6.1 Laplace Transform. Inverse Transform. Linearity. *s*-Shifting

(Unilateral)

If $f(t)$ is a function defined for all $t \geqq 0$, its **Laplace transform**[1] is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$; thus

$s \in \mathbb{C}$

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

Improper Integral

Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value). This assumption is usually satisfied in applications—we shall discuss this near the end of the section.

Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform.** It is an **"integral transform"**

$$F(s) = \int_0^\infty k(s, t) f(t)\, dt$$

with **"kernel"** $k(s, t) = e^{-st}$.

Furthermore, the given function $f(t)$ in (1) is called the **inverse transform** of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F). \tag{1*}$$

Note that (1) and (1*) together imply $\mathcal{L}^{-1}(\mathcal{L}(f)) = f$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$.

Notation

Original functions depend on t and their transforms on s—keep this in mind! Original functions are denoted by ***lowercase letters*** and their transforms by the same ***letters in capital,*** so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

EXAMPLE 1 Laplace Transform

Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

Re $s > 0$

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad (s > 0).$$

[1]PIERRE SIMON MARQUIS DE LAPLACE (1749–1827), great French mathematician, was a professor in Paris. He developed the foundation of potential theory and made important contributions to celestial mechanics, astronomy in general, special functions, and probability theory. Napoléon Bonaparte was his student for a year. For Laplace's interesting political involvements, see Ref. [GR2], listed in App. 1.

The powerful practical Laplace transform techniques were developed over a century later by the English electrical engineer OLIVER HEAVISIDE (1850–1925) and were often called "Heaviside calculus."

We shall drop variables when this simplifies formulas without causing confusion. For instance, in (1) we wrote $\mathcal{L}(f)$ instead of $\mathcal{L}(f)(s)$ and in (1*) $\mathcal{L}^{-1}(F)$ instead of $\mathcal{L}^{-1}(F)(t)$.

(Remark)

$$\begin{cases} F(s) \triangleq \int_{-\infty}^{\infty} e^{-st} f(t)\, dt & : \text{Bilateral Laplace Transf.} \\ F(s) \triangleq \int_{0}^{\infty} e^{-st} f(t)\, dt & : \text{Unilateral Laplace Transf.} \end{cases}$$

Our notation is convenient, but we should say a word about it. The interval of integration in (1) is infinite. Such an integral is called an **improper integral** and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt.$$

Hence our convenient notation means

(Re $s > 0$)

$$\int_0^\infty e^{-st}\, dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^0 \right] = \frac{1}{s} \qquad (s > 0).$$

We shall use this notation throughout this chapter. ■

EXAMPLE 2 Laplace Transform $\mathcal{L}(e^{at})$ of the Exponential Function e^{at}

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $\mathcal{L}(f)$.

Solution. Again by (1),

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{a - s} e^{-(s-a)t} \Big|_0^\infty ;$$

hence, when $s - a > 0$,

(Re $s > a$)

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}.$$ ■

Must we go on in this fashion and obtain the transform of one function after another directly from the definition? The answer is no. And the reason is that new transforms can be found from known ones by the use of the many general properties of the Laplace transform. Above all, the Laplace transform is a "linear operation," just as differentiation and integration. By this we mean the following.

THEOREM 1 Linearity of the Laplace Transform

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

PROOF By the definition in (1),

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}[af(t) + bg(t)]\, dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$ ■

EXAMPLE 3 Application of Theorem 1: Hyperbolic Functions

Find the transforms of $\cosh at$ and $\sinh at$.

Solution. Since $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we obtain from Example 2 and Theorem 1

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}.$$ ■

Riemann Integral

Given a bounded function $f : [a,b] \to R$, $\exists\ m, M \ni$
$m \leq f(t) \leq M$, $\forall\ t \in [a,b]$

Let P be a partition of the interval $[a,b]$:
$P = \{ [t_{k-1}, t_k], k = 1, \cdots, n\}$ where $t_0 = a$, $t_n = b$

Let $M_i = \sup \{ f(t) : t_{i-1} \leq t \leq t_i \}$
$m_i = \operatorname{Inf} \{ f(t) : t_{i-1} \leq t \leq t_i \}$

Let $U(P,f) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$
$L(P,f) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$

Then $m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

$$\Rightarrow \begin{cases} \exists \overline{\int_a^b} f\,dt \triangleq \inf U(P,f) \\ \exists \underline{\int_a^b} f\,dt \triangleq \sup L(P,f) \end{cases}$$

and

(*) $\underline{\int_a^b} f\,dt \leq \overline{\int_a^b} f\,dt$ since $L(P,f) \leq U(Q,f)$

<u>Definition</u>: If the equality holds in (*), f is Riemann-Integrable

<u>Definition</u>: If f is R-integrable on $[a,x]$ for each $x \geq a$ and $\exists \lim_{x\to\infty} \int_a^x f\,dt$, then f is Improper Riemann-Integrable

Theorem (Apostol, p354) Let f, $D_2 f$ be continuous on $[a,b]\times[c,d]$. Let p and q be diff. on $[c,d]$ where $p(y)\in[a,b]$ and $q(y)\in[a,b]$ for each y in $[c,d]$. Define F by

$$(1) \qquad F(y)=\int_{p(y)}^{q(y)} f(x,y)\,dx \quad \text{if } y\in[c,d].$$

Then, $F'(y)$ exists for each y in (c,d) and is given by

$$(2) \qquad F'(y)=\int_{p(y)}^{q(y)} D_2 f(x,y)\,dx + f(q(y),y)\,q'(y)-f(p(y),y)\,p'(y)$$

Theorem (Apostol, p167) Let f be continuous on $[a,b]\times[c,d]$. Then,

$$\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

Remark ① If f is piecewise-continuous,

$$\int_{t_0}^{t_n} f(t)\,dt=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} f(t)\,dt,$$ where

t_k, $k=1,2,\dots$ are the points at which jumps occur.

Theorem (Interchanging the integration order)

Let X, Y be two subintervals of $\mathbb{R}$ and let $k : X \times Y \to \mathbb{R}$ be <u>continuous</u> and <u>bounded</u> on $X \times Y$. Then the two integrals $\int_X f(x) \int_Y g(y)\, k(x, y)\, dy\, dx$ and $\int_Y g(y) \int_X f(x)\, k(x, y)\, dx\, dy$ exist and they are equal, if $f \in L(X)$ and $g \in L(Y)$ □

Theorem (continuous function)

Let $f : X \to Y$ be continuous at $x \in X$.

$\Leftrightarrow$ for any sequence $\{x_k\}_1^\infty$ converging to x,

$$\lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n)$$

□

Theorem (Improper Integral)

Suppose that f is Riemann-integrable on $[a, b]$ for every $b \geq a$ and that $\exists\, M > 0 \ni$

$$\int_a^b |f(x)|\, dx \leq M(a) \text{ for every } b \geq a$$

Then, f and $|f|$ are improper Riemann-integrable on $(-\infty, +\infty)$.

□

EXAMPLE 4 **Cosine and Sine**

Derive the formulas

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}, \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Solution by Calculus. We write $L_c = \mathcal{L}(\cos \omega t)$ and $L_s = \mathcal{L}(\sin \omega t)$. Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain

$(\mathrm{Re}\, s > 0)$

$$L_c = \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{1}{s} - \frac{\omega}{s} L_s,$$

$$L_s = \int_0^\infty e^{-st} \sin \omega t \, dt = \frac{e^{-st}}{-s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{\omega}{s} L_c.$$

By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right), \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s}, \qquad L_c = \frac{s}{s^2 + \omega^2},$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s} L_s\right), \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}, \qquad L_s = \frac{\omega}{s^2 + \omega^2}.$$

Solution by Transforms Using Derivatives. See next section.

Solution by Complex Methods. In Example 2, if we set $a = i\omega$ with $i = \sqrt{-1}$, we obtain

$$\mathcal{L}(\sin \omega t) \Leftarrow \quad \mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{(s - i\omega)(s + i\omega)} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i\,\frac{\omega}{s^2 + \omega^2}. \;(?)$$

Note that $s \in \mathbb{C}$

Now by Theorem 1 and $e^{i\omega t} = \cos \omega t + i \sin \omega t$ [see (11) in Sec. 2.2 with ωt instead of t] we have

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos \omega t + i \sin \omega t) = \mathcal{L}(\cos \omega t) + i\mathcal{L}(\sin \omega t).$$

If we equate the real and imaginary parts of this and the previous equation, the result follows. (This formal calculation can be justified in the theory of complex integration.) ■

$$= \mathcal{L}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2j}\right) = \frac{1}{2j}\mathcal{L}(e^{j\omega t}) + \frac{1}{2j}\mathcal{L}(e^{-j\omega t}) = \frac{1}{2j}\,\frac{1}{s - j\omega} + \frac{1}{2j}\,\frac{1}{s + j\omega} = \frac{\omega}{s^2 + \omega^2} \quad (\mathrm{Re}\, s > 0)$$

Basic transforms are listed in Table 6.1. We shall see that from these almost all the others can be obtained by the use of the general properties of the Laplace transform. Formulas 1–3 are special cases of formula 4, which is proved by induction. Indeed, it is true for $n = 0$ because of Example 1 and $0! = 1$. We make the induction hypothesis that it holds for any integer $n \geqq 0$ and then get it for $n + 1$ directly from (1). Indeed, integration by parts first gives

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1} \, dt = -\frac{1}{s} e^{-st} t^{n+1} \Big|_0^\infty + \frac{n+1}{s} \int_0^\infty e^{-st} t^n \, dt.$$

Now the integral-free part is zero and the last part is $(n + 1)/s$ times $\mathcal{L}(t^n)$. From this and the induction hypothesis,

$$\mathcal{L}(t^{n+1}) = \frac{n+1}{s} \mathcal{L}(t^n) = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}. \qquad (\mathrm{Re}\, s > 0)$$

This proves formula 4.

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

$\Gamma(a+1)$ in formula 5 is the so-called *gamma function* [(15) in Sec. 5.5 or (24) in App. A3.1]. We get formula 5 from (1), setting $st = x$:

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx$$

where $s > 0$. The last integral is precisely that defining $\Gamma(a+1)$, so we have $\Gamma(a+1)/s^{a+1}$, as claimed. (CAUTION! $\Gamma(a+1)$ has x^a in the integral, not x^{a+1}.)

Note the formula 4 also follows from 5 because $\Gamma(n+1) = n!$ for integer $n \geqq 0$.

Formulas 6–10 were proved in Examples 2–4. Formulas 11 and 12 will follow from 7 and 8 by "shifting," to which we turn next.

s-Shifting: Replacing s by $s - a$ in the Transform

The Laplace transform has the very useful property that if we know the transform of $f(t)$, we can immediately get that of $e^{at}f(t)$, as follows.

THEOREM 2 **First Shifting Theorem, s-Shifting**

($s \to \operatorname{Re} s$)

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform $F(s-a)$ (where $s - a > k$). In formulas,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

or, if we take the inverse on both sides,

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}.$$

PROOF We obtain $F(s-a)$ by replacing s with $s-a$ in the integral in (1), so that

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st}[e^{at} f(t)]\, dt = \mathcal{L}\{e^{at} f(t)\}.$$

If $F(s)$ exists (i.e., is finite) for s greater than some k, then our first integral exists for $s - a > k$. Now take the inverse on both sides of this formula to obtain the second formula in the theorem. (CAUTION! $-a$ in $F(s-a)$ but $+a$ in $e^{at} f(t)$.) ■

EXAMPLE 5 **s-Shifting: Damped Vibrations. Completing the Square**

From Example 4 and the first shifting theorem we immediately obtain formulas 11 and 12 in Table 6.1,

$$\mathcal{L}\{e^{at}\cos \omega t\} = \frac{s-a}{(s-a)^2+\omega^2}, \qquad \mathcal{L}\{e^{at}\sin \omega t\} = \frac{\omega}{(s-a)^2+\omega^2}.$$

For instance, use these formulas to find the inverse of the transform

$$\mathcal{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}.$$

Solution. Applying the inverse transform, using its linearity (Prob. 28), and completing the square, we obtain

$$f = \mathcal{L}^{-1}\left\{\frac{3(s+1) - 140}{(s+1)^2 + 400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2 + 20^2}\right\}.$$

We now see that the inverse of the right side is the damped vibration (Fig. 113)

$$f(t) = e^{-t}(3 \cos 20t - 7 \sin 20t).$$ ■

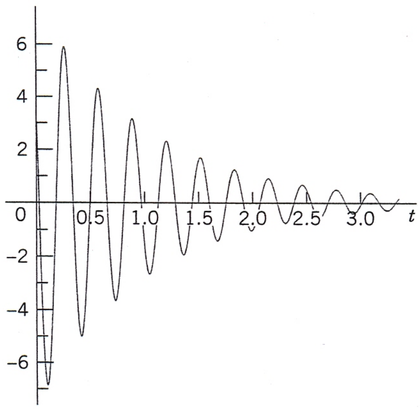


Fig. 113. Vibrations in Example 5

Existence and Uniqueness of Laplace Transforms

This is not a big *practical* problem because in most cases we can check the solution of an ODE without too much trouble. Nevertheless we should be aware of some basic facts.

A function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geqq 0$ and some constants M and k it satisfies the "**growth restriction**"

(2) $$|f(t)| \leqq Me^{kt}.$$

(The growth restriction (2) is sometimes called "growth of exponential order," which may be misleading since it hides that the exponent must be kt, not kt^2 or similar.)

$f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is *piecewise continuity*. $f(t)$ is **piecewise continuous** on a finite interval $a \leqq t \leqq b$ where f is defined, if this interval can be divided into *finitely many* subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior. This then gives **finite jumps** as in Fig. 114 as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.

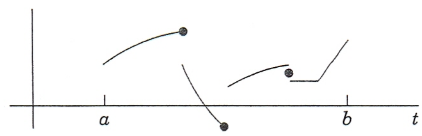


Fig. 114. Example of a piecewise continuous function $f(t)$. (The dots mark the function values at the jumps.)

THEOREM 3

$|f(t)| \leq Me^{kt}$

Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.

PROOF

See the attached.

Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathcal{L}(f)$ from

$$|\mathcal{L}(f)| = \left|\int_0^\infty e^{-st} f(t)\, dt\right| \leqq \int_0^\infty |f(t)| e^{-st}\, dt \leqq \int_0^\infty Me^{kt} e^{-st}\, dt = \frac{M}{s-k}. \qquad \blacksquare$$

Note that (2) can be readily checked. For instance, $\cosh t < e^t$, $t^n < n!e^t$ (because $t^n/n!$ is a single term of the Maclaurin series), and so on. A function that does not satisfy (2) for any M and k is e^{t^2} (take logarithms to see it). We mention that the conditions in Theorem 3 are sufficient rather than necessary (see Prob. 22).

$\mathcal{L}(f_1) = \mathcal{L}(f_2)$
$\Downarrow$
$f_1 = f_2$
(a.e.)

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (see Ref. [A14] in App. 1). Hence we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical.

PROBLEM SET 6.1

1–20 LAPLACE TRANSFORMS

Find the Laplace transforms of the following functions. Show the details of your work. (a, b, k, ω, θ are constants.)

1. $t^2 - 2t$
2. $(t^2 - 3)^2$
3. $\cos 2\pi t$
4. $\sin^2 4t$
5. $e^{2t} \cosh t$
6. $e^{-t} \sinh 5t$
7. $\cos(\omega t + \theta)$
8. $\sin(3t - \frac{1}{2})$
9. e^{3a-2bt}
10. $-8 \sin 0.2t$

HW

Prob. 6.1-15
Prob. 6.1-24
Prob. 6.1-31
Prob. 6.1-46

Proof of Theorem 3, p. 226

$$\int_0^T |e^{-st} f(t)|\, dt = \sum_{j=1}^{n} \int_{b_{j-1}}^{b_j} |e^{-st} f(t)|\, dt$$

$$\leq \sum_{j=1}^{n} \int_{b_{j-1}}^{b_j} M e^{kt} e^{-(\mathrm{Re}\, s)t}\, dt$$

$$= M \sum_{j=1}^{n} \int_{b_{j-1}}^{b_j} e^{-(\mathrm{Re}\, s - k)t}\, dt$$

$$= \frac{M}{(\mathrm{Re}\, s - k)} \sum_{j=1}^{n} \left[e^{-(\mathrm{Re}\, s - k) b_{j-1}} - e^{-(\mathrm{Re}\, s - k) b_j} \right]$$

$$= \frac{M}{(\mathrm{Re}\, s - k)} \left[1 - e^{-(\mathrm{Re}\, s - k)T} \right]$$

$$\leq \frac{M}{\mathrm{Re}\, s - k} \quad \text{for every } T \geq 0.$$

Here, $b_0 \triangleq 0$, $b_n \triangleq T$. And other points $b_1, \cdots, b_{n-1}$ are finite-jump discontinuities.

Hence, the Laplace Transform in (1), p. 221 exits if $\mathrm{Re}\, s - k > 0$.

$\blacksquare$

THEOREM 1

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{1}$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Formula (1) *holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction* (2) *in Sec.* 6.1 *and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly,* (2) *holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.*

PROOF We prove (1) first under the *additional assumption* that f' is continuous. Then by the definition and integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st}f'(t)\,dt = [e^{-st}f(t)]\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt.$$

Since f satisfies (2) in Sec. 6.1, the integrated part on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$. It exists for $s > k$ because of Theorem 3 in Sec. 6.1. Hence $\mathcal{L}(f')$ exists when $s > k$ and (1) holds.

If f' is merely piecewise continuous, the proof is similar. In this case the interval of integration of f' must be broken up into parts such that f' is continuous in each such part.

The proof of (2) now follows by applying (1) to f'' and then substituting (1), that is

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] = s^2\mathcal{L}(f) - sf(0) - f'(0). \qquad \blacksquare$$

Continuing by substitution as in the proof of (2) and using induction, we obtain the following extension of Theorem 1.

THEOREM 2

Laplace Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction (2) *in Sec.* 6.1. *Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies*

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

EXAMPLE 1 **Transform of a Resonance Term (Sec. 2.8)**

Let $f(t) = t \sin \omega t$. Then $f(0) = 0$, $f'(t) = \sin \omega t + \omega t \cos \omega t$, $f'(0) = 0$, $f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (2),

$$\mathcal{L}(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f), \qquad \text{thus} \qquad \mathcal{L}(f) = \mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}. \qquad \blacksquare$$

EXAMPLE 2 Formulas 7 and 8 in Table 6.1, Sec. 6.1

This is a third derivation of $\mathcal{L}(\cos \omega t)$ and $\mathcal{L}(\sin \omega t)$; cf. Example 4 in Sec. 6.1. Let $f(t) = \cos \omega t$. Then $f(0) = 1$, $f'(0) = 0$, $f''(t) = -\omega^2 \cos \omega t$. From this and (2) we obtain

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f). \qquad \text{By algebra,} \qquad \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Similarly, let $g = \sin \omega t$. Then $g(0) = 0$, $g' = \omega \cos \omega t$. From this and (1) we obtain

$$\mathcal{L}(g') = s\mathcal{L}(g) = \omega\mathcal{L}(\cos \omega t). \qquad \text{Hence} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s}\mathcal{L}(\cos \omega t) = \frac{\omega}{s^2 + \omega^2}.$$ ∎

Laplace Transform of the Integral of a Function

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

THEOREM 3

$|f(t)| \leq M e^{kt}$

Laplace Transform of Integral

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction (2), Sec. 6.1. Then, for $s > 0$, $s > k$, and $t > 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}. \tag{4}$$

PROOF Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives

$$|g(t)| = \left|\int_0^t f(\tau)\, d\tau\right| \leqq \int_0^t |f(\tau)|\, d\tau \leqq M \int_0^t e^{k\tau}\, d\tau = \frac{M}{k}(e^{kt} - 1) \leqq \frac{M}{k}e^{kt} \qquad (k > 0).$$

This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t) = f(t)$, except at points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(0) = 0$ (the integral from 0 to 0 is zero)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}.$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. ∎

EXAMPLE 3 Application of Theorem 3: Formulas 19 and 20 in the Table of Sec. 6.9

Using Theorem 3, find the inverse of $\dfrac{1}{s(s^2 + \omega^2)}$ and $\dfrac{1}{s^2(s^2 + \omega^2)}$.

Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) (second formula with the sides interchanged) we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + \omega^2)}\right\} = \int_0^t \frac{\sin \omega\tau}{\omega}\, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

This is formula 19 in Sec. 6.9. Integrating this result again and using (4) as before, we obtain formula 20 in Sec. 6.9:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}\right]_0^t = \frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}.$$

It is typical that results such as these can be found in several ways. In this example, try partial fraction reduction. ■

Differential Equations, Initial Value Problems

We shall now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$\text{(5)} \qquad y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are constant. Here $r(t)$ is the given **input** (*driving force*) applied to the mechanical or electrical system and $y(t)$ is the **output** (*response to the input*) to be obtained. In Laplace's method we do three steps:

Step 1. Setting up the subsidiary equation. This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming (5) by means of (1) and (2), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Solution of the subsidiary equation by algebra. We divide by $s^2 + as + b$ and use the so-called **transfer function**

$$\text{(6)} \qquad Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

(Q is often denoted by H, but we need H much more frequently for other purposes.) This gives the solution

$$\text{(7)} \qquad Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$: Input-Output Transferfunction

and this explains the name of Q. Note that ***Q depends neither on r(t) nor on the initial conditions*** (but only on a and b).

Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$. We reduce (7) (usually by ***partial fractions*** as in calculus) to a sum of terms whose inverses can be found from the tables (e.g., in Sec. 6.1 or Sec. 6.9) or by a CAS, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ *of* (5).

EXAMPLE 4 **Initial Value Problem: The Basic Laplace Steps**

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution. ***Step 1.*** From (2) and Table 6.1 we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s + 1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}.$$

Simplification and **partial fraction expansion** gives

$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

The diagram in Fig. 115 summarizes our approach. ■

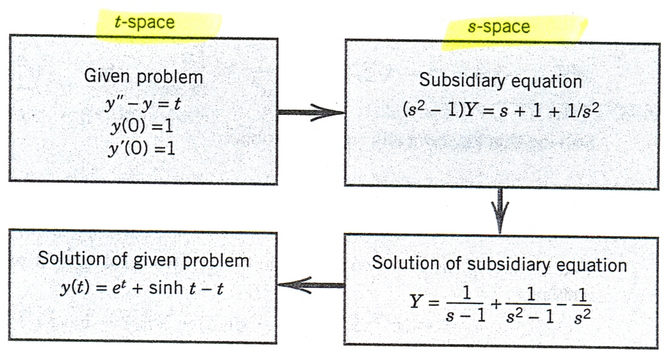


Fig. 115. Laplace transform method

EXAMPLE 5 **Comparison with the Usual Method**

Solve the initial value problem

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

Solution. From (1) and (2) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, \qquad \text{thus} \qquad (s^2 + s + 9)Y = 0.16(s + 1).$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}.$$

Hence by the first shifting theorem and the formulas for cos and sin in Table 6.1 we obtain

$$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16 \cos\sqrt{\frac{35}{4}}\,t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin\sqrt{\frac{35}{4}}\,t\right)$$
$$= e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t).$$

This agrees with Example 2, Case (III) in Sec. 2.4. The work was less. ■

Advantages of the Laplace Method

1. *Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.* See Example 4.
2. *Initial values are automatically taken care of.* See Examples 4 and 5.
3. *Complicated inputs* $r(t)$ (right sides of linear ODEs) *can be handled very efficiently,* as we show in the next sections.

EXAMPLE 6 Shifted Data Problems

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Let $\tilde{y}(\tilde{t}) \triangleq y(\tilde{t} + \frac{1}{4}\pi)$

$t = \tilde{t} + \frac{1}{4}\pi$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s\cdot\tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\begin{aligned} \tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) &= 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t} \\ &= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}. \end{aligned}$$

Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin\tilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t.$$ ■

$y(t) = y(\tilde{t} + \frac{1}{4}\pi) = \tilde{y}(\tilde{t}) = \tilde{y}(t - \frac{1}{4}\pi)$

PROBLEM SET 6.2

1–8 OBTAINING TRANSFORMS BY DIFFERENTIATION

Using (1) or (2), find $\mathcal{L}(f)$ if $f(t)$ equals:

1. te^{kt}
2. $t\cos 5t$
3. $\sin^2 \omega t$
4. $\cos^2 \pi t$
5. $\sinh^2 at$
6. $\cosh^2 \frac{1}{2}t$
7. $t\sin\frac{1}{2}\pi t$
8. $\sin^4 t$ (Use Prob. 3.)

9. (Derivation by different methods) It is typical that various transforms can be obtained by several methods. Show this for Prob. 1. Show it for $\mathcal{L}(\cos^2 \frac{1}{2}t)$ (a) by expressing $\cos^2 \frac{1}{2}t$ in terms of $\cos t$, (b) by using Prob. 3.

10–24 INITIAL VALUE PROBLEMS

Solve the following initial value problems by the Laplace transform. (If necessary, use partial fraction expansion as in Example 4. Show all details.)

10. $y' + 4y = 0, \quad y(0) = 2.8$
11. $y' + \frac{1}{2}y = 17\sin 2t, \quad y(0) = -1$
12. $y'' - y' - 6y = 0, \quad y(0) = 6, \quad y'(0) = 13$

HW

Prob. 6.2-9
Prob. 6.2-24
Prob. 6.2-25 (b)

Unit Step Function (Heaviside Function) $u(t - a)$

The **unit step function** or **Heaviside function** $u(t - a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geqq 0). \tag{1}$$

Figure 117 shows the special case $u(t)$, which has its jump at zero, and Fig. 118 the general case $u(t - a)$ for an arbitrary positive a. (For Heaviside see Sec. 6.1.)

The transform of $u(t - a)$ follows directly from the defining integral in Sec. 6.1,

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\cdot 1\,dt = -\left.\frac{e^{-st}}{s}\right|_{t=a}^{\infty};$$

here the integration begins at $t = a$ ($\geqq 0$) because $u(t - a)$ is 0 for $t < a$. Hence

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad (s > 0). \tag{2}$$

The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either "off" or "on." Multiplying functions $f(t)$ with $u(t - a)$, we can produce all sorts of effects. The simple basic idea is illustrated in Figs. 119 and 120. In Fig. 119 the given function is shown in (A). In (B) it is switched off between $t = 0$ and $t = 2$ (because $u(t - 2) = 0$ when $t < 2$) and is switched on beginning at $t = 2$. In (C) it is shifted to the right by 2 units, say, for instance, by 2 secs, so that it begins 2 secs later in the same fashion as before. More generally we have the following.

Let $f(t) = 0$ for all negative t. Then $f(t - a)u(t - a)$ with $a > 0$ is f(t) ***shifted*** *(translated) to the right by the amount a.*

Figure 120 shows the effect of many unit step functions, three of them in (A) and infinitely many in (B) when continued periodically to the right; this is the effect of a rectifier that clips off the negative half-waves of a sinusoidal voltage. CAUTION! Make sure that you fully understand these figures, in particular the difference between parts (B) and (C) of Figure 119. Figure 119(C) will be applied next.

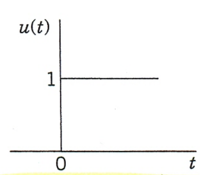


Fig. 117. Unit step function $u(t)$

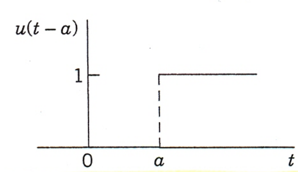
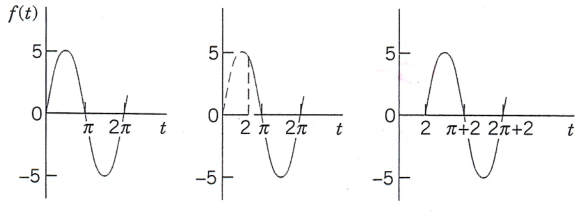
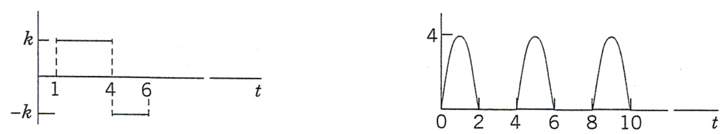


Fig. 118. Unit step function $u(t - a)$



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Fig. 119. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.



(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 120. Use of many unit step functions.

Time Shifting (t-Shifting): Replacing t by $t - a$ in $f(t)$

The first shifting theorem ("s-shifting") in Sec. 6.1 concerned transforms $F(s) = \mathcal{L}\{f(t)\}$ and $F(s-a) = \mathcal{L}\{e^{at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t-a)$. Unit step functions are just tools, and the theorem will be needed to apply them in connection with any other functions.

THEOREM 1 **Second Shifting Theorem; Time Shifting**

If $f(t)$ has the transform $F(s)$, then the **"shifted function"**

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases} \tag{3}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s). \tag{4}$$

Or, if we take the inverse on both sides, we can write

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}. \tag{4*}$$

Practically speaking, if we know $F(s)$, we can obtain the transform of (3) by multiplying $F(s)$ by e^{-as}. In Fig. 119, the transform of $5 \sin t$ is $F(s) = 5/(s^2 + 1)$, hence the shifted function $5 \sin(t-2)\, u(t-2)$ shown in Fig. 119(C) has the transform

$$e^{-2s}F(s) = 5e^{-2s}/(s^2 + 1).$$

PROOF We prove Theorem 1. In (4) on the right we use the definition of the Laplace transform, writing τ for t (to have t available later). Then, taking e^{-as} inside the integral, we have

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau.$$

Substituting $\tau + a = t$, thus $\tau = t - a$, $d\tau = dt$, in the integral (CAUTION, the lower limit changes!), we obtain

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt.$$

To make the right side into a Laplace transform, we must have an integral from 0 to ∞, not from a to ∞. But this is easy. We multiply the integrand by $u(t-a)$. Then for t from 0 to a the integrand is 0, and we can write, with $\widetilde{f}$ as in (3),

$$e^{-as}F(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt = \int_0^\infty e^{-st}\widetilde{f}(t)\,dt.$$

(Do you now see why $u(t-a)$ appears?) This integral is the left side of (4), the Laplace transform of $\widetilde{f}(t)$ in (3). This completes the proof. ∎

EXAMPLE 1 Application of Theorem 1. Use of Unit Step Functions

Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } \quad t > \frac{1}{2}\pi. \end{cases} \qquad \text{(Fig. 121)}$$

Solution. *Step 1.* In terms of unit step functions,

$$f(t) = 2(\overset{u(t)}{\cancel{1}} - u(t-1)) + \tfrac{1}{2}t^2(u(t-1) - u(t-\tfrac{1}{2}\pi)) + (\cos t)u(t-\tfrac{1}{2}\pi).$$

Indeed, $2(1 - u(t-1))$ gives $f(t)$ for $0 < t < 1$, and so on.

Step 2. To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t-a)u(t-a)$. Thus, $2(1 - u(t-1))$ remains as it is and gives the transform $2(1 - e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t-\frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t-\frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right)u\left(t-\frac{1}{2}\pi\right)\right\}$$

$$= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

$$\mathcal{L}\left\{(\cos t)u\left(t-\frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t-\frac{1}{2}\pi\right)\right)u\left(t-\frac{1}{2}\pi\right)\right\} = -\frac{1}{s^2+1}e^{-\pi s/2}.$$

Together,

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2+1}e^{-\pi s/2}.$$

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

If the conversion of $f(t)$ to $f(t-a)$ is inconvenient, replace it by

$$\text{(4**)} \qquad \mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

$\mathcal{L}[f(t-a)\,u(t-a)]$
$= e^{-as}F(s)$ (4)

(4**) follows from (4) by writing $f(t-a) = g(t)$, hence $f(t) = g(t+a)$ and then again writing f for g. Thus,

p.235

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2 + t + \frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

as before. Similarly for $\mathcal{L}\{\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\}$. Finally, by (4**),

$$\mathcal{L}\left\{\cos t\, u\left(t - \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t + \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}. \quad ■$$

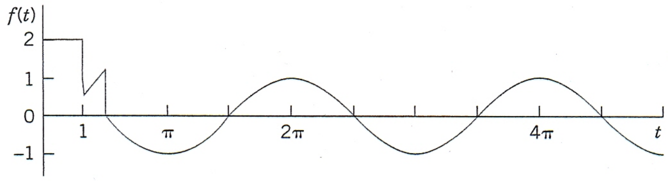


Fig. 121. $f(t)$ in Example 1

EXAMPLE 2 Application of Both Shifting Theorems. Inverse Transform

Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

Solution. Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and te^{-2t} because $1/s^2$ has the inverse t, so that $1/(s+2)^2$ has the inverse te^{-2t} by the first shifting theorem in Sec. 6.1. Hence by the second shifting theorem (t-shifting),

$$f(t) = \frac{1}{\pi}\sin(\pi(t-1))\,u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))\,u(t-2) + (t-3)e^{-2(t-3)}\,u(t-3).$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so that the second and third terms cancel each other when $t > 2$. Hence we obtain $f(t) = 0$ if $0 < t < 1$, $-(\sin \pi t)/\pi$ if $1 < t < 2$, 0 if $2 < t < 3$, and $(t-3)e^{-2(t-3)}$ if $t > 3$. See Fig. 122. ■

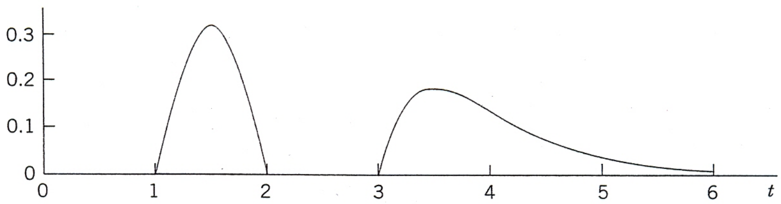
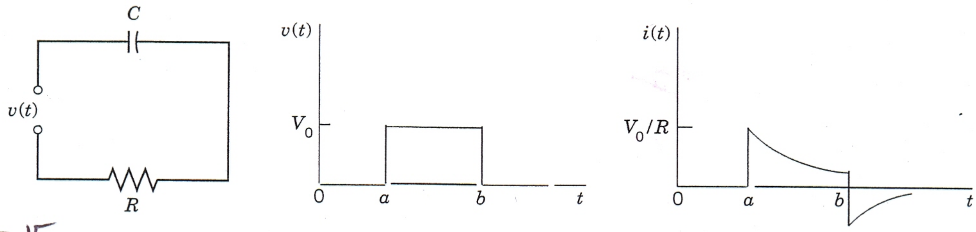


Fig. 122. $f(t)$ in Example 2

EXAMPLE 3 Response of an *RC*-Circuit to a Single Rectangular Wave

Find the current $i(t)$ in the RC-circuit in Fig. 123 if a single rectangular wave with voltage V_0 is applied. The circuit is assumed to be quiescent before the wave is applied.



$\begin{cases} R\dfrac{dq}{dt} + \dfrac{1}{c}q = v \\ q(0) = q_0 \end{cases}$

Fig. 123. *RC*-circuit, electromotive force $v(t)$, and current in Example 3

Solution. The input is $V_0[u(t-a) - u(t-b)]$. Hence the circuit is modeled by the integro-differential equation (see Sec. 2.9 and Fig. 123)

* O.K. only for $q_0 = 0$

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = v(t) = V_0[u(t-a) - u(t-b)].$$

Using Theorem 3 in Sec. 6.2 and formula (1) in this section, we obtain the subsidiary equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}[e^{-as} - e^{-bs}].$$

Solving this equation algebraically for $I(s)$, we get

$$I(s) = F(s)(e^{-as} - e^{-bs}) \qquad \text{where} \qquad F(s) = \frac{V_0/R}{s + 1/(RC)} \qquad \text{and} \qquad \mathcal{L}^{-1}(F) = \frac{V_0}{R}e^{-t/(RC)},$$

the last expression being obtained from Table 6.1 in Sec. 6.1. Hence Theorem 1 yields the solution (Fig. 123)

$$i(t) = \mathcal{L}^{-1}(I) = \mathcal{L}^{-1}\{e^{-as}F(s)\} - \mathcal{L}^{-1}\{e^{-bs}F(s)\} = \frac{V_0}{R}\left[e^{-(t-a)/(RC)}u(t-a) - e^{-(t-b)/(RC)}u(t-b)\right];$$

that is, $i(t) = 0$ if $t < a$, and

$$i(t) = \begin{cases} K_1 e^{-t/(RC)} & \text{if } a < t < b \\ (K_1 - K_2)e^{-t/(RC)} & \text{if } a > b \end{cases}$$

where $K_1 = V_0 e^{a/(RC)}/R$ and $K_2 = V_0 e^{b/(RC)}/R$. ■

EXAMPLE 4 **Response of an *RLC*-Circuit to a Sinusoidal Input Acting Over a Time Interval**

Find the response (the current) of the *RLC*-circuit in Fig. 124, where $E(t)$ is sinusoidal, acting for a short time interval only, say,

$$E(t) = 100 \sin 400t \quad \text{if } 0 < t < 2\pi \qquad \text{and} \qquad E(t) = 0 \text{ if } t > 2\pi$$

and current and charge are initially zero.

Solution. The electromotive force $E(t)$ can be represented by $(100 \sin 400t)(1 - u(t - 2\pi))$. Hence the model for the current $i(t)$ in the circuit is the integro-differential equation (see Sec. 2.9)

$$0.1i' + 11i + 100\int_0^t i(\tau)\,d\tau = (100 \sin 400t)(1 - u(t-2\pi)), \qquad i(0) = 0, \quad i'(0) = 0.$$

From Theorems 2 and 3 in Sec. 6.2 we obtain the subsidiary equation for $I(s) = \mathcal{L}(i)$

$$0.1sI + 11I + 100\frac{I}{s} = \frac{100 \cdot 400s}{s^2 + 400^2}\left(\frac{1}{s} - \frac{e^{-2\pi s}}{s}\right).$$

HW

Prob. 6.3-28
Prob. 6.3-34

force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 130 has area 1, the impulse of f_k in (1) is

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1. \tag{2}$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0$ $(k > 0)$. This limit is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

$\delta(t-a)$ is called the **Dirac delta function**[2] or the **unit impulse function.**

$\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but a so-called *generalized function*.[2] To see this, we note that the impulse I_k of f_k is 1, so that from (1) and (2) by taking the limit as $k \to 0$ we obtain

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-a)\,dt = 1, \tag{3}$$

but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function. In particular, for a continuous function $g(t)$ one uses the property [often called the **sifting property** of $\delta(t-a)$, not to be confused with *shifting*]

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a) \tag{4}$$

See the attached for the proof

which is plausible by (2).

To obtain the Laplace transform of $\delta(t-a)$, we write

$$f_k(t-a) = \frac{1}{k}\,[u(t-a) - u(t-(a+k))]$$

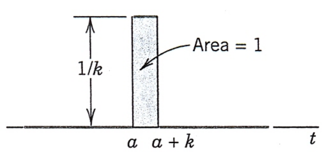


Fig. 130. The function $f_k(t-a)$ in (1)

[2]PAUL DIRAC (1902–1984), English physicist, was awarded the Nobel Prize [jointly with the Austrian ERWIN SCHRÖDINGER (1887–1961)] in 1933 for his work in quantum mechanics.

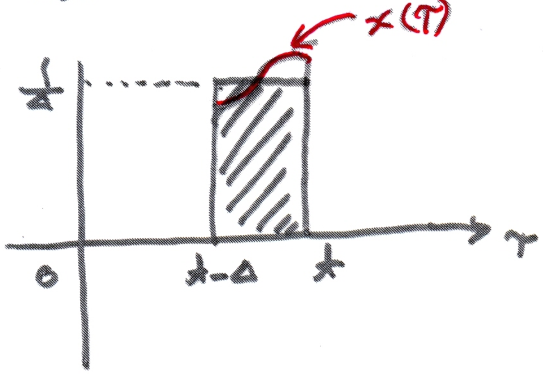
Generalized functions are also called **distributions.** Their theory was created in 1936 by the Russian mathematician SERGEI L'VOVICH SOBOLEV (1908–1989), and in 1945, under wider aspects, by the French mathematician LAURENT SCHWARTZ (1915–2002).

Theorem

Let $t>0$. Suppose that x is integrable on $(-\infty,\infty)$, and continuous at t. Then,

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t) \quad \left(\text{i.e. } x(t) * \delta(t) = x(t)\right)$$

<Proof>



$\delta_\Delta(t-\tau)$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta_\Delta(t-\tau)\,d\tau = \frac{1}{\Delta}\int_{t-\Delta}^{t} x(\tau)\,d\tau \qquad (1)$$

By the mean value theorem for integrals, $\exists$ $t_0 \in [t-\Delta, t] \ni$

$$\int_{t-\Delta}^{t} x(\tau)\,d\tau = \Delta\, x(t_0) \qquad (2)$$

By (1) and (2),

$$\int_{-\infty}^{\infty} x(\tau)\,\delta_\Delta(t-\tau)\,d\tau = x(t_0) \qquad (3)$$

Take the limit of both sides of eq. (3) as $\Delta \to 0^+$.

Then,

$$\begin{cases} \lim\limits_{\Delta\to 0^+} t_0 = t & \text{since } t_0 \in [t-\Delta, t] \\ \lim\limits_{\Delta\to 0^+} x(t_0) = x(t) & \text{since } x \text{ is continuous at } t \\ \lim\limits_{\Delta\to 0^+} \delta_\Delta(t-\tau) = \delta(t-\tau) & \text{by definition.} \end{cases}$$

///

and take the transform [see (2)]

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as}\,\frac{1-e^{-ks}}{ks}.$$

We now take the limit as $k \to 0$. By l'Hôpital's rule the quotient on the right has the limit 1 (differentiate the numerator and the denominator separately with respect to k, obtaining se^{-ks} and s, respectively, and use $se^{-ks}/s \to 1$ as $k \to 0$). Hence the right side has the limit e^{-as}. This suggests defining the transform of $\delta(t-a)$ by this limit, that is,

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}. \tag{5}$$

The unit step and unit impulse functions can now be used on the right side of ODEs modeling mechanical or electrical systems, as we illustrate next.

EXAMPLE 1 Mass–Spring System Under a Square Wave

Determine the response of the damped mass–spring system (see Sec. 2.8) under a square wave, modeled by (see Fig. 131)

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution. From (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = \frac{1}{s}(e^{-s} - e^{-2s}). \qquad \text{Solution} \qquad Y(s) = \frac{1}{s(s^2+3s+2)}(e^{-s} - e^{-2s}).$$

Using the notation $F(s)$ and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}.$$

From Table 6.1 in Sec. 6.1, we see that the inverse is

$$f(t) = \mathcal{L}^{-1}(F) = \tfrac{1}{2} - e^{-t} + \tfrac{1}{2}e^{-2t}.$$

Therefore, by Theorem 1 in Sec. 6.3 (t-shifting) we obtain the square-wave response shown in Fig. 131,

$$\begin{aligned} y &= \mathcal{L}^{-1}(F(s)e^{-s} - F(s)e^{-2s}) \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \\ &= \begin{cases} 0 & (0 < t < 1) \\ \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{-(t-2)} + \tfrac{1}{2}e^{-2(t-1)} - \tfrac{1}{2}e^{-2(t-2)} & (t > 2). \end{cases} \end{aligned}$$

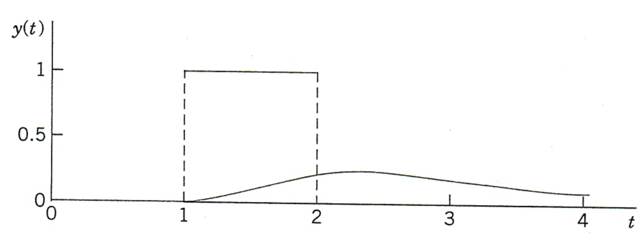


Fig. 131. Square wave and response in Example 1

(Remark)

$$\int_0^\infty \lim_{k\to\infty} f_k(t-a)\,e^{-st}\,dt = \lim_{k\to\infty}\int_0^\infty f_k(t-a)\,e^{-st}\,dt$$

(by uniform convergence)

$$= \lim_{k\to\infty} e^{-as}\left[\frac{1-e^{-ks}}{ks}\right] = e^{-as} \quad (*)$$

PROBLEM SET 6.4

1–12 EFFECT OF DELTA FUNCTION ON VIBRATING SYSTEMS

Showing the details, find, graph, and discuss the solution.

1. $y'' + y = \delta(t - 2\pi)$, $y(0) = 10$, $y'(0) = 0$
2. $y'' + 2y' + 2y = e^{-t} + 5\,\delta(t - 2)$, $y(0) = 0$, $y'(0) = 1$
3. $y'' - y = 10\delta(t - \frac{1}{2}) - 100\delta(t - 1)$, $y(0) = 10$, $y'(0) = 1$
4. $y'' + 3y' + 2y = 10(\sin t + \delta(t - 1))$, $y(0) = 1$, $y'(0) = -1$
5. $y'' + 4y' + 5y = [1 - u(t - 10)]e^t - e^{10}\delta(t - 10)$, $y(0) = 0$, $y'(0) = 1$
6. $y'' + 2y' - 3y = 100\delta(t - 2) + 100\delta(t - 3)$, $y(0) = 1$, $y'(0) = 0$
7. $y'' + 2y' + 10y = 10[1 - u(t - 4)] - 10\delta(t - 5)$, $y(0) = 1$, $y'(0) = 1$
8. $y'' + 5y' + 6y = \delta(t - \frac{1}{2}\pi) + u(t - \pi)\cos t$, $y(0) = 0$, $y'(0) = 0$
9. $y'' + 2y' + 5y = 25t - 100\delta(t - \pi)$, $y(0) = -2$, $y'(0) = 5$
10. $y'' + 5y = 25t - 100\delta(t - \pi)$, $y(0) = -2$, $y'(0) = 5$. (Compare with Prob. 9.)
11. $y'' + 3y' - 4y = 2e^t - 8e^2\delta(t - 2)$, $y(0) = 2$, $y'(0) = 0$
12. $y'' + y = -2\sin t + 10\delta(t - \pi)$, $y(0) = 0$, $y'(0) = 1$

13. **CAS PROJECT. Effect of Damping.** Consider a vibrating system of your choice modeled by

$$y'' + cy' + ky = r(t)$$

with $r(t)$ involving a δ-function. **(a)** Using graphs of the solution, describe the effect of continuously decreasing the damping to 0, keeping k constant.

(b) What happens if c is kept constant and k is continuously increased, starting from 0?

(c) Extend your results to a system with two δ-functions on the right, acting at different times.

14. **CAS PROJECT. Limit of a Rectangular Wave. Effects of Impulse.**

(a) In Example 1, take a rectangular wave of area 1 from 1 to $1 + k$. Graph the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 132. *Hint:* If your CAS gives no solution for the differential equation. involving k, take specific k's from the beginning.

(b) Experiment on the response of the ODE in Example 1 (or of another ODE of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting graphs?

15. **PROJECT. Heaviside Formulas. (a)** Show that for a simple root a and fraction $A/(s - a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s - a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s - a)^m} + \frac{A_{m-1}}{(s - a)^{m-1}} + \cdots + \frac{A_1}{s - a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s - a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m - k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s - a)^m F(s)}{G(s)} \right], \quad k = 1, \cdots, m - 1.$$

16. **TEAM PROJECT. Laplace Transform of Periodic Functions**

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0). \tag{11}$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n - 1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of $\sin \omega t$ in Fig. 135 has the Laplace transform

$$\mathcal{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 136.)

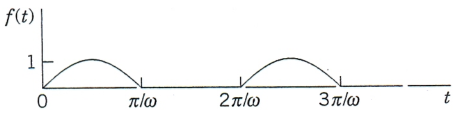


Fig. 135. Half-wave rectification

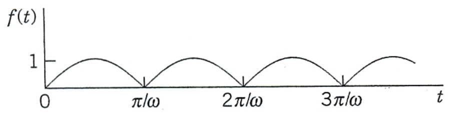


Fig. 136. Full-wave rectification

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 137.

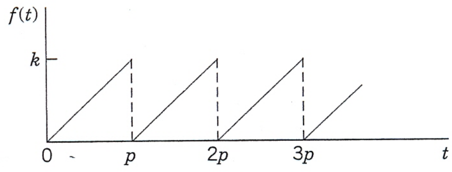


Fig. 137. Saw-tooth wave

(e) Staircase function. Find the Laplace transform of the staircase function in Fig. 138 by noting that it is the difference of kt/p and the function in (d).

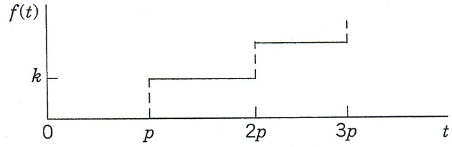


Fig. 138. Staircase function

6.5 Convolution. Integral Equations

Convolution has to do with the multiplication of transforms. The situation is as follows. *Addition* of transforms provides no problem; we know that $\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g)$. Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathcal{L}(f)$ and $\mathcal{L}(g)$ and would like to know the function whose transform is the product $\mathcal{L}(f)\mathcal{L}(g)$. We might perhaps guess that it is fg, but this is false. *The transform of a product is generally different from the product of the transforms of the factors,*

$$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g) \qquad \text{in general.}$$

To see this take $f = e^t$ and $g = 1$. Then $fg = e^t$, $\mathcal{L}(fg) = 1/(s - 1)$, but $\mathcal{L}(f) = 1/(s - 1)$ and $\mathcal{L}(1) = 1/s$ give $\mathcal{L}(f)\mathcal{L}(g) = 1/(s^2 - s)$.

According to the next theorem, the correct answer is that $\mathcal{L}(f)\mathcal{L}(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f * g$ and defined by the integral

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau. \tag{1}$$

THEOREM 1 **Convolution Theorem** (piecewise-continuity + growth restriction)

If two functions f and g satisfy the assumption in the existence theorem in Sec. 6.1, so that their transforms F and G exist, the product $H = FG$ is the transform of h given by (1). (Proof after Example 2.)

EXAMPLE 1 **Convolution**

Let $H(s) = 1/[(s-a)s]$. Find $h(t)$.

Solution. $1/(s-a)$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t-\tau) \equiv 1$ we thus obtain from (1) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 \, d\tau = \frac{1}{a}(e^{at} - 1).$$

To check, calculate

$$H(s) = \mathcal{L}(h)(s) = \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{a} \cdot \frac{a}{s^2 - as} = \frac{1}{s-a} \cdot \frac{1}{s} = \mathcal{L}(e^{at})\,\mathcal{L}(1).$$ ■

EXAMPLE 2 **Convolution**

Let $H(s) = 1/(s^2 + \omega^2)^2$. Find $h(t)$.

Solution. The inverse of $1/(s^2 + \omega^2)$ is $(\sin \omega t)/\omega$. Hence from (1) and the trigonometric formula (11) in App. 3.1 with $x = \frac{1}{2}(\omega t + \omega\tau)$ and $y = \frac{1}{2}(\omega t - \omega\tau)$ we obtain

$$\begin{aligned} h(t) = \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} &= \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t-\tau)\, d\tau \\ &= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos \omega\tau]\, d\tau \\ &= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t} \\ &= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right] \end{aligned}$$

in agreement with formula 21 in the table in Sec. 6.9. ■

PROOF We prove the Convolution Theorem 1. CAUTION! Note which ones are the variables of integration! We can denote them as we want, for instance, by τ and p, and write

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \qquad \text{and} \qquad G(s) = \int_0^\infty e^{-sp} g(p)\, dp.$$

We now set $t = p + \tau$, where τ is at first constant. Then $p = t - \tau$, and t varies from τ to ∞. Thus

$$G(s) = \int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\, dt = e^{s\tau}\int_\tau^\infty e^{-st} g(t-\tau)\, dt.$$

$$\begin{aligned} \Rightarrow F(s)\,G(s) &= \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \; G(s) = \int_0^\infty e^{-s\tau} f(\tau)\, G(s)\, d\tau \\ &= \int_0^\infty e^{-s\tau} f(\tau)\, e^{s\tau}\int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau \\ &= \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau \end{aligned}$$

τ in F and t in G vary independently. Hence we can insert the G-integral into the F-integral. Cancellation of $e^{-s\tau}$ and $e^{s\tau}$ then gives

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\, dt\, d\tau.$$

Here we integrate for fixed τ over t from τ to ∞ and then over τ from 0 to ∞. This is the blue region in Fig. 139. Under the assumption on f and g the order of integration can be reversed (see Ref. [A5] for a proof using uniform convergence). We then integrate first over τ from 0 to t and then over t from 0 to ∞, that is,

$$F(s)G(s) = \int_0^\infty e^{-st} \int_0^t f(\tau) g(t-\tau)\, d\tau\, dt = \int_0^\infty e^{-st} h(t)\, dt = \mathcal{L}(h) = H(s).$$

This completes the proof. ■

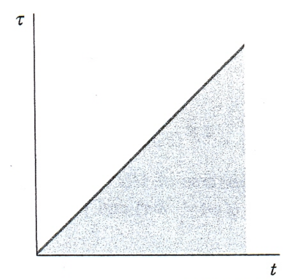


Fig. 139. Region of integration in the $t\tau$-plane in the proof of Theorem 1

From the definition it follows almost immediately that convolution has the properties

$$f * g = g * f \qquad \text{(commutative law)}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)}$$

$$(f * g) * v = f * (g * v) \qquad \text{(associative law)}$$

$$f * 0 = 0 * f = 0$$

similar to those of the multiplication of numbers. Unusual are the following two properties.

EXAMPLE 3 Unusual Properties of Convolution

$f * 1 \neq f$ in general. For instance,

$$t * 1 = \int_0^t \tau \cdot 1\, d\tau = \tfrac{1}{2}t^2 \neq t.$$

$(f * f)(t) \geqq 0$ may not hold. For instance, Example 2 with $\omega = 1$ gives

$$\sin t * \sin t = -\tfrac{1}{2} t \cos t + \tfrac{1}{2} \sin t \qquad \text{(Fig. 140).}$$ ■

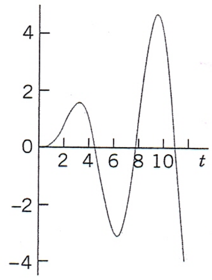


Fig. 140. Example 3

We shall now take up the case of a complex double root (left aside in the last section in connection with partial fractions) and find the solution (the inverse transform) directly by convolution.

EXAMPLE 4 Repeated Complex Factors. Resonance

In an undamped mass–spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system. Then the model is (see Sec. 2.8)

$$y'' + \omega_0^{\ 2} y = K \sin \omega_0 t$$

where $\omega_0^{\ 2} = k/m$, k is the spring constant, and m is the mass of the body attached to the spring. We assume $y(0) = 0$ and $y'(0) = 0$, for simplicity. Then the subsidiary equation is

$$s^2 Y + \omega_0^{\ 2} Y = \frac{K\omega_0}{s^2 + \omega_0^{\ 2}}. \qquad \text{Its solution is} \qquad Y = \frac{K\omega_0}{(s^2 + \omega_0^{\ 2})^2}.$$

This is a transform as in Example 2 with $\omega = \omega_0$ and multiplied by $K\omega_0$. Hence from Example 2 we can see directly that the solution of our problem is p. 249

$$y(t) = \frac{K\omega_0}{2\omega_0^{\ 2}}\left(-t \cos \omega_0 t + \frac{\sin \omega_0 t}{\omega_0}\right) = \frac{K}{2\omega_0^{\ 2}}(-\omega_0 t \cos \omega_0 t + \sin \omega_0 t).$$

We see that the first term grows without bound. Clearly, in the case of resonance such a term must occur. (See also a similar kind of solution in Fig. 54 in Sec. 2.8.) ■

Application to Nonhomogeneous Linear ODEs

Nonhomogeneous linear ODEs can now be solved by a general method based on convolution by which the solution is obtained in the form of an integral. To see this, recall from Sec. 6.2 that the subsidiary equation of the ODE

$$y'' + ay' + by = r(t) \qquad (a, b \text{ constant}) \tag{2}$$

has the solution [(7) in Sec. 6.2]

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function. Inversion of the first term $[\cdots]$ provides no difficulty; depending on whether $\frac{1}{4}a^2 - b$ is positive, zero, or negative, its inverse will be a linear combination of two exponential functions, or of the form $(c_1 + c_2 t)e^{-at/2}$, or a damped oscillation, respectively. The interesting term is $R(s)Q(s)$ because $r(t)$ can have various forms of practical importance, as we shall see. If $y(0) = 0$ and $y'(0) = 0$, then $Y = RQ$, and the convolution theorem gives the solution

$$y(t) = \int_0^t q(t - \tau) r(\tau)\, d\tau. \tag{3}$$

EXAMPLE 5 **Response of a Damped Vibrating System to a Single Square Wave**

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and 0 otherwise}, \qquad y(0) = y'(0) = 0.$$

This system with an **input** (a driving force) ***that acts for some time only*** (Fig. 141) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

$Y(s) = Q(s)R(s)$

$\Downarrow$

$y(t) = \int_0^t q(t-\tau) r(\tau)\, d\tau$

$= \int_0^t [e^{-(t-\tau)} - e^{-2(t-\tau)}] r(\tau)\, d\tau$

where

$r(\tau) = \begin{cases} 1, & 1 \le \tau \le 2 \\ 0, & \text{otherwise} \end{cases}$

Solution by Convolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t - \tau) \cdot 1\, d\tau = \int \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau = e^{-(t-\tau)} - \tfrac{1}{2}e^{-2(t-\tau)}.$$

Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to t. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}).$$

Figure 141 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why?), and finally decreases to zero in a monotone fashion. ■

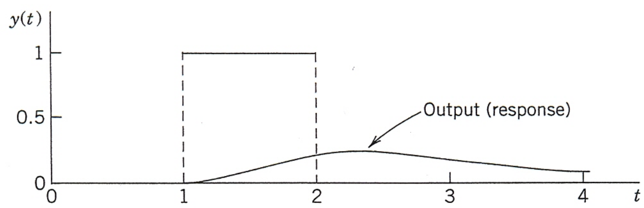


Fig. 141. Square wave and response in Example 5

Integral Equations

Convolution also helps in solving certain **integral equations,** that is, equations in which the unknown function $y(t)$ appears in an integral (and perhaps also outside of it). This concerns equations with an integral of the form of a convolution. Hence these are special and it suffices to explain the idea in terms of two examples and add a few problems in the problem set.

EXAMPLE 6 **A Volterra Integral Equation of the Second Kind**

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau) \sin(t - \tau)\, d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad \text{and gives the answer} \qquad y(t) = t + \frac{t^3}{6}.$$

Check the result by a CAS or by substitution and repeated integration by parts (which will need patience). ■

EXAMPLE 7 **Another Volterra Integral Equation of the Second Kind**

Solve the Volterra integral equation

$$y(t) - \int_0^t (1+\tau)\, y(t-\tau)\, d\tau = 1 - \sinh t.$$

Solution. By (1) we can write $y - (1 + t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1}, \qquad \text{hence} \qquad Y(s)\cdot\frac{s^2-s-1}{s^2} = \frac{s^2-1-s}{s(s^2-1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2-1} \qquad \text{and the solution is} \qquad y(t) = \cosh t.$$ ■

PROBLEM SET 6.5

1–8 CONVOLUTIONS BY INTEGRATION

Find by integration:

1. $1 * 1$

2. $t * t$

3. $t * e^t$

4. $e^{at} * e^{bt}$ $(a \neq b)$

5. $1 * \cos \omega t$

6. $1 * f(t)$

7. $e^{kt} * e^{-kt}$

8. $\sin t * \cos t$

9–16 INVERSE TRANSFORMS BY CONVOLUTION

Find $f(t)$ if $\mathcal{L}(f)$ equals:

9. $\dfrac{1}{(s-3)(s+5)}$

10. $\dfrac{1}{s(s-1)}$

11. $\dfrac{1}{s(s^2+4)}$

12. $\dfrac{1}{s^2(s-2)}$

13. $\dfrac{1}{s^2(s^2+1)}$

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{1}{s(s^2-9)}$

16. $\dfrac{5}{(s^2+1)(s^2+25)}$

17. **(Partial fractions)** Solve Probs. 9, 11, and 13 by using partial fractions. Comment on the amount of work.

18–25 SOLVING INITIAL VALUE PROBLEMS

Using the convolution theorem, solve:

18. $y'' + y = \sin t, \quad y(0) = 0, \quad y'(0) = 0$

19. $y'' + 4y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0$

20. $y'' + 5y' + 4y = 2e^{-2t}, \quad y(0) = 0,$ $y'(0) = 0$

[3] If the upper limit of integration is *variable,* the equation is named after the Italian mathematician VITO VOLTERRA (1860–1940), and if that limit is *constant,* the equation is named after the Swedish mathematician IVAR FREDHOLM (1866–1927). "Of the second kind (first kind)" indicates that y occurs (does not occur) outside of the integral.

HW

Prob. 6.5–13
Prob. 6.5–17 for 13
Prob. 6.5–21
Prob. 6.5–26 (a) – (d)

Consequently, if $\mathcal{L}(f) = F(s)$, then

$$\text{(1)} \qquad \mathcal{L}\{tf(t)\} = -F'(s), \qquad \text{hence} \qquad \mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

where the second formula is obtained by applying $\mathcal{L}^{-1}$ on both sides of the first formula. In this way, *differentiation of the transform of a function corresponds to the multiplication of the function by* $-t$.

EXAMPLE 1 Differentiation of Transforms. Formulas 21–23 in Sec. 6.9

We shall derive the following three formulas.

	$\mathcal{L}(f)$	$f(t)$
(2)	$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
(3)	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
(4)	$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

Solution. From (1) and formula 8 (with $\omega = \beta$) in Table 6.1 of Sec. 6.1 we obtain by differentiation (CAUTION! Chain rule!)

$$\mathcal{L}(t\sin\beta t) = \frac{2\beta s}{(s^2+\beta^2)^2}.$$

Dividing by 2β and using the linearity of $\mathcal{L}$, we obtain (3).

Formulas (2) and (4) are obtained as follows. From (1) and formula 7 (with $\omega = \beta$) in Table 6.1 we find

$$\text{(5)} \qquad \mathcal{L}(t\cos\beta t) = -\frac{(s^2+\beta^2) - 2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}.$$

From this and formula 8 (with $\omega = \beta$) in Table 6.1 we have

$$\mathcal{L}\left(t\cos\beta t \pm \frac{1}{\beta}\sin\beta t\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2} \pm \frac{1}{s^2+\beta^2}.$$

On the right we now take the common denominator. Then we see that for the plus sign the numerator becomes $s^2 - \beta^2 + s^2 + \beta^2 = 2s^2$, so that (4) follows by division by 2. Similarly, for the minus sign the numerator takes the form $s^2 - \beta^2 - s^2 - \beta^2 = -2\beta^2$, and we obtain (2). This agrees with Example 2 in Sec. 6.5. ■

Integration of Transforms

$g(t) \doteq f(t)/t \Leftarrow$ is piecewise-continuous and satisfies the growth restriction $|g(t)| \le \tilde{M} e^{\tilde{k}t}$

Similarly, if $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1 and the limit of $f(t)/t$, as t approaches 0 from the right, exists, then for $s > k$,

$$\text{(6)} \qquad \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s} \qquad \text{hence} \qquad \mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}.$$

In this way, *integration of the transform of a function $f(t)$ corresponds to the division of $f(t)$ by t.*

(Remark)

We indicate how (6) is obtained. From the definition it follows that

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)\, dt\right] d\tilde{s},$$

and it can be shown (see Ref. [GR4] in App. 1) that under the above assumptions we may reverse the order of integration, that is,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t} f(t)\, d\tilde{s}\right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-\tilde{s}t}\, d\tilde{s}\right] dt.$$

Integration of $e^{-\tilde{s}t}$ with respect to $\tilde{s}$ gives $e^{-\tilde{s}t}/(-t)$. Here the integral over $\tilde{s}$ on the right equals e^{-st}/t. Therefore,

$$\int_s^\infty F(\tilde{s})\, d\tilde{s} = \int_0^\infty e^{-st} \frac{f(t)}{t}\, dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\} \qquad (s > k). \qquad \blacksquare$$

EXAMPLE 2 Differentiation and Integration of Transforms

Find the inverse transform of $\ln\left(1 + \frac{\omega^2}{s^2}\right) = \ln \frac{s^2 + \omega^2}{s^2}$.

Solution. Denote the given transform by $F(s)$. Its derivative is

$$F'(s) = \frac{d}{ds}\left(\ln(s^2 + \omega^2) - \ln s^2\right) = \frac{2s}{s^2 + \omega^2} - \frac{2s}{s^2}.$$

Taking the inverse transform and using (1), we obtain

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2 + \omega^2} - \frac{2}{s}\right\} = 2\cos\omega t - 2 = -tf(t).$$

Hence the inverse $f(t)$ of $F(s)$ is $f(t) = 2(1 - \cos\omega t)/t$. This agrees with formula 42 in Sec. 6.9.

Alternatively, if we let

$$G(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}, \qquad \text{then} \qquad g(t) = \mathcal{L}^{-1}(G) = 2(\cos\omega t - 1).$$

From this and (6) we get, in agreement with the answer just obtained,

$$\ln \frac{s^2 + \omega^2}{s^2} = \int_s^\infty G(s)\, ds = -\frac{g(t)}{t} = \frac{2}{t}(1 - \cos\omega t),$$

the minus occurring since s is the lower limit of integration.

In a similar way we obtain formula 43 in Sec. 6.9,

$$\mathcal{L}^{-1}\left\{\ln\left(1 - \frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1 - \cosh at). \qquad \blacksquare$$

Special Linear ODEs with Variable Coefficients

Formula (1) can be used to solve certain ODEs with variable coefficients. The idea is this. Let $\mathcal{L}(y) = Y$. Then $\mathcal{L}(y') = sY - y(0)$ (see Sec. 6.2). Hence by (1),

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}. \tag{7}$$

$$(1) \begin{cases} \mathcal{L}\{tf(t)\} = -F'(s) \\ f(t) \hat{=} y'(t) \end{cases}$$

Similarly, $\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$ and by (1)

$$\mathcal{L}(ty'') = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] = -2sY - s^2\frac{dY}{ds} + y(0). \tag{8}$$

Hence if an ODE has coefficients such as $at + b$, the subsidiary equation is a first-order ODE for Y, which is sometimes simpler than the given second-order ODE. But if the latter has coefficients $at^2 + bt + c$, then two applications of (1) would give a second-order ODE for Y, and this shows that the present method works well only for rather special ODEs with variable coefficients. An important ODE for which the method is advantageous is the following.

EXAMPLE 3 Laguerre's Equation. Laguerre Polynomials

Laguerre's ODE is

$$ty'' + (1 - t)y' + ny = 0. \tag{9}$$

We determine a solution of (9) with $n = 0, 1, 2, \cdots$. From (7)–(9) we get the subsidiary equation

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0.$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken zero), and taking exponentials, we get

$$\frac{dY}{Y} = -\frac{n + 1 - s}{s - s^2}\,ds = \left(\frac{n}{s - 1} - \frac{n + 1}{s}\right)ds \quad \text{and} \quad Y = \frac{(s - 1)^n}{s^{n+1}}. \tag{10*}$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and prove **Rodrigues's formula**

$$l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots. \tag{10}$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (see Problem Set 5.7, but we continue to reserve capital letters for transforms). We prove (10). By Table 6.1 and the first shifting theorem (s-shifting),

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s + 1)^{n+1}}, \qquad \text{hence by (3) in Sec. 6.2} \qquad \mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n!\,s^n}{(s + 1)^{n+1}}$$

because the derivatives up to the order $n - 1$ are zero at 0. Now make another shift and divide by $n!$ to get [see (10) and then (10*)]

$$\mathcal{L}(l_n) = \frac{(s - 1)^n}{s^{n+1}} = Y. \qquad ■$$

PROBLEM SET 6.6

1–12 TRANSFORMS BY DIFFERENTIATION

Showing the details of your work, find $\mathcal{L}(f)$ if $f(t)$ equals:

1. $4te^t$
2. $-t \cosh 2t$
3. $t \sin \omega t$
4. $t \cos (t + k)$
5. $te^{-2t} \sin t$
6. $t^2 \sin 3t$
7. $t^2 \sinh 4t$
8. $t^n e^{kt}$
9. $t^2 \sin \omega t$
10. $t \cos \omega t$
11. $t \sin (t + k)$
12. $te^{-kt} \sin t$

HW

Prob. 6.6-4
Prob. 6.6-14

13–20 INVERSE TRANSFORMS

Using differentiation, integration, s-shifting, or convolution (and showing the details), find $f(t)$ if $\mathcal{L}(f)$ equals:

13. $\dfrac{6}{(s+1)^2}$

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{2(s+2)}{[(s+2)^2+1]^2}$

16. $\dfrac{s}{(s^2-1)^2}$

17. $\dfrac{2}{(s-k)^3}$

18. $\ln \dfrac{s+a}{s+b}$

19. $\ln \dfrac{s}{s-1}$

20. $\operatorname{arccot} \dfrac{s}{\omega}$

21. WRITING PROJECT. Differentiation and Integration of Functions and Transforms. Make a short draft of these four operations from memory. Then compare your notes with the text and write a report of 2–3 pages on these operations and their significance in applications.

22. CAS PROJECT. Laguerre Polynomials. (a) Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

(b) Show that

$$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n}{m} t^m$$

and calculate $l_0, \cdots, l_{10}$ from this formula.

(c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

$$(n+1)l_{n+1} = (2n+1-t)l_n - nl_{n-1}.$$

(d) Experiment with the graphs of $l_0, \cdots, l_{10}$, finding out empirically how the first maximum, first minimum, $\cdots$ is moving with respect to its location as a function of n. Write a short report on this.

(e) A **generating function** (definition in Problem Set 5.3) for the Laguerre polynomials is

$$\sum_{n=0}^{\infty} l_n(t)x^n = (1-x)^{-1}e^{tx/(x-1)}.$$

Obtain $l_0, \cdots, l_{10}$ from the corresponding partial sum of this power series in x and compare the l_n with those in (a), (b), or (c).

6.7 Systems of ODEs

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients (as discussed in Sec. 4.1)

$y'(t) = Ay(t) + g(t)$ where $y(t) \triangleq \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$, $g(t) \triangleq \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$

$$\text{(1)} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned}$$

Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) in Sec. 6.2 the subsidiary system

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned}$$

By collecting the Y_1- and Y_2-terms we have

$$\text{(2)} \qquad \begin{aligned} (a_{11} - s)Y_1 + \quad a_{12}Y_2 \quad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \quad + (a_{22} - s)Y_2 &= -y_2(0) - G_2(s). \end{aligned}$$

By solving this system algebraically for $Y_1(s)$, $Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$ of the given system (1).

Let $Y(s) = \begin{bmatrix} Y_1(s) \\ \vdots \\ Y_n(s) \end{bmatrix}$. Then, $sY(s) - y(0) = \mathcal{L}[Ay(t)] + \mathcal{L}[g(t)]$

$= AY(s) + G(s)$

$G(s) = \begin{bmatrix} G_1(s) \\ \vdots \\ G_n(s) \end{bmatrix}$

$\Rightarrow Y(s) = (sI - A)^{-1}y(0) + (sI - A)^{-1}G(s)$

$\Rightarrow y(t) = \Phi(t)\,y(0) + \Phi * g(t)$ where $\Phi(t) \triangleq \mathcal{L}^{-1}[(sI-A)^{-1}]$

Note that $\dot{\Phi}(t) = A\Phi(t)$, $\Phi(0) = I_n$.

Also, compare with the result, p. 161

6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$ $= \int_0^t f(t - \tau) g(\tau)\, d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\, u(t - a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref.* [A9] *in Appendix 1.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!}\, t^{n-1}e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}\, t^{k-1}e^{at}$	
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{(a-b)}\,(ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2 + \omega^2}$	$\dfrac{1}{\omega} \sin \omega t$	6.1
14	$\dfrac{s}{s^2 + \omega^2}$	$\cos \omega t$	
15	$\dfrac{1}{s^2 - a^2}$	$\dfrac{1}{a} \sinh at$	
16	$\dfrac{s}{s^2 - a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2 + \omega^2}$	$\dfrac{1}{\omega} e^{at} \sin \omega t$	
18	$\dfrac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	
19	$\dfrac{1}{s(s^2 + \omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$	6.2
20	$\dfrac{1}{s^2(s^2 + \omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$	
21	$\dfrac{1}{(s^2 + \omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
22	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\sin \omega t$	6.6
23	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	6.6
24	$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$	
25	$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2}\sin kt \sinh kt$	
27	$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2}t\right)$	5.6
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	5.5
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}} e^{at}(1+2at)$	
33	$\dfrac{1}{(s^2-a^2)^k} \quad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	5.6
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	
40	$\dfrac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	5.6

(*continued*)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
41	$\ln \dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \dfrac{s^2+\omega^2}{s^2}$	$\dfrac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \dfrac{s^2-a^2}{s^2}$	$\dfrac{2}{t}(1 - \cosh at)$	
44	$\arctan \dfrac{\omega}{s}$	$\dfrac{1}{t} \sin \omega t$	
45	$\dfrac{1}{s} \operatorname{arccot} s$	$\mathrm{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

1. What do we mean by operational calculus?
2. What are the steps needed in solving an ODE by Laplace transform? What is the subsidiary equation?
3. The Laplace transform is a linear operation. What does this mean? Why is it important?
4. For what problems is the Laplace transform preferable over the usual method? Explain.
5. What are the unit step and Dirac's delta functions? Give examples.
6. What is the difference between the two shifting theorems? When do they apply?
7. Is $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
8. Can a discontinuous function have a Laplace transform? Does every continuous function have a Laplace transform? Give reasons.
9. State the transforms of a few simple functions from memory.
10. If two different continuous functions have transforms, the latter are different. Why is this practically important?

11–22 LAPLACE TRANSFORMS

Find the transform (showing the details of your work and indicating the method or formula you are using):

11. te^{3t} **12.** $e^{-t} \sin 2t$

13. $\sin^2 t$ **14.** $\cos^2 4t$

15. $tu(t - \pi)$ **16.** $u(t - 2\pi) \sin t$

17. $e^t * \cos 2t$ **18.** $(\sin \omega t) * (\cos \omega t)$

19. $\sin t + \sinh t$ **20.** $\cosh t - \cos t$

21. $e^{at} - e^{bt} \quad (a \neq b)$ **22.** $\cosh 2t - \cosh t$

23–34 INVERSE LAPLACE TRANSFORMS

Find the inverse transform (showing the details of your work and indicating the method or formula used):

23. $\dfrac{10s}{s^2 + 2}$ **24.** $\dfrac{15}{s^2 - 4}$

25. $\dfrac{12}{s^2 + 4s + 20}$ **26.** $\dfrac{3s}{s^2 - 2s + 2}$

27. $\dfrac{5s + 4}{s^2} e^{-2s}$ **28.** $\dfrac{2s - 10}{s^3} e^{-5s}$

29. $\dfrac{2s + 4}{(s^2 + 4s + 5)^2}$ **30.** $\dfrac{s^2 - 16}{(s^2 + 16)^2}$

31. $\left(\dfrac{2}{s^2} + \dfrac{2}{s^3}\right) e^{-s}$ **32.** $\dfrac{180 + 18s^2 + 3s^4}{s^7}$

33. $\dfrac{\pi}{s^2(s^2 + \omega^2)}$ **34.** $\dfrac{2}{2s^2 + 2s + 1}$

HW

Prob. 6.7–15
Prob. 6.7–20
Prob. 6.7–21