x in I,

$$k_1 y_1 + \cdots + k_n y_n = 0. \tag{7}$$

By $n - 1$ differentiations of (7) we obtain for all x in I

$$\begin{aligned} k_1 y_1' + \cdots + k_n y_n' &= 0 \\ &\vdots \\ k_1 y_1^{(n-1)} + \cdots + k_n y_n^{(n-1)} &= 0. \end{aligned} \tag{8}$$

$$k_1 \begin{bmatrix} y_1(x) \\ y_1'(x) \\ \vdots \\ y_1^{(n)}(x_0) \end{bmatrix} + \cdots + k_n \begin{bmatrix} y_n(x) \\ y_n'(x) \\ \vdots \\ y_n^{(n)}(x) \end{bmatrix} = 0 \text{ for all } x \in I$$

(7), (8) is a homogeneous linear system of algebraic equations with a nontrivial solution $k_1, \cdots, k_n$. Hence its coefficient determinant must be zero for every x on I, by Cramer's theorem (Sec. 7.7). But that determinant is the Wronskian W, as we see from (6). Hence W is zero for every x on I.

Sufficiency ($\Leftarrow$) Proof

(b) Conversely, if W is zero at an x_0 in I, then the system (7), (8) with $x = x_0$ has a solution $k_1{}^*, \cdots, k_n{}^*$, not all zero, by the same theorem. With these constants we define the solution $y^* = k_1{}^* y_1 + \cdots + k_n{}^* y_n$ of (2) on I. By (7), (8) this solution satisfies the initial conditions $y^*(x_0) = 0, \cdots, y^{*(n-1)}(x_0) = 0$. But another solution satisfying the same conditions is $y \equiv 0$. Hence $y^* \equiv y$ by Theorem 2, which applies since the coefficients of (2) are continuous. Together, $y^* = k_1{}^* y_1 + \cdots + k_n{}^* y_n \equiv 0$ on I. This means linear dependence of $y_1, \cdots, y_n$ on I. (7)', (8)'

(c) If W is zero at an x_0 in I, we have linear dependence by (b) and then $W \equiv 0$ by (a). Hence if W is not zero at an x_1 in I, the solutions $y_1, \cdots, y_n$ must be linearly independent on I. ■

EXAMPLE 5 **Basis, Wronskian**

We can now prove that in Example 3 we do have a basis. In evaluating W, pull out the exponential functions columnwise. In the result, subtract Column 1 from Columns 2, 3, 4 (without changing Column 1). Then expand by Row 1. In the resulting third-order determinant, subtract Column 1 from Column 2 and expand the result by Row 2:

$\times e^{-2x} \times e^{-x} \times e^{x} \times e^{2x}$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ -3 & -3 & 0 \\ 7 & 9 & 16 \end{vmatrix} = 72. \quad ■$$

A General Solution of (2) Includes All Solutions

Let us first show that general solutions always exist. Indeed, Theorem 3 in Sec. 2.6 extends as follows.

THEOREM 4 **Existence of a General Solution**

piecewise-

If the coefficients $p_0(x), \cdots, p_{n-1}(x)$ of (2) are continuous on some open interval I, then (2) has a general solution on I.

PROOF We choose any fixed x_0 in I. By Theorem 2 the ODE (2) has n solutions $y_1, \cdots, y_n$, where y_j satisfies initial conditions (5) with $K_{j-1} = 1$ and all other K's equal to zero. Their Wronskian at x_0 equals 1. For instance, when $n = 3$, then $y_1(x_0) = 1$, $y_2'(x_0) = 1$, $y_3''(x_0) = 1$, and the other initial values are zero. Thus, as claimed,

$$W(y_1(x_0), y_2(x_0), y_3(x_0)) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & y_3(x_0) \\ y_1'(x_0) & y_2'(x_0) & y_3'(x_0) \\ y_1''(x_0) & y_2''(x_0) & y_3''(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Hence for any n those solutions $y_1, \cdots, y_n$ are linearly independent on I, by Theorem 3. They form a basis on I, and $y = c_1y_1 + \cdots + c_ny_n$ is a general solution of (2) on I. ■

We can now prove the basic property that from a general solution of (2) every solution of (2) can be obtained by choosing suitable values of the arbitrary constants. Hence an nth order ***linear*** ODE has no **singular solutions,** that is, solutions that cannot be obtained from a general solution.

THEOREM 5

General Solution Includes All Solutions

piecewise-

If the ODE (2) *has continuous coefficients* $p_0(x), \cdots, p_{n-1}(x)$ *on some open interval* I, *then every solution* $y = Y(x)$ *of* (2) *on* I *is of the form*

$$Y(x) = C_1y_1(x) + \cdots + C_ny_n(x) \tag{9}$$

where $y_1, \cdots, y_n$ *is a basis of solutions of* (2) *on* I *and* $C_1, \cdots, C_n$ *are suitable constants.*

PROOF Let Y be a given solution and $y = c_1y_1 + \cdots + c_ny_n$ a general solution of (2) on I. We choose any fixed x_0 in I and show that we can find constants $c_1, \cdots, c_n$ for which y and its first $n - 1$ derivatives agree with Y and its corresponding derivatives at x_0. That is, we should have at $x = x_0$

$$\begin{aligned} c_1y_1 + \cdots + c_ny_n &= Y \\ c_1y_1' + \cdots + c_ny_n' &= Y' \\ &\ \ \vdots \\ c_1y_1^{(n-1)} + \cdots + c_ny_n^{(n-1)} &= Y^{(n-1)}. \end{aligned} \tag{10}$$

$$\begin{bmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & & \vdots \\ y_1^{(n)}(x_0) & \cdots & y_n^{(n)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ \vdots \\ Y^{(n-1)}(x_0) \end{bmatrix}$$

But this is a linear system of equations in the unknowns $c_1, \cdots, c_n$. Its coefficient determinant is the Wronskian W of $y_1, \cdots, y_n$ at x_0. Since $y_1, \cdots, y_n$ form a basis, they are linearly independent, so that W is not zero by Theorem 3. Hence (10) has a unique solution $c_1 = C_1, \cdots, c_n = C_n$ (by Cramer's theorem in Sec. 7.7). With these values we obtain the particular solution

$$y^*(x) = C_1y_1(x) + \cdots + C_ny_n(x)$$

on I. Equation (10) shows that y^* and its first $n - 1$ derivatives agree at x_0 with Y and its corresponding derivatives. That is, y^* and Y satisfy at x_0 the same initial conditions.

The uniqueness theorem (Theorem 2) now implies that $y^* \equiv Y$ on I. This proves the theorem. ■

This completes our theory of the homogeneous linear ODE (2). Note that for $n = 2$ it is identical with that in Sec. 2.6. This had to be expected.

PROBLEM SET 3.1

1–5 TYPICAL EXAMPLES OF BASES

To get a feel for higher order ODEs, show that the given functions are solutions and form a basis on any interval. Use Wronskians. (In Prob. 2, $x > 0$.)

1. $1, x, x^2, x^3, \quad y^{\text{iv}} = 0$
2. $1, x^2, x^4, \quad x^2y''' - 3xy'' + 3y' = 0$
3. $e^x, xe^x, x^2e^x, \quad y''' - 3y'' + 3y' - y = 0$
4. $e^{2x}\cos x, e^{2x}\sin x, e^{-2x}\cos x, e^{-2x}\sin x,$ $y^{\text{iv}} - 6y'' + 25y = 0.$
5. $1, x, \cos 3x, \sin 3x, \quad y^{\text{iv}} + 9y'' = 0$

6. **TEAM PROJECT. General Properties of Solutions of Linear ODEs.** These properties are important in obtaining new solutions from given ones. Therefore extend Team Project 34 in Sec. 2.2 to nth-order ODEs. Explore statements on sums and multiples of solutions of (1) and (2) systematically and with proofs. Recognize clearly that no new ideas are needed in this extension from $n = 2$ to general n.

7–19 LINEAR INDEPENDENCE AND DEPENDENCE

Are the given functions linearly independent or dependent on the positive x-axis? (Give a reason.)

7. $1, e^x, e^{-x}$
8. $x + 1, x + 2, x$
9. $\ln x, \ln x^2, (\ln x)^2$
10. $e^x, e^{-x}, \sinh 2x$
11. $x^2, x|x|, x$
12. $x, 1/x, 0$
13. $\sin 2x, \sin x, \cos x$
14. $\cos^2 x, \sin^2 x, \cos 2x$
15. $\tan x, \cot x, 1$
16. $(x - 1)^2, (x + 1)^2, x$
17. $\sin x, \sin \frac{1}{2}x$
18. $\cosh x, \sinh x, \cosh^2 x$
19. $\cos^2 x, \sin^2 x, 2\pi$

20. **TEAM PROJECT. Linear Independence and Dependence. (a)** Investigate the given question about a set S of functions on an interval I. Give an example. Prove your answer.

(1) If S contains the zero function, can S be linearly independent?

(2) If S is linearly independent on a subinterval J of I, is it linearly independent on I?

(3) If S is linearly dependent on a subinterval J of I, is it linearly dependent on I?

(4) If S is linearly independent on I, is it linearly independent on a subinterval J?

(5) If S is linearly dependent on I, is it linearly independent on a subinterval J?

(6) If S is linearly dependent on I, and if T contains S, is T linearly dependent on I?

(b) In what cases can you use the Wronskian for testing linear independence? By what other means can you perform such a test?

3.2 Homogeneous Linear ODEs with Constant Coefficients

In this section we consider nth-order homogeneous linear ODEs with constant coefficients, which we write in the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \tag{1}$$

where $y^{(n)} = d^ny/dx^n$, etc. We shall see that this extends the case $n = 2$ discussed in Sec. 2.2. Substituting $y = e^{\lambda x}$ (as in Sec. 2.2), we obtain the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 = 0 \tag{2}$$

of (1). If λ is a root of (2), then $y = e^{\lambda x}$ is a solution of (1). To find these roots, you may need a numeric method, such as Newton's in Sec. 19.2, also available on the usual CASs. For general n there are more cases than for $n = 2$. We shall discuss all of them and illustrate them with typical examples.

Distinct Real Roots

If all the n roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the n solutions

$$y_1 = e^{\lambda_1 x}, \quad \cdots, \quad y_n = e^{\lambda_n x} \tag{3}$$

constitute a basis for all x. The corresponding general solution of (1) is

$$y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}. \tag{4}$$

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.

EXAMPLE 1 Distinct Real Roots

Solve the ODE $y''' - 2y'' - y' + 2y = 0$.

Solution. The characteristic equation is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$. It has the roots $-1, 1, 2$; if you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation (explain!). The corresponding general solution (4) is $y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$. ■

Linear Independence of (3). Students familiar with nth-order determinants may verify that by pulling out all exponential functions from the columns and denoting their product by E, thus $E = \exp[(\lambda_1 + \cdots + \lambda_n)x]$, the Wronskian of the solutions in (3) becomes

$$\begin{aligned} W &= \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \lambda_1^2 e^{\lambda_1 x} & \lambda_2^2 e^{\lambda_2 x} & \cdots & \lambda_n^2 e^{\lambda_n x} \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} \\ &= E \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}. \end{aligned} \tag{5}$$

The exponential function E is never zero. Hence $W = 0$ if and only if the determinant on the right is zero. This is a so-called **Vandermonde** or **Cauchy determinant**[1]. It can be shown that it equals

[1]ALEXANDRE THÉOPHILE VANDERMONDE (1735–1796), French mathematician, who worked on solution of equations by determinants. For CAUCHY see footnote 4, in Sec. 2.5.

$$(-1)^{n(n-1)/2}V \tag{6}$$

where V is the product of all factors $\lambda_j - \lambda_k$ with $j < k$ ($\leqq n$); for instance, when $n = 3$ we get $-V = -(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. This shows that the Wronskian is not zero if and only if all the n roots of (2) are different and thus gives the following.

THEOREM 1

Basis

Solutions $y_1 = e^{\lambda_1 x}, \cdots, y_n = e^{\lambda_n x}$ of (1) (with any real or complex λ_j's) form a basis of solutions of (1) on any open interval if and only if all n roots of (2) are different.

Actually, Theorem 1 is an important special case of our more general result obtained from (5) and (6):

THEOREM 2

Linear Independence

Any number of solutions of (1) of the form $e^{\lambda x}$ are linearly independent on an open interval I if and only if the corresponding λ are all different.

Simple Complex Roots

If complex roots occur, they must occur in conjugate pairs since the coefficients of (1) are real. Thus, if $\lambda = \gamma + i\omega$ is a simple root of (2), so is the conjugate $\bar{\lambda} = \gamma - i\omega$, and two corresponding linearly independent solutions are (as in Sec. 2.2, except for notation)

$$y_1 = e^{\gamma x} \cos \omega x, \qquad y_2 = e^{\gamma x} \sin \omega x.$$

EXAMPLE 2 Simple Complex Roots. Initial Value Problem

Solve the initial value problem

$$y''' - y'' + 100y' - 100y = 0, \qquad y(0) = 4, \qquad y'(0) = 11, \qquad y''(0) = -299.$$

Solution. The characteristic equation is $\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$. It has the root 1, as can perhaps be seen by inspection. Then division by $\lambda - 1$ shows that the other roots are $\pm 10i$. Hence a general solution and its derivatives (obtained by differentiation) are

$$y = c_1 e^x + A \cos 10x + B \sin 10x,$$
$$y' = c_1 e^x - 10A \sin 10x + 10B \cos 10x,$$
$$y'' = c_1 e^x - 100A \cos 10x - 100B \sin 10x.$$

From this and the initial conditions we obtain by setting $x = 0$

$$\text{(a)} \quad c_1 + A = 4, \qquad \text{(b)} \quad c_1 + 10B = 11, \qquad \text{(c)} \quad c_1 - 100A = -299.$$

We solve this system for the unknowns A, B, c_1. Equation (a) minus Equation (c) gives $101A = 303$, $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is (Fig. 72)

$$y = e^x + 3 \cos 10x + \sin 10x.$$

This gives the solution curve, which oscillates about e^x (dashed in Fig. 72 on p. 114). ■

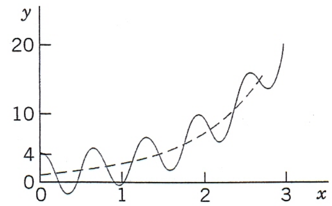

Fig. 72. Solution in Example 2

Multiple Real Roots

If a real double root occurs, say, $\lambda_1 = \lambda_2$, then $y_1 = y_2$ in (3), and we take y_1 and xy_1 as corresponding linearly independent solutions. This is as in Sec. 2.2.

More generally, if λ is a real root of order m, then m corresponding linearly independent solutions are

$$e^{\lambda x}, \quad xe^{\lambda x}, \quad x^2e^{\lambda x}, \quad \cdots, \quad x^{m-1}e^{\lambda x}. \tag{7}$$

We derive these solutions after the next example and indicate how to prove their linear independence.

EXAMPLE 3 Real Double and Triple Roots

Solve the ODE $y^{v} - 3y^{iv} + 3y''' - y'' = 0$.

Solution. The characteristic equation $\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$ has the roots $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 1$, and the answer is

$$y = c_1 + c_2x + (c_3 + c_4x + c_5x^2)e^x. \tag{8}$$ ∎

See the attached.

***Derivation of* (7).** We write the left side of (1) as

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y.$$

Let $y = e^{\lambda x}$. Then by performing the differentiations we have

$$L[e^{\lambda x}] = (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0)e^{\lambda x}.$$

Now let λ_1 be a root of mth order of the polynomial on the right, where $m \leqq n$. For $m < n$ let $\lambda_{m+1}, \cdots, \lambda_n$ be the other roots, all different from λ_1. Writing the polynomial in product form, we then have

$$L[e^{\lambda x}] = (\lambda - \lambda_1)^m h(\lambda)e^{\lambda x}$$

with $h(\lambda) = 1$ if $m = n$, and $h(\lambda) = (\lambda - \lambda_{m+1}) \cdots (\lambda - \lambda_n)$ if $m < n$. Now comes the key idea: We differentiate on both sides with respect to λ,

$$\frac{\partial}{\partial\lambda} L[e^{\lambda x}] = m(\lambda - \lambda_1)^{m-1}h(\lambda)e^{\lambda x} + (\lambda - \lambda_1)^m \frac{\partial}{\partial\lambda}[h(\lambda)e^{\lambda x}]. \tag{9}$$

Substituting $y(x) = e^{\lambda x}$ into

(1) $$y^{(n)}(x) + \sum_{i=0}^{n-1} a_i y^{i}(x) = 0$$

we have

(2) $$\lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i = 0.$$

Case I : distinct $\begin{cases} \text{real roots} & \lambda_i,\ i=1,\dots,n-2m \\ \text{complex roots} & P_i \pm j\, q_i,\ i=1,\dots,m \end{cases}$

$$y = \sum_{i=1}^{m} \left[c_i e^{\lambda_i t} + \bar{c}_i e^{\bar{\lambda}_i t}\right] + \sum_{i=1}^{n-2m} c_i e^{\lambda_i t}$$

$$= \sum_{i=1}^{m} e^{P_i x}(A_i \cos q_i x + B_i \sin q_i x) + \sum_{i=1}^{n-2m} c_i e^{\lambda_i t}$$

(By Theorem 2, p. 113.)

Case II multiple roots $(\lambda-\lambda_i)^{m_i}$

(1) can be written as

$$Q(D)(D-\lambda_i)^{m_i} y = 0.$$

Let $y = x^k e^{\lambda_i x}$. Then, we claim that

(3)' $$(D-\lambda_i)^{l} y = k P_l\, x^{k-l} e^{\lambda_i x}, \quad \forall\, l = 1,\dots,k.$$

Compute first for inductive proof of (3)'.

$$(D-\lambda_i) y = k x^{k-1} e^{\lambda_i x} + \lambda_i x^k e^{\lambda_i x} - \lambda_i x^k e^{\lambda_i x} = k x^{k-1} e^{\lambda_i}$$

Hence, (3)' holds for $l=1$. Suppose : (3)' holds for $l = r$. That is,

$$(D-\lambda_i)^r y = k P_r\, x^{k-r} e^{\lambda_i x}$$

(3)' $(D-\lambda_i)^l y(x) = {}_kP_l\, x^{k-l} e^{\lambda_i x}, \quad \forall l = 1, \cdots, k$

for $y(x) = x^k e^{\lambda_i x}$

Then,

$$(D-\lambda_i)^{r+1} y = (D-\lambda_i)(D-\lambda_i)^r y = (D-\lambda_i)\, {}_kP_r\, x^{k-r} e^{\lambda_i x}$$
$$= {}_kP_{r+1}\, x^{k-r-1} e^{\lambda_i x}.$$

Hence, (3)' holds for $l = 1, \cdots, k$. Consequently,

$$Q(D)(D-\lambda_i)^{m_i} y = Q(D)(D-\lambda_i)^{m_i-k}(D-\lambda_i)^k y$$
$$= Q(D)(D-\lambda_i)^{m_i-k}\, k!\, e^{\lambda_i x} = 0, \quad \forall k = 0, \cdots, (m_i-1)$$

Thus, $x^k e^{\lambda_i x}$, $k = 0, \cdots, (m_i-1)$ are solutions.

Hence, $y = \left(\sum_{k=0}^{m_i-1} c_k x^k\right) e^{\lambda_i x}$ is a general sol.

If λ_i is a complex pole, $\lambda_i = \gamma_i + j\omega_i$,

$$y = \left(\sum_{k=0}^{m_i-1} c_k x^k\right) e^{\lambda_i x} + \left(\sum_{k=0}^{m_i-1} \bar{c}_k x^k\right) e^{\bar{\lambda}_i x}$$

(12)' $$= e^{\gamma_i x}\left\{\left(\sum_{k=0}^{m_i-1} A_k x^k\right)\cos\omega_i x + \left(\sum_{k=0}^{m_i-1} B_k x^k\right)\sin\omega_i x\right\}$$

(Remark)

$|y(x)| \to 0$ as $x \to \infty$ if $\gamma_i < 0$.

HW

- Prob. 3.2.10
- Prob. 3.2.14
- Prob. 3.2.20

13–18 **INITIAL VALUE PROBLEMS**

Solve by a CAS, giving a general solution and the particular solution and its graph.

13. $y^{iv} + 0.45y''' - 0.165y'' + 0.0045y' - 0.00175y = 0$, $y(0) = 17.4$, $y'(0) = -2.82$, $y''(0) = 2.0485$, $y'''(0) = -1.458675$

14. $4y''' + 8y'' + 41y' + 37y = 0$, $y(0) = 9$, $y'(0) = -6.5$, $y''(0) = -39.75$

15. $y''' + 3.2y'' + 4.81y' = 0$, $y(0) = 3.4$, $y'(0) = -4.6$, $y''(0) = 9.91$

16. $y^{iv} + 4y = 0$, $y(0) = \frac{1}{2}$, $y'(0) = -\frac{3}{2}$, $y''(0) = \frac{5}{2}$, $y'''(0) = -\frac{7}{2}$

17. $y^{iv} - 9y'' - 400y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 41$, $y'''(0) = 0$

18. $y''' + 7.5y'' + 14.25y' - 9.125y = 0$, $y(0) = 10.05$, $y'(0) = -54.975$, $y''(0) = 257.5125$

19. CAS PROJECT. Wronskians. Euler–Cauchy Equations of Higher Order. Although Euler–Cauchy equations have *variable* coefficients (powers of x), we include them here because they fit quite well into the present methods.

(a) Write a program for calculating Wronskians.

(b) Apply the program to some bases of third-order and fourth-order constant-coefficient ODEs. Compare the results with those obtained by the program most likely available for Wronskians in your CAS.

(c) Extend the solution method in Sec. 2.5 to any order n. Solve $x^3y''' + 2x^2y'' - 4xy' + 4y = 0$ and another ODE of your choice. In each case calculate the Wronskian.

20. PROJECT. Reduction of Order. This is of practical interest since a single solution of an ODE can often be guessed. For second order, see Example 7 in Sec. 2.1.

(a) How could you reduce the order of a linear constant-coefficient ODE if a solution is known?

(b) Extend the method to a variable-coefficient ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0.$$

Assuming a solution y_1 to be known, show that another solution is $y_2(x) = u(x)y_1(x)$ with $u(x) = \int z(x)\,dx$ and z obtained by solving

$$y_1z'' + (3y_1' + p_2y_1)z' + (3y_1'' + 2p_2y_1' + p_1y_1)z = 0.$$

(c) Reduce

$$x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0,$$

using $y_1 = x$ (perhaps obtainable by inspection).

21. CAS EXPERIMENT. Reduction of Order. Starting with a basis, find third-order ODEs with variable coefficients for which the reduction to second order turns out to be relatively simple.

3.3 Nonhomogeneous Linear ODEs

We now turn from homogeneous to nonhomogeneous linear ODEs of nth order. We write them in standard form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \tag{1}$$

with $y^{(n)} = d^ny/dx^n$ as the first term, which is practical, and $r(x) \not\equiv 0$. As for second-order ODEs, a general solution of (1) on an open interval I of the x-axis is of the form

$$y(x) = y_h(x) + y_p(x). \tag{2}$$

Here $y_h(x) = c_1y_1(x) + \cdots + c_ny_n(x)$ is a general solution of the corresponding homogeneous ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \tag{3}$$

on I. Also, y_p is any solution of (1) on I containing no arbitrary constants. If (1) has piecewise- continuous coefficients and a piecewise continuous $r(x)$ on I, then a general solution of (1) exists and includes all solutions. Thus (1) has no singular solutions.

An **initial value problem** for (1) consists of (1) and n **initial conditions**

$$y(x_0) = K_0, \quad y'(x_0) = K_1, \quad \cdots, \quad y^{(n-1)}(x_0) = K_{n-1} \tag{4}$$

piecewise-

with x_0 in I. Under those continuity assumptions it has a unique solution. The ideas of proof are the same as those for $n = 2$ in Sec. 2.7.

Method of Undetermined Coefficients

Equation (2) shows that for solving (1) we have to determine a particular solution of (1). For a constant-coefficient equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x) \tag{5}$$

($a_0, \cdots, a_{n-1}$ constant) and special $r(x)$ as in Sec. 2.7, such a $y_p(x)$ can be determined by the **method of undetermined coefficients,** as in Sec. 2.7, using the following rules.

(A) Basic Rule as in Sec. 2.7.

(B) Modification Rule. *If a term in your choice for $y_p(x)$ is a solution of the homogeneous equation* (3), *then multiply $y_p(x)$ by x^k, where k is the smallest positive integer such that no term of $x^k y_p(x)$ is a solution of* (3).

(C) Sum Rule as in Sec. 2.7.

The practical application of the method is the same as that in Sec. 2.7. It suffices to illustrate the typical steps of solving an initial value problem and, in particular, the new Modification Rule, which includes the old Modification Rule as a particular case (with $k = 1$ or 2). We shall see that the technicalities are the same as for $n = 2$, perhaps except for the more involved determination of the constants.

EXAMPLE 1 Initial Value Problem. Modification Rule

Solve the initial value problem

$$y''' + 3y'' + 3y' + y = 30e^{-x}, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47. \tag{6}$$

Solution. *Step 1.* The characteristic equation is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0$. It has the triple root $\lambda = -1$. Hence a general solution of the homogeneous ODE is

$$\begin{aligned} y_h &= c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} \\ &= (c_1 + c_2x + c_3x^2)e^{-x}. \quad (*) \end{aligned}$$

Step 2. If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has no solution. Try Cxe^{-x} and Cx^2e^{-x}. The Modification Rule calls for

$$y_p = Cx^3e^{-x}.$$

Then

$$\begin{aligned} y_p' &= C(3x^2 - x^3)e^{-x}, \\ y_p'' &= C(6x - 6x^2 + x^3)e^{-x}, \\ y_p''' &= C(6 - 18x + 9x^2 - x^3)e^{-x}. \end{aligned}$$

(Remark)

$r(x) = e^{-x} \Rightarrow (D+1)r = 0$

$\Rightarrow p(D)y = (D+1)(D^3+3D^2+3D+1)y$

$= (D+1)(D+1)^3 y = (D+1)r = 0$

$\Rightarrow y(x) = (c_1 + c_2x + c_3x^2 + c_4x^3)e^{-x}$

$\Rightarrow y_p(x) = cx^3e^{-x}$ by $(*)$

Substitution of these expressions into (6) and omission of the common factor e^{-x} gives

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$. This gives $y_p = 5x^3e^{-x}$.

Step 3. We now write down $y = y_h + y_p$, the general solution of the given ODE. From it we find c_1 by the first initial condition. We insert the value, differentiate, and determine c_2 from the second initial condition, insert the value, and finally determine c_3 from $y''(0)$ and the third initial condition:

$$\begin{aligned}
y &= y_h + y_p = (c_1 + c_2x + c_3x^2)e^{-x} + 5x^3e^{-x}, && y(0) = c_1 = 3 \\
y' &= [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, && y'(0) = -3 + c_2 = -3, \quad c_2 = 0 \\
y'' &= [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, && y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.
\end{aligned}$$

Hence the *answer* to our problem is (Fig. 73)

$$y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}.$$

The curve of y begins at $(0, 3)$ with a negative slope, as expected from the initial values, and approaches zero as $x \to \infty$. The dashed curve in Fig. 73 is y_p. ■

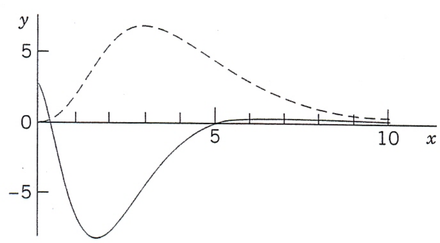

Fig. 73. y and y_p (dashed) in Example 1

Method of Variation of Parameters

The method of variation of parameters (see Sec. 2.10) also extends to arbitrary order n. It gives a particular solution y_p for the nonhomogeneous equation (1) (in standard form with $y^{(n)}$ as the first term!) by the formula

$$\begin{aligned}
y_p(x) &= \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\, dx \\
&= y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)} r(x)\, dx
\end{aligned} \tag{7}$$

on an open interval I on which the coefficients of (1) and $r(x)$ are continuous. In (7) the functions $y_1, \cdots, y_n$ form a basis of the homogeneous ODE (3), with Wronskian W, and W_j ($j = 1, \cdots, n$) is obtained from W by replacing the jth column of W by the column $[0 \quad 0 \quad \cdots \quad 0 \quad 1]^{\mathsf{T}}$. Thus, when $n = 2$, this becomes identical with (2) in Sec. 2.10,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1.$$

The proof of (7) uses an extension of the idea of the proof of (2) in Sec. 2.10 and can be found in Ref [A11] listed in App. 1.

EXAMPLE 2 Variation of Parameters. Nonhomogeneous Euler–Cauchy Equation

Solve the nonhomogeneous Euler–Cauchy equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x \qquad (x > 0).$$

Solution. *Step 1. General solution of the homogeneous ODE.* Substitution of $y = x^m$ and the derivatives into the homogeneous ODE and deletion of the factor x^m gives

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0.$$

The roots are 1, 2, 3 and give as a basis

$$y_1 = x, \qquad y_2 = x^2, \qquad y_3 = x^3.$$

Hence the corresponding general solution of the homogeneous ODE is

$$y_h = c_1x + c_2x^2 + c_3x^3.$$

Step 2. Determinants needed in (7). These are

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0 \text{ if } x > 0$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

Step 3. Integration. In (7) we also need the right side $r(x)$ of our ODE in standard form, obtained by division of the given equation by the coefficient x^3 of y'''; thus, $r(x) = (x^4 \ln x)/x^3 = x \ln x$. In (7) we have the simple quotients $W_1/W = x/2$, $W_2/W = -1$, $W_3/W = 1/(2x)$. Hence (7) becomes

$$y_p = x \int \frac{x}{2} x \ln x \, dx - x^2 \int x \ln x \, dx + x^3 \int \frac{1}{2x} x \ln x \, dx$$

$$= \frac{x}{2}\left(\frac{x^3}{3} \ln x - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2} \ln x - \frac{x^2}{4}\right) + \frac{x^3}{2}(x \ln x - x).$$

Simplification gives $y_p = \frac{1}{6}x^4 (\ln x - \frac{11}{6})$. Hence the answer is

$$y = y_h + y_p = c_1x + c_2x^2 + c_3x^3 + \tfrac{1}{6}x^4 (\ln x - \tfrac{11}{6}).$$

Figure 74 shows y_p. Can you explain the shape of this curve? Its behavior near $x = 0$? The occurrence of a minimum? Its rapid increase? Why would the method of undetermined coefficients not have given the solution? ■

HW

Prob. 3.3.4

Prob. 3.3.12