Ocean Environment Information System

Course: 414.311A

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Chapter 0. Introduction:

Fundamentals of Fluid Flows

1. Concept of Fluid

► What is Fluid ?

A fluid may be defined as a substance that deforms when subjected to a shear stress, no matter how small that stress may be.

► Two Types of Fluid: Liquid and Gas

- Concerning the effect of cohesive forces

(1) Liquid
  - Close-packed molecules with strong cohesive force
  - Tends to retain its volume

(2) Gas
  - Molecules are widely spaced with negligible cohesive forces.
  - Free to expand until it is retained by wall

- Liquids form free surface.

► Fluids are considered as *continuum material* in classical engineering.

2. Physical Parameters and Units

- SI Unit: International System of Units
- BG Unit: British Gravitational Units

Parameter	Dimension	SI Unit	BG Unit	Conversion
Mass	M	Kilogram (kg)	Slug Pound	1 slug = 14.5939 kg 1 pound = 0.454 kg
Length	L	Meter (m)	Foot (ft) (=12 in)	1 ft = 0.3048 m
Time	T	Second (s)	Second (s)	1 s = 1 s
Temperature	Θ	Kelvin (K)	Rankine (^oR)	1 K = 1.8 ^oR
Density	ML^{-3}	Kg / m^3	Slug / ft^3	1 Slug/ft^3 =515.4 kg/m^3
Area	M^2	m^2	ft^2	1 m^2 =10.764 ft^2
Volume	M^3	m^3	ft^3	1 m^3 =35.315 ft^3
Velocity	MT^{-1}	m/s	ft/s	1 ft/s = 0.3048 m/s
Angular Velocity	T^{-1}	1/s	1/s	1 /s = 1 /s
Acceleration	MT^{-2}	m/s^2	t/s^2	1 ft/s^2 = 0.3048 m/s^2
Pressure / Stress	ML^{-1} T^{-2}	N/m^2 (Pa)	lbf/ft^2 lbf/in^2 (psi)	1 lbf/ft^2 = 47.88 Pa 1 psi = 6895 Pa
Force (Weight)	MLT^{-2}	kg-m/s^2 (N)	lbf=slug-ft/s^2	1 lbf = 4.4482 N
Energy / Heat / Work	ML^2L^{-2}	J=N-m	ft-lbf	1 ft-lbf = 1.3558 J
Power	ML^2L^{-3}	W = J/s	Ft-lbf/s	1 ft-lbf/s = 1.3558 W
Viscosity	$ML^{-1}T^{-1}$	kg/(m-s)	slug/(ft-s)	1 slug/(ft-s)= 47.88 kg/(m-s)

Misc.

1 knots = 0.515 m/s

1 mile = 1.609344 km

1 pound = 12 ounces

1 gallon (US) = 0.00379 m^3

1 nautical mile = 1.852 km

^oC = (5/9)*(^oF -32) (C: Celsius, F:Fehrenheit)

3. Methods of Describing Fluid Motion

Two ways to describe the fluid motion

(i) **Lagrangian** description: follows all fluid particles and describes the variation around each fluid particles along its trajectory
(ii) **Eulerian** description: the variations are described at all fixed stations as a function of time

3.1 Lagrangian description

Let's specify a fluid particle, say *k*-th particle. A certain physical property $q(\vec{x}_k, t)$ (e.g. position, velocity, density,…) can be written as

$$q(\vec{x}_k, t),$$

where $\vec{x}_k$ is the (*x*,*y*,*z*) coordinate of the particle. $\vec{x}_k$ should be traced all through the fluid motion, i.e.

$$\vec{x}_k = \vec{x}_k(\vec{x}_{o,k}, t)$$

where the subscript *o* indicates the quantity at $t=0$.

For instance, if $q(\vec{x}_k, t)$ is the coordinate of the particle, the velocity at time *t* becomes

$$\vec{v}_k = \left.\frac{d\vec{x}}{dt}\right|_{o,k}.$$

3.2 Eulerian description

Let's specify a fluid volume. A certain physical property in the fluid volume is written as

$$q(\vec{x}, t),$$

i.e. defined in the fixed coordinate system, (x,y,z), and time. We can define a variation with respect to space or time, s.t.

$$\left.\frac{\partial q}{\partial t}\right|_{fixed\ \vec{x}} \equiv \frac{\partial q}{\partial t}, \quad \left.\frac{\partial q}{\partial \vec{x}}\right|_{fixed\ \vec{x}} \equiv \frac{\partial q}{\partial \vec{x}}$$

3.3 Conversion of Variations between Lagrangian and Eulerian

Let's defined a certain function (or quantity) $F(\vec{x},t)$ in Eulerian frame. After a short time Δt, $F(\vec{x},t)$ becomes $F(\vec{x}+\vec{v}\Delta t, t+\Delta t)$. Then

$$\begin{aligned}\Delta F &= F(\vec{x}+\vec{v}\Delta t, t+\Delta t) - F(\vec{x},t) \\ &= F(\vec{x},t) + \left(\vec{v}\cdot\nabla F + \frac{\partial F}{\partial t}\right)\Delta t - F(\vec{x},t) + O(\Delta t^2)\end{aligned}$$

Therefore,

$$\underbrace{\lim_{\Delta t\to\infty}\frac{\Delta F}{\Delta t}}_{\text{Lagrangian}} = \underbrace{\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F}_{\text{Eulerian}}$$

Lagrangian Eulerian

Total Derivative: Conversion between two frames

$$\frac{D}{Dt}(\ ) = \frac{\partial}{\partial t}(\ ) + \vec{v}\cdot\nabla(\ )$$

Steady Flow: no difference in time => $\frac{D}{Dt}(\ ) = \vec{v}\cdot\nabla(\ )$

3.4 Velocity Field

Velocity: $\vec{v} = \left(u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)\right)$

Acceleration: $\frac{D}{Dt}\vec{v} = \frac{\partial}{\partial t}\vec{v} + \underline{\vec{v} \cdot \nabla \vec{v}}$

Convection

3.5 Continuous Flow Field

For a fluid flow to be continuous, we require that the velocity $\vec{v}$ is a finite and continuous function of space $\vec{x}$ and time *t*. That is, $\frac{\partial}{\partial t}\vec{v}$ and $\nabla \cdot \vec{v}$ are finite but not necessary continuous. If $\frac{\partial}{\partial t}\vec{v}$ and $\nabla \cdot \vec{v}$ are not finite, it is non-physical as long as any singularity does not exist.

(i) Material volume remains material. No segment of fluid can be joined or broken apart.

(ii) Material surface remains material. The interface between two material volumes always exists.

(iii) Material line remains material. The interface of two material surfaces always exists.

3.6 Flow Lines

- Streamline: A line everywhere tangent to the fluid velocity $\vec{v}$ at a given time. In an Eulerian description, it would be a `snapshot' of the flow.
- Pathline: The trajectory of a given particle P in time. The photograph analogy would be a long time exposure of a given particle.
- Streakline: Instantaneous locus of all particles that pass a given point. In an Eulerian description, it would be a `snapshot's of certain particles.
- Timeline: a set of adjacent fluid particles that were marked at the same (earlier) instant in time

Chapter 1. Basic Equations of Fluid Flows

► Einstein's Notation: Repeated indices are summed by implication over all values of the index ***i.***

$$\begin{aligned}\vec{v} &= u\hat{i} + v\hat{j} + w\hat{k} \\ &= u_1\hat{x_1} + u_2\hat{x_2} + u_3\hat{x_3} \\ &= \sum_i^3 u_i\vec{x_i} \\ &= u_i\vec{x_i}\end{aligned}$$

In this example, the summation is over *i =1, 2, 3.*

1. Kinematics of Fluid Motion in the Euler Frame

$$\frac{\partial q_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)}_{= E_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right)}_{= \Omega_{ij}}$$

Rate-of-Strain Tensor   Vorticity Tensor

=> $$\delta q_i = \delta x_j E_{ij} + \delta x_j \Omega_{ij}$$

1.1 Rate-of-Strain Tensor

In matrix form,

$$
E_{ij} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\dfrac{\partial q_1}{\partial x_1} & \dfrac{\partial q_1}{\partial x_2} + \dfrac{\partial q_2}{\partial x_1} & \dfrac{\partial q_1}{\partial x_3} + \dfrac{\partial q_3}{\partial x_1} \\ \dfrac{\partial q_2}{\partial x_1} + \dfrac{\partial q_1}{\partial x_2} & 2\dfrac{\partial q_2}{\partial x_2} & \dfrac{\partial q_2}{\partial x_3} + \dfrac{\partial q_3}{\partial x_2} \\ \dfrac{\partial q_3}{\partial x_1} + \dfrac{\partial q_1}{\partial x_3} & \dfrac{\partial q_3}{\partial x_2} + \dfrac{\partial q_2}{\partial x_3} & 2\dfrac{\partial q_3}{\partial x_3} \end{bmatrix}
$$

Diagonal terms

These indicate the rate of stretch per unit length in the direction of *(x,y,z)*

In particular, when q is the velocity of fluid flow,

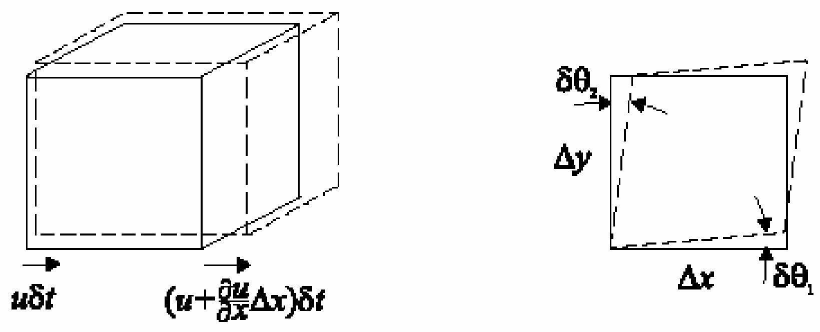
$$
\begin{aligned} e_{ii} &= e_{11} + e_{22} + e_{33} \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \end{aligned} \qquad : \quad \text{rate of volume dilatation}
$$

Proof: Consider a small volume $V(t) = \Delta x \Delta y \Delta z$. At $t + \delta t$, the expansion volume becomes

$$
\begin{aligned} V(t+\delta t) &= \Delta x\left(1 + \frac{\partial u}{\partial x}\delta t\right)\Delta y\left(1 + \frac{\partial v}{\partial y}\delta t\right)\Delta z\left(1 + \frac{\partial w}{\partial z}\delta t\right) \\ &= \Delta x\ \Delta y\ \Delta z\left[1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\delta t + O(\delta t^2)\right] \\ &= V(t)\left[1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\delta t + O(\delta t^2)\right] \end{aligned}
$$

Then, the rate of volume change becomes

$$
\frac{1}{V(t)}\frac{dV}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{u}
$$



(a) Diagonal component (b) Off-diagonal component

Off-diagonal terms

These indicate the rate of angular deformation. As above figure shows,

$\delta\theta_1 = \frac{\delta(\Delta y)}{\Delta x} = \frac{\Delta v \delta t}{\Delta x} = \frac{\partial v}{\partial x}\delta t$ and $-\delta\theta_2 = \frac{\delta(\Delta x)}{\Delta y} = \frac{\Delta u \delta t}{\Delta y} = \frac{\partial u}{\partial y}\delta t$ (note the direction of angle).

Then,

$$\frac{\delta\theta_1}{\delta t} - \frac{\delta\theta_2}{\delta t} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

This is called the rate of shear strain.

Note that the rate-of-strain tensor is symmetry, i.e.

$$e_{ij} = e_{ji}$$

1.2 Vorticity Tensor

In matrix form,

$$\Omega_{ij} = \begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} \\ \varpi_{21} & \varpi_{22} & \varpi_{23} \\ \varpi_{31} & \varpi_{32} & \varpi_{33} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial q_1}{\partial x_2} - \frac{\partial q_2}{\partial x_1} & \frac{\partial q_1}{\partial x_3} - \frac{\partial q_3}{\partial x_1} \\ \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} & 0 & \frac{\partial q_2}{\partial x_3} - \frac{\partial q_3}{\partial x_2} \\ \frac{\partial q_3}{\partial x_1} - \frac{\partial q_1}{\partial x_3} & \frac{\partial q_3}{\partial x_2} - \frac{\partial q_2}{\partial x_3} & 0 \end{bmatrix}$$

Note that the vorticity tensor is anti-symmetry, i.e.

$$\varpi_{ij} = -\varpi_{ji}$$

There are only component, and sometimes these are written as

$$\nabla \times \vec{q} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ q_1 & q_2 & q_3 \end{vmatrix}$$

when q is the velocity, this tensor indicates the rate of rotation of velocity.

(Line integral of this tensor becomes circulation component.)

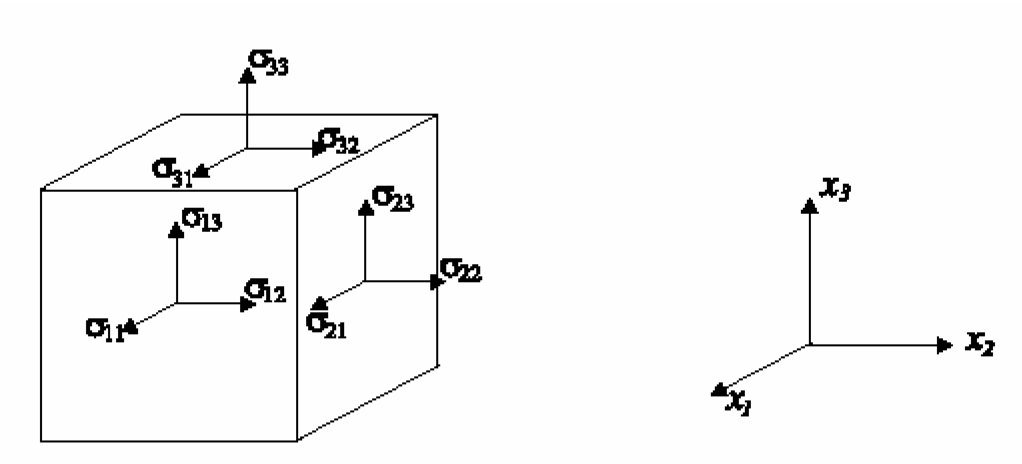
2. Surface Forces and Stresses

Let's consider a fluid volume (as shown in figure). Then we can define a stress tensor of surface force and stress,

$$\{\sigma_{ij}\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

where the fist subscript indicates the index of surface and the second means the direction of acting force/stress. The diagonal terms are the normal stress components, while the off-diagonal terms are the shear stress components. Based on the conservation of angular momentum, we can find that

$$\sigma_{ij} = \sigma_{ji}$$



3. Stress Tensor and Rate-of-Strain Tensor

When we assume a small volume of fluid (not in macroscopic scale), the stress that the volume experiences is written as

$$\sigma_{ij} = -p\delta_{ji} + \tau_{ij}$$

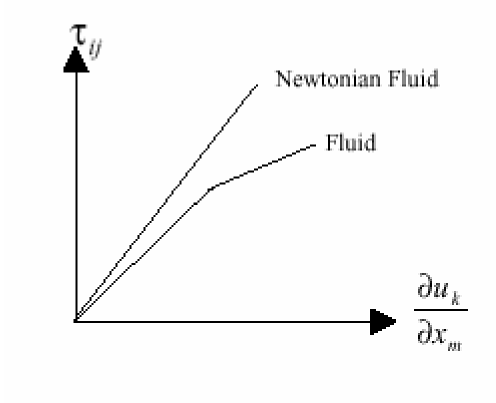
where p is the normal pressure and τ_{ij} is the viscous stress which depends on gradients of velocity.

3.1 Newtonian Fluid

Newtonian fluid is the fluid which satisfies with

$$\tau_{ij} \propto \frac{\partial u_l}{\partial x_m}$$
$$= \alpha_{ijlm} \frac{\partial u_l}{\partial x_m}$$

where α_{ijlm} is a coefficient tensor. In principle, there are 3^4=81 α_{ijlm} coefficients.



For a isotropic fluid (no change in direction), this reduces to

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda \frac{\partial u_m}{\partial x_m}\delta_{ij}$$

where

μ : dynamic viscosity

λ : bulk elasticity, 'second' coefficient of viscosity

In particular case of *impressible Newtonian fluid*,

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

3.2 Non-Newtonian Fluid

Non-Newtonian fluids are the fluids that viscous stress shows nonlinear behavior w.r.t.

$$\frac{\partial u_l}{\partial x_m}.$$

Many fluids, e.g. toothphase, honey, heavy oil, flows like a fluid if the shear stress is above a critical value. In this case, we can use a popular non-Newtonian fluid modeling, the Bingham plastic model. This model is written as

$$\frac{\partial u}{\partial y} = \begin{cases} 0 & \tau \le \tau_c \\ \frac{1}{\mu}(\tau - \tau_c) & \tau > \tau_c \end{cases}$$

where τ_c is yield stress and μ is the Bingham viscosity.

4. Kinematic Transport Theorem

Theorem 1

Let $G(\vec{x},t)$ be the a certain fluid property per unit volume, then

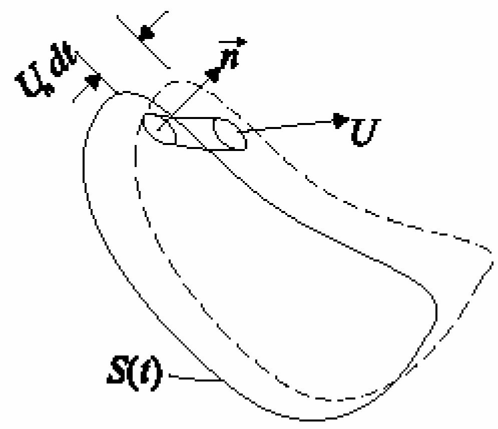
$$\frac{d}{dt}\iiint_V G\,dV = \iiint_V \frac{\partial}{\partial t} G\,dV + \iint_S G\,U_n dS$$

where U_n is the normal component of the velocity of a point on surface S.

Theorem 2

If *V(t)* is a material volume containing the same moving fluid particles, then

$$\frac{D}{Dt}\iiint_V G\,dV = \iiint_V \frac{\partial}{\partial t} G\,dV + \iint_S G\,\vec{u}\cdot\vec{n}dS$$



5. Mass Conservation: Continuity Equation

Let $G(\vec{x},t)$ be the fluid density, ρ. Then, as long as we stay in a material volume in which there is no mass source or sink, we know that

$$\frac{D}{Dt}\iiint_V \rho\,dV = 0$$

by mass conservation. Using the Theorem 2, we can get

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \iint_S \rho\,\vec{u}\cdot\vec{n}dS = \iiint_V \frac{\partial \rho}{\partial t} dV + \iiint_V \nabla\cdot(\rho\vec{u})dV = 0 \qquad \text{(by divergence theorem)}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$ => Differential form of mass conservation

Alternative form:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = \frac{D\rho}{Dt} + u \cdot \nabla \rho = 0$$

In the case of an incompressible fluid, we can get the continuity equation:

$$\nabla \cdot \vec{u} = 0$$

6. Momentum Conservation: Euler & Navier-Stokes Equations

By Newton's second law, the force acting on a certain fluid volume should be in an equilibrium condition. This can be expressed as

$$\frac{d}{dt}(\text{momentum of fluid}) = \iint_S (\text{surface force}) dS + \iiint_V (\text{body force on fluid}) dV$$

(1) $\frac{d}{dt}(\text{momentum of fluid})$

$$\begin{aligned}\frac{d}{dt}&(\text{momentum of fluid in } i\text{-th direction})\\ &= \frac{D}{Dt}\iiint_V \rho u_i \, dV = \iiint_V \frac{\partial}{\partial t}(\rho u_i) \, dV + \iint_S (\rho u_i) \vec{u} \cdot \vec{n} \, dS \\ &= \iiint_V \left[\frac{\partial}{\partial t}(\rho u_i) + \nabla \cdot (\rho u_i \vec{u}) \right] dV \\ &= \iiint_V \left[\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i \nabla \cdot (\rho \vec{u}) + \rho \vec{u} \cdot \nabla \cdot (u_i) \right] dV\end{aligned}$$

From mass conservation, the second and third term inside above integral becomes zero. Hence,

$$\iiint_V \left[\rho \frac{\partial u_i}{\partial t} + \rho \vec{u} \cdot \nabla \cdot (u_i) \right] dV = \iiint_V \rho \frac{Du_i}{Dt} dV$$

(2) $\iint_S (\text{surface force}) dS$

$$\iint_S (\text{surface force in } i\text{ - th direction}) dS = \iint_S \sigma_{ij} n_j dS$$

$$= \iiint_V \frac{\partial \sigma_{ij}}{\partial x_j} dV \qquad \text{(by divergence theorem)}$$

(3) $\iiint_V (\text{body force on fluid}) dV$

$$\iiint_V (\text{body force on fluid in } i\text{ - th direction}) dV = \iiint_V \rho f_i dV$$

where f_i is defined as a body force component.

From (1), (2), (3), we can get

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i \qquad i=1,2,3$$

For incompressible Newtonian fluid, we can have the Navier-Stokes equation such that

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\rho} f_i \quad \text{Tensor form}$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} + \frac{1}{\rho}\vec{f} \quad \text{Vector form}$$

where

$$\nu = \frac{\mu}{\rho} \quad : \text{kinematic viscosity } [L^2/T]$$

When there is no viscosity, we will get the Euler Equation,

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho} f_i \quad \text{Tensor form}$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\vec{f} \quad \text{Vector form}$$

- Unknowns: u, v, w, p => 4 unknowns

- Equations: Continuity + Navier-Stokes equation 3 (x,y,z) => 4 equations

- Knowns: material parameter, body force

7. Boundary Conditions

7.1 Kinematic Boundary Condition

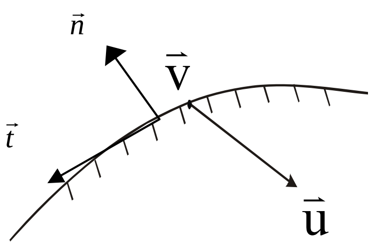
The kinematic boundary condition specifies the position, velocity, or their behaviors on fluid boundary.

- On rigid body

No-flux condition

$$\vec{u} \cdot \vec{n} = \vec{V} \cdot \vec{n}$$

where $\vec{V}$ is the velocity of moving boundary



No-slip condition (viscous flow)

$$\vec{u} \cdot \vec{t} = \vec{V} \cdot \vec{t}$$

where $\vec{t}$ is the tangential vector on fluid boundary

- On free surface

The water particles on free surface stay on free surface.

$$\frac{D\vec{x}}{Dt} = \frac{D\vec{x}_{f.s.}}{Dt}$$

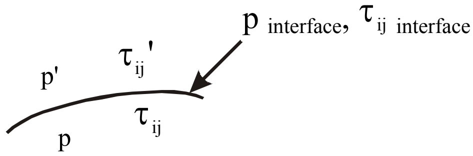
where $\vec{x}_{f.s.}$ indicates the position of free surface.

7.2 Dynamic Boundary Condition

The dynamic boundary condition specifies the pressure, stress, or their behaviors on fluid boundary.

$$p = p' + P_{interface}$$

$$\tau_{ij} = \tau_{ij} + \tau_{ij,interface}$$

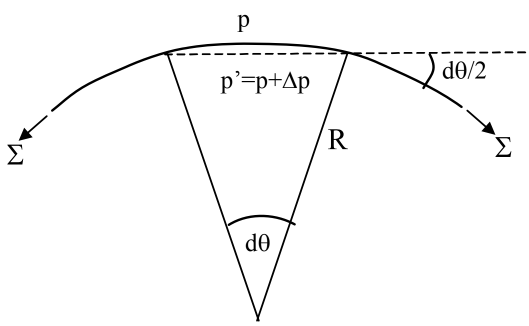


For instance, on a free surface boundary in the absence of surface tension, the dynamic boundary conditions become $p = P_{air}$ and $\tau_{ij} = 0$ where P_{air} is the pressure above free surface.

When surface tension is not ignorable, we have to consider the stress across boundary. Define Σ is a tension force on surface. Then, for the 2-D case as shown in figure, the force equilibrium says

$$\underbrace{\cos\frac{d\theta}{2}}_{\approx 1} \cdot \Delta p \cdot R d\theta = 2\Sigma \underbrace{\sin\frac{d\theta}{2}}_{\approx \frac{d\theta}{2}} \approx 2\Sigma\frac{d\theta}{2}$$

$$\therefore \Delta p = \frac{\Sigma}{R}$$



In the case of 3-D case,

$$\Delta p = \Sigma\left(\frac{1}{R_x} + \frac{1}{R_y}\right).$$

If the boundary profile is $z = \eta(x, y)$,

$$\frac{1}{R_x} + \frac{1}{R_y} = \frac{\dfrac{\partial^2 \eta}{\partial x^2} + \dfrac{\partial^2 \eta}{\partial y^2}}{\left[1 + \left(\dfrac{\partial \eta}{\partial x}\right)^2 + \left(\dfrac{\partial \eta}{\partial y}\right)^2\right]}$$

Chapter 2. Similarity

2.1 Why Similarity is important?

•To carry out model test
•To understand the physical parameters involved
•To check the sensitivity to each parameters

2.2 Three similarities

(i) Geometric similarity
-Shape
(ii) Kinematic similarity
-Velocity, flow pattern
(iii) Dynamic similarity
− Force, Pressure
↓
In experiment, we assume that if
(i) & (ii) are satisfied, (iii) is satisfied
⇒This is the fundamental assumption for model test.

2.3 Non-dimensional parameters

(i) $\frac{\vec{u}}{\sqrt{gL}} = Fr$: Froude number ~ $\frac{F_{inertia}}{F_{gravity}}$

(ii) $\frac{\vec{u}L}{\nu} = \text{Re}$: Reynolds number ~ $\frac{F_{inertia}}{F_{viscous}}$

(iii) $\frac{P}{\frac{1}{2}\rho u^2} = Eu$: Euler number ~ $\frac{F_{pressure}}{F_{inertia}}$

(iv) $\frac{L}{\vec{u}T} = S$: Strouhal number ~ $\dfrac{Eulerian \quad inertia \quad \frac{\partial \vec{u}}{\partial t}}{convection \quad inertia \quad (\vec{v} \bullet \nabla)\vec{v}}$

↓

4 key parameters

Variation

- $\dfrac{P - P_V}{\frac{1}{2}\rho u^2} = \sigma$: cavitation number (where Pv : vapor pressure)

- $\dfrac{uT}{L} = Kc$: Keulegan-Carpenter Number

•When surface tension is involved,

$\dfrac{u^3 L}{\Sigma / \rho} = We$: Weber number ~ $\dfrac{inertia \quad force}{Surface \quad tension \quad force}$

2.4 Buckingham's π theorem

- Total number of parameters involved in the physical problem : m
- Total number of independent parameters : n

⇒ Total number of non-dimensional parameters

= m-n

2.5 For continuity eq. & Navier-Stokes eq.

•Parameters involved

•Length : L

•Time : T

•Velocity : M/L

•Pressure : M/LT^2

•Density : M/L^3

•Viscosity : M/LT

•Body force ⇒ gravity : L/T^2

…… m = 7

- Independent Parameters : L, T, M : n = 3

Number of non-dimensional parameters = 4

Chapter 3. Ideal Fluid flow

3.1 Ideal fluid

① inviscid ($\nu = 0$)

② incompressible ($\dfrac{\partial \rho}{\partial t} = 0$)

This is a good approximation when viscous effect << inertia effect

3.2 Governing equations

• Continuity equation

$$\nabla \bullet \vec{u} = 0$$

• momentum equation : Euler equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \bullet \nabla \vec{u} = -\frac{1}{\rho}\nabla p - \vec{f}$$ [1] (where f = body force)

• Boundary Condition

(i) Kinematic boundary condition

$\vec{u} \bullet \vec{n} = \vec{V} \bullet \vec{n}$: No-flux condition

Where $\vec{V}$: goven on boundary

(ii) Dynamic boundary condition

p = specified

※ shear stress $\tau = 0$ since $\nu = 0$

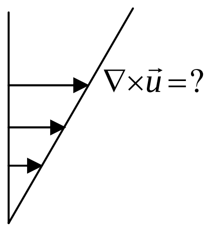
3.3 Irrotational flow

• Vorticity : $\nabla \times \vec{u} = \vec{\omega}$

• Irrotational flow

$$\nabla \times \vec{u} = 0$$

(frictionless flow)



$$\bar{\omega} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

3.4 Velocity Potential

[1] First order P.D.E. (and N-S eq. is 2^{nd} order P.D.E.)

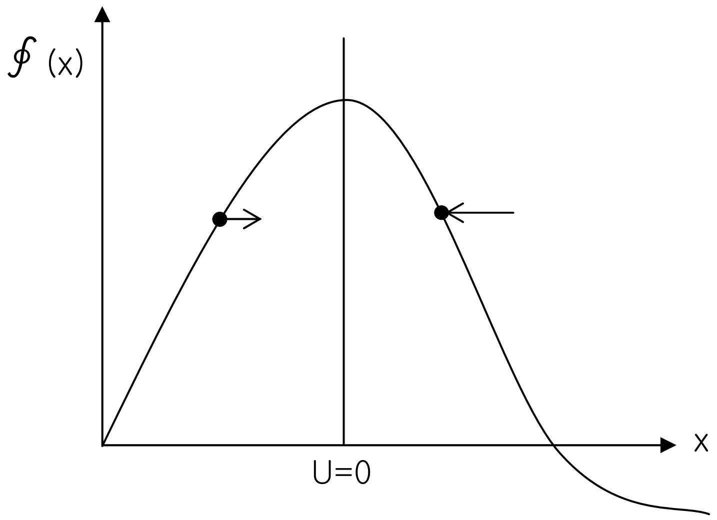
If (i) ideal fluid, in $\nu = 0, \dfrac{\partial \rho}{\partial t} = const.$

(ii) irrotational flow,

We can define the velocity potential Φ

$$\vec{u} = \nabla \Phi$$

- Φ is a scalar quantity
- The velocity vector $\vec{v}$ always points towards higher value of the velocity potential



- Continuity equation

$$\nabla \bullet \vec{u} = \nabla \bullet (\nabla \Phi) = \nabla^2 \Phi = 0 \quad \Rightarrow \text{Laplace Equation}$$

3.5 Laplace equation

$\nabla^2 \Phi = 0$

Indicates the conversion of

(1) mass,

(2) momentum,

(3) energy

Unknown : Φ

Condition : $\nabla^2\Phi = 0$
$\Rightarrow$We can solve the problem

Pressure, p, is not involved in Laplace equation

3.6 Bernoulli's equation

• Euler equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \bullet \nabla \vec{u} = -\frac{1}{\rho}\nabla p - g\vec{k}$$

Substituting $\vec{u} = \nabla\Phi$, we can get

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 + \frac{p}{\rho} + gz = f(t) = const.$$ $\Rightarrow$ Bernoulli's equation

• Steady flow

$$p = -\rho(\frac{1}{2}\vec{v}^2 + g\eta)$$
$$= -\rho(\frac{1}{2}|\nabla\Phi|^2 + g\eta)$$

• Hydrostatics $(\vec{u} = 0, \frac{\partial}{\partial t} = 0)$

3.7 stream function

$$\vec{u} = \nabla \times \Psi$$

3.8 Simple Potential Flows

(1) uniform stream
$\Phi = \vec{u} \bullet \vec{x} + const$
(2) source
A. 2D : $\Phi \sim \ln r \Rightarrow \Phi = \frac{m}{2\pi}\ln r$
B. 3D : $\Phi \sim \frac{4}{r} \Rightarrow \Phi = -\frac{m}{4\pi r}$

(3) Vortex

$$\Phi = -\frac{\Gamma}{2\pi}\theta$$

(4) Dipole(doublet)

A. 2D : $\Phi = \frac{m}{2\pi}\frac{\cos\theta}{r}$

B. 3D : $\Phi = \frac{m}{4\pi}\frac{\cos\theta}{r^2}$

3.9 Superposition

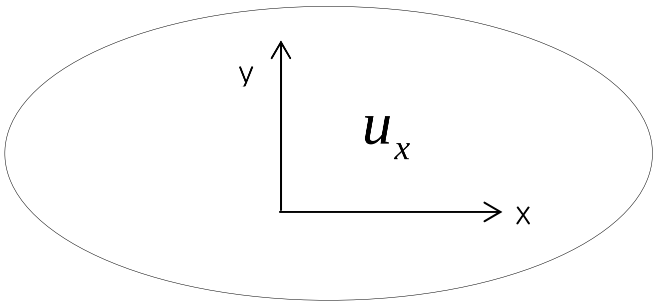
$$\Phi = \sum_i \Phi_i \quad \text{if} \quad \nabla^2\Phi_i = 0$$

3.10 Added Mass

•Artificial mass

•Total Momentum due to the body in motion

$$L_x = mU_x + \iiint_{V} \rho u_x \, dV$$



$$L_x \equiv (m + m_{ax})U_x$$

m_{ax} : added mass

$m + m_{ax}$: virtual (or total) mass

$$m_a = \iiint_{\mathcal{V}} \rho \frac{u_x}{U_x} d\mathcal{V} \qquad \text{where} \quad u_x = \frac{\partial}{\partial x}$$

$$= \iint_S \rho \frac{1}{U} \frac{\partial \phi}{\partial x} n_x dS$$

$$= \iint_S \rho \frac{\partial \Phi}{\partial x} n_x dS$$

Where Φ is velocity potential due to "UNIT" velocity

•In a general form

$$m_{aij} = \iint_S \rho \frac{\partial \Phi_i}{\partial x} n_j dS$$

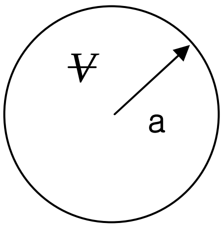
Where Φ_i : velocity potential due to "UNIT" velocity in i-direction.

n_j : normal vector component of j-direction

m_{aij} : added mass for j-direction due to the body motion to i-th direction

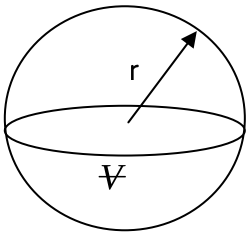
•Some examples

2D cylinder,
radius : a
Volume : $\mathcal{V}$



$\Rightarrow m_a = \rho \mathcal{V}$

3D sphere
Radius : r
Volume : $\mathcal{V}$



$\Rightarrow m_a = \frac{1}{2} \rho \mathcal{V}$

3.11 Other important concepts in Fluid Dynamics

(viscous Flow)

•Laminar Flow
•Turbulent Flow (and From Laminar flow/Turbulent flow, Boundary Layer)
•Boundary Layer
•Separation
•Instability, Transition

3.12 Summary

Viscous Fluid	Ideal Fluid	
		Potential Flow
Navier-Stokes Equation	Euler's Equation	$\nabla^2\Phi = 0$
Unknown : u, v, ω, ρ Number of equations : 4	Unknown : u, v, ω, ρ Number of equations : 4	Φ Number of equations : 1
Boundary Condition No-slip / No-flux	No-flux	No-flux

Chapter 4. Linear Waves: Introduction

4.1 Primary Mechanisms involved

•Source of wave generation

-Wind : primary source

-Earthquake : especially for tsunami

-Moving bodies : ship waves

-Meteorite

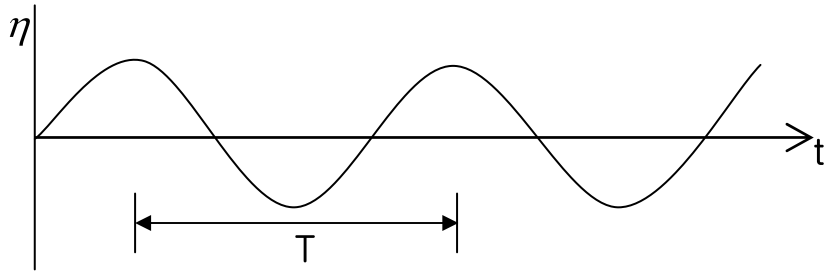
•Source of restoring to create oscillatory motion : Gravity

⇒(Ocean) Waves are gravity waves (mostly)

Exception : capillary wave

4.2 Two characteristics of waves

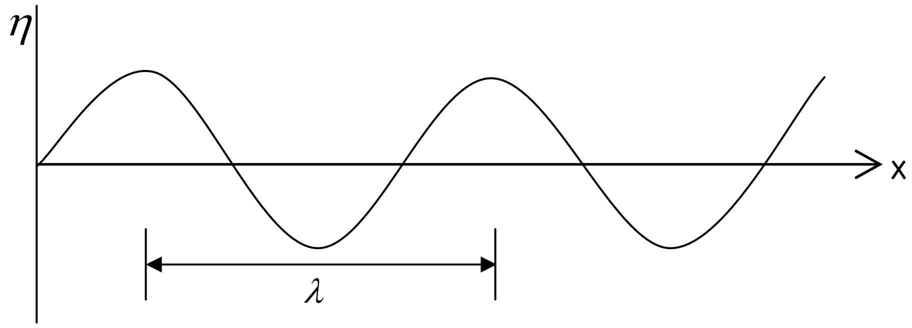
(1) Time characteristics



$\omega = \dfrac{2\pi}{T}$:wave frequency

⇒Change to frequency-domain quantity

(2) Space characteristics



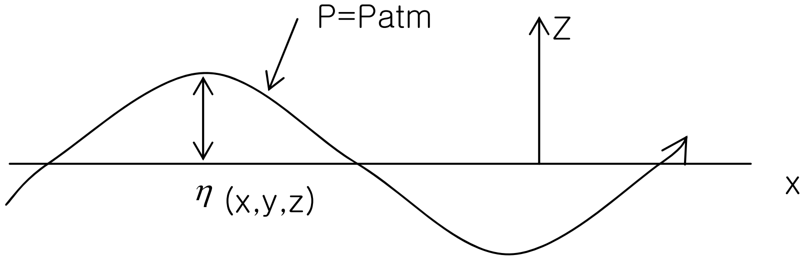
• $k = \dfrac{2\pi}{\lambda}$: wave number

•wave slope : $\dfrac{A}{\lambda}$ or kA

•depth effect : $\dfrac{\lambda}{h}, \dfrac{A}{h}$

(1)↔ (2) : Dispersion relation

4.3 Free Surface Boundary Condition



(1) Kinematic FSBC

If a geometric surface is written to

F(x,y,z,t) = 0, (e.g. $x^2 + y^2 + z^2 - a^2 = 0$)

On the moving surface

$$\frac{dF}{dt} = 0 \quad \text{all the time}$$

$$\Rightarrow \frac{\partial F}{\partial t} + \vec{u} \bullet \nabla \vec{F} = 0$$ (where $\vec{u}$ is moving speed)

On free surface

$$F = z - \eta = 0$$

$$\Rightarrow \begin{aligned} \frac{DF}{Dt} &= \frac{d}{dt}(z - \eta) = 0 \\ &= \frac{\partial}{\partial t}(z - \eta) + \vec{u} \bullet \nabla (z - \eta) = 0 \end{aligned}$$

$$\frac{\partial z}{\partial t}=0, \frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0, \frac{\partial \eta}{\partial z}=0(\because \eta = f(x,y,t)), \frac{\partial z}{\partial z}=1$$

Thus, $\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y}-\frac{\partial \phi}{\partial z}=0$

(2) Dynamic FSBC

Bernoulli's Equation

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\nabla\phi\bullet\nabla\phi+\frac{P_{atm}}{\rho}+g\eta=C(t)$$

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\nabla\phi\bullet\nabla\phi+g\eta=C(t)-\frac{P_{atm}}{\rho}$$

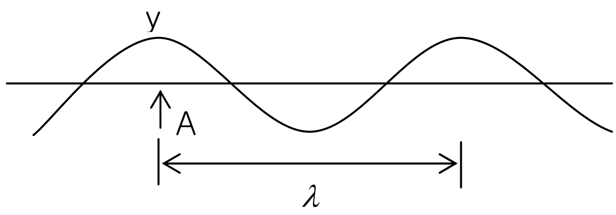
When $\phi=0\Rightarrow\eta=0, then$

$$C(t)-\frac{P_{atm}}{\rho}=0$$

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\nabla\phi\bullet\nabla\phi+g\eta=0$$

4.4 Small Amplitude Waves

•Assumption : kA << 1 : small slope



$\frac{A}{\lambda}<<1$

⇒Small disturbance

•Linearization

Taylor Series Expansion

$$f(x) \;=\; f(x_0) \;+ (\text{x-x}_0)\frac{\partial}{\partial x} f(x_0) \;+\; \frac{(x-x_0)^2}{2}\frac{\partial^2}{\partial x^2} f(x_0) \;+ \ldots\ldots$$

(1) Kinematic F.S.B.C.

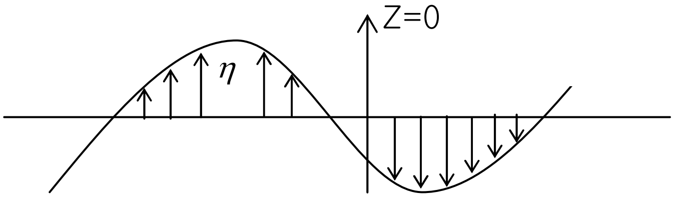
$$\left(\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y}-\frac{\partial \phi}{\partial z}\right)_{z=0}$$

$$+\;\eta \bullet \frac{\partial}{\partial z}\left(\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y}-\frac{\partial \phi}{\partial z}\right)_{z=0}$$

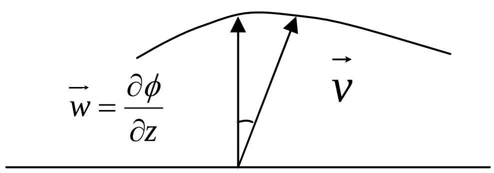
$$+\;\frac{\eta^2}{2} \bullet \frac{\partial^2}{\partial z^2}\left(\frac{\partial \eta}{\partial t}+\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y}-\frac{\partial \phi}{\partial z}\right)_{z=0} \quad = 0$$

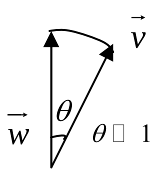
$O(\varepsilon): \dfrac{\partial \eta}{\partial t}-\dfrac{\partial \phi}{\partial z}=0$ on z=0

Physical interpretation



The velocity of wave elevation is equal to the vertical velocity at z=0





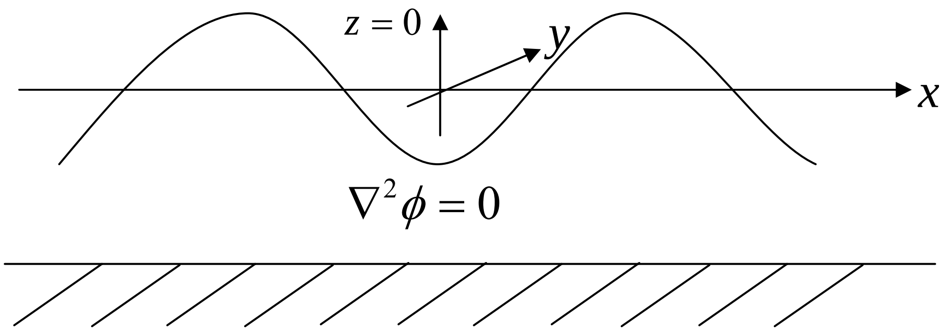
Small slope

(2) Dynamic F.S.B.C.

$$\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\bullet\nabla\phi+g\eta\right)_{z=0}+\eta\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\bullet\nabla\phi+g\eta\right)_{z=0}$$
$$+\eta^2........=0$$

$$O(\varepsilon):\frac{\partial\phi}{\partial t}+g\eta=0 \quad \text{at z=0}$$

•Boundary Value Problem



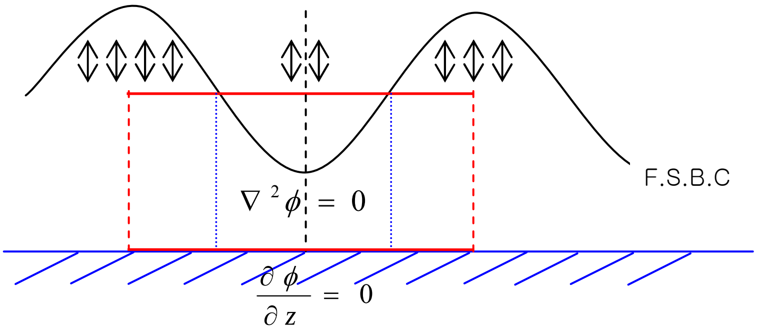
$$\frac{\partial\phi}{\partial t}+g\eta=0 \quad \text{or} \quad \eta=-\frac{1}{g}\frac{\partial\phi}{\partial t}$$

$$\frac{\partial\eta}{\partial t}-\frac{\partial\phi}{\partial z}=0 \quad \text{or} \quad \frac{\partial\eta}{\partial t}=\frac{\partial\phi}{\partial z}$$

Chapter 5. Linear Waves

5.1 2-D standing Wave

•Boundary Value Problem



•Periodicity

-Time : $\eta(x,t)=\eta(x,t+T), \phi(x,t)=\phi(x,t+T)$

-Space : $\eta(x,t)=\eta(x+\lambda,t), \phi(x,t)=\phi(x+\lambda,t)$

•Separation of Variables

$$\phi = X(x)Z(z)T(t)$$

•Periodicity in Time

$$\text{T(t)} = \sin\omega t \quad \text{where} \quad \omega = \frac{2\pi}{T}$$

(1) Laplace Eq.

$$\frac{\partial X^2}{\partial x^2}ZT + \frac{\partial Z^2}{\partial z^2}XT = 0 \qquad \text{or} \qquad \frac{\frac{\partial X^2}{\partial x^2}}{X} + \frac{\frac{\partial Z^2}{\partial z^2}}{Z} = 0$$

Put $\frac{\frac{\partial Z^2}{\partial z^2}}{Z} = k^2,\ \frac{\frac{\partial X^2}{\partial x^2}}{X} = -k^2$ => $\begin{aligned} Z_{zz} - k^2 Z = 0 \\ X_{xx} + k^2 X = 0 \end{aligned}$

Three possible cases for k is
- real
- 0

0 imaginary

(Reference : Dean & Darlymple. pp.55)

※ for ODE, $F_{xx} + c^2 F = 0$

If c : real, F is oscillatory
imaginary : F is exponentially decrease or increase

•Periodicity in space $\Rightarrow$ k is real THEN,

$$\begin{matrix} Z_{zz} - k^2 Z = 0 \\ X_{xx} + k^2 X = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} X = A\cos kx + B\sin kx \\ Z = Ce^{kz} + De^{-kz} \end{matrix}$$

We will consider $X = A\cos kx$ (we will return the other case)

$$\phi = A\cos kx(Ce^{kz} + De^{-kz})\sin \omega t$$

(2) Bottom Boundary Condition

$\frac{\partial \phi}{\partial z} = 0$ on z=-h (바닥)

$k\{Ce^{kz} - De^{-kz}\} = 0$ thus, $Ce^{kz} - De^{-kz} = 0$

Or $C = De^{2kh}$

$$\begin{aligned} \phi &= A\cos kxD(e^{2kh+kz} + e^{-kz})\sin \omega t \\ &= A\cos kxDe^{kh}(e^{k(z+h)} + e^{-k(z+h)})\sin \omega t \\ &= 2AD\cos kxe^{kh}\cosh k(z+h)\sin \omega t \end{aligned}$$

(3) Dynamic F.S.B.C.

$\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t}$ when z=0

$$-\frac{1}{g}\frac{\partial \phi}{\partial t} = -\frac{\omega}{g}\{2AD\cos kx e^{kh}\cosh k(z+h)\}_{z=0}\cos \omega t$$

$$\eta = -\frac{\omega}{g}\{2ADe^{kh}\cosh k(z+h)\}\cos kx \cos \omega t$$

$$-\frac{\omega}{g}\{2ADe^{kh}\cosh k(z+h)\} = \eta_0 \quad \text{wave amplitude}$$

$$-2ADe^{kh} = \frac{\eta_0}{\cosh kh}\frac{g}{\omega}$$

$$\phi = -\frac{g\eta_0}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos kx \sin \omega t$$

Velocity potential of 2-D standing waves.

(4) Kinematic F.S.B.C.

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on z=0}$$

$$\frac{\partial \eta}{\partial t} = -\omega\eta_0 \cos kx \sin \omega t = \frac{\partial \phi}{\partial z} = -\frac{g\eta_0}{\omega}k\frac{\sinh kh}{\cosh kh}\cos kx \sin \omega t$$

$$\omega^2 = gk \tanh kh$$: Dispersion Relation

Summary

$$\phi = -\frac{g\eta_0}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos kx \sin \omega t$$

$$\eta = A\cos kx \cos \omega t$$

$$\omega^2 = gk \tanh kh$$

$$\frac{\partial \phi}{\partial x} = u = \frac{gAk}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin kx \sin \omega t$$
$$\frac{\partial \phi}{\partial x} = w = -\frac{gAk}{\omega}\frac{\sinh k(z+h)}{\cosh kh}\cos kx \sin \omega t$$

$$\frac{P}{\rho} = -\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \bullet \nabla\phi\right)$$

Linear pressure $\frac{P}{\rho} = -\frac{\partial \phi}{\partial t} = -\frac{gA_0}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos kx \cos \omega t$

5.2 2D Progressive Waves

5.2.1 Velocity Potential

•Now consider another standing waves s.t.

$$\phi = -\frac{g\eta_0}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin kx \cos \omega t$$
$$\eta = A \sin kx \sin \omega t$$

•Add two standing waves

$$\phi = \frac{gA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\left(\sin kx \cos \omega t - \cos kx \sin \omega t\right)$$
$$= \frac{gA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin(kx - \omega t)$$

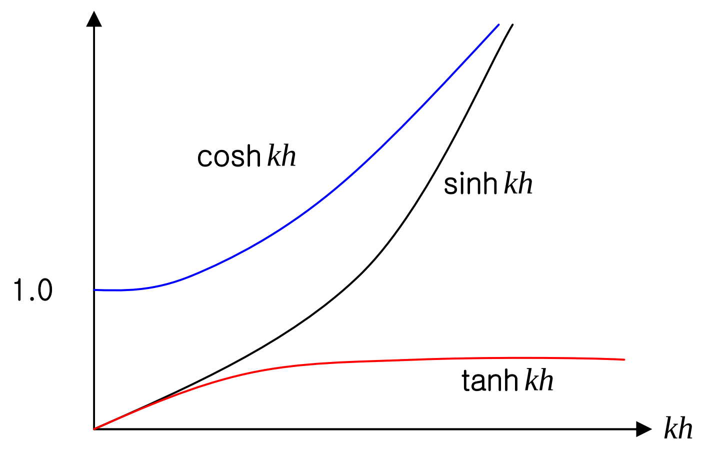
$$\eta = A\{\cos kx \cos \omega t - \sin kx \sin \omega t\}$$
$$= A\cos(kx - \omega t)$$

Becomes a progressive wave

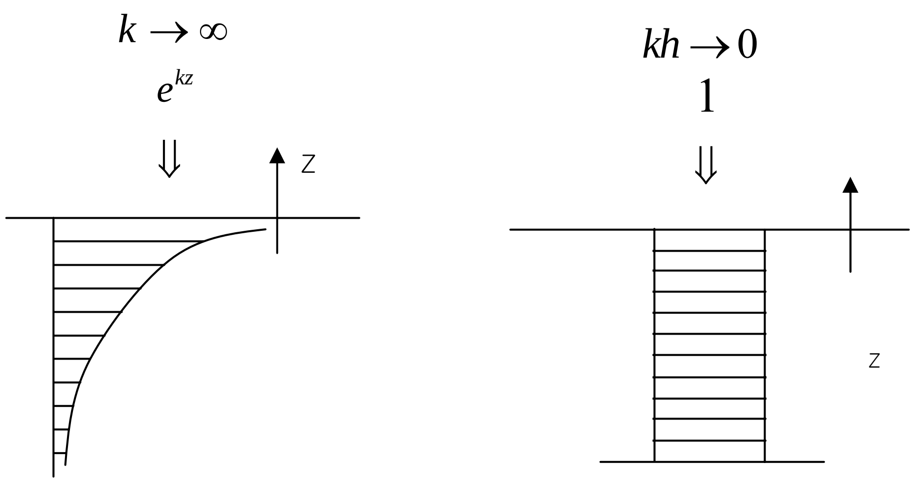
$$\phi = \frac{gA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t)$$
$$\eta = A\cos(kx-\omega t)$$

Put K(z) = $\dfrac{\cosh k(z+h)}{\cosh kh}$

	$k \to \infty$	$k \to 0$
$\cosh kh$	$e^{kh}/2$	1
$\sinh kh$	$e^{kh}/2$	kh
$\tanh kh$	1	kh
	Deep water	shallow water



K(Z)



Exponentially decay constant

Same trend in ϕ since

$$\phi = \frac{gA}{\omega} K(z) \sin(kx - \omega t)$$

5.2.2 velocity component

u = $\dfrac{\partial \phi}{\partial x} = \dfrac{gAk}{\omega} \dfrac{\cosh k(z+h)}{\cosh kh} \cos(kx - wt)$

w= $\dfrac{\partial \phi}{\partial z} = \dfrac{gAk}{\omega} \dfrac{\sinh k(z+h)}{\cosh kh} \sin(kx - wt)$

using dispersion relation $\omega^2 = gk \tanh kh$

u = $A\omega \dfrac{\cosh k(z+h)}{\cosh kh} \cos(kx - wt)$

w= $A\omega \dfrac{\sinh k(z+h)}{\cosh kh} \sin(kx - wt)$

• On z=0

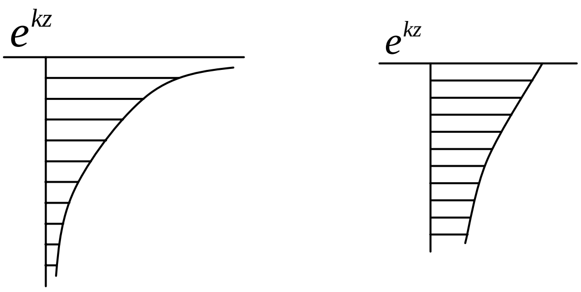
u = $u_0 = A\omega \coth kh \cos(kx - \omega t)$

w = $w_0 = A\omega \sin(kx - \omega t)$ $\qquad \left(= \dfrac{\partial \eta}{\partial t} \right)$

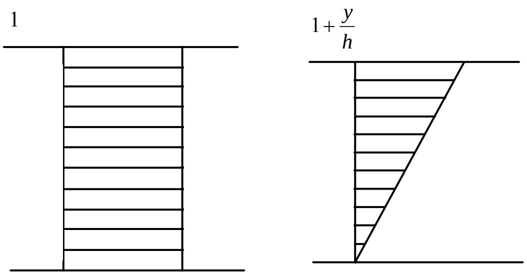
$$\frac{u}{u_0} = \frac{\cosh k(y+h)}{\cosh kh}$$

$$\frac{w}{w_0} = \frac{\sinh k(y+h)}{\cosh kh}$$

$k \to \infty$ 의 경우, exponentially decay



$k \to 0$ 의 경우, constant



5.2.3 Pressure : Bernoulli's equation

$$P = -\rho \frac{\partial \phi}{\partial t} - \rho g z$$

$-\rho \dfrac{\partial \phi}{\partial t}$: dynamic pressure P_d

$$P_d = \rho g A \frac{\cosh k(z+h)}{\cosh kh} \cos(kx - wt)$$

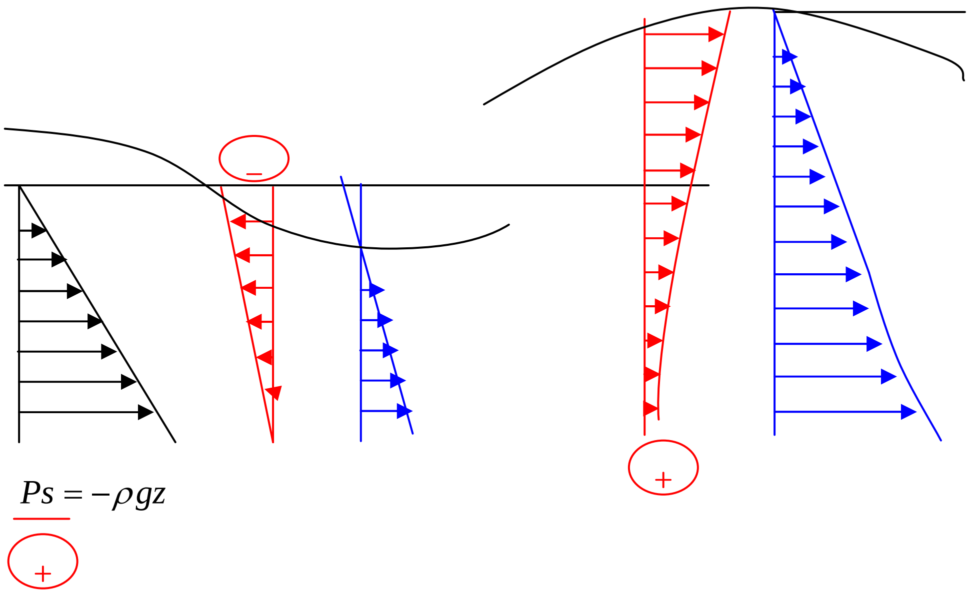
In Deep water… $k \to \infty$

$$P_d = \rho g A e^{kz} \cos(kx - wt) = \rho g e^{kz} \eta$$

$$P_{total} \approx \rho g \left[\eta e^{kz} - z \right]$$

Shallow water… $k \to 0$

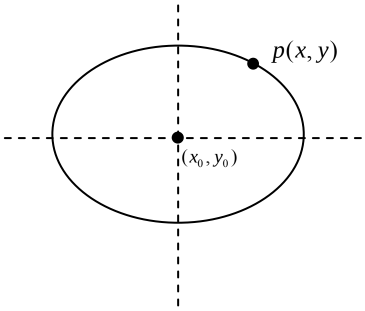
$$P_d = \rho g A \cos(kx - wt) = \rho g \eta$$
$$P_{total} \approx \rho g [\eta - z]$$



5.2.4 Particle Orbit (Lagrangian)

$$\vec{x} = \int_0^t u\,dt$$

e.g. $u = \dfrac{dx}{dt} = u(x_0,t) + (x - x_0)\dfrac{\partial u}{\partial x}(x_0,t) + \dfrac{(x-x_0)^2}{2}\dfrac{\partial^2 u}{\partial x^2}(x_0,t) + \ldots.$



In linear theory, $u \approx u(x_0, t)$

$$\vec{x} = \int_0^t u(x_0, t)dt$$

• $x_p = -A\dfrac{\cosh k(z_0 + h)}{\sinh kh}\sin(kx_0 - \omega t)$

• $z_p = A\dfrac{\sinh k(z_0 + h)}{\sinh kh}\cos(kx_0 - \omega t)$

x_0, z_0 is mean position of particle

• $\dfrac{(x_p - x_0)^2}{a^2} + \dfrac{(z_p - z_0)^2}{b^2} = 1$

Where

$$a = A\frac{\cosh k(z_0 + h)}{\sinh kh}, b = A\frac{\sinh k(z_0 + h)}{\sinh kh}$$

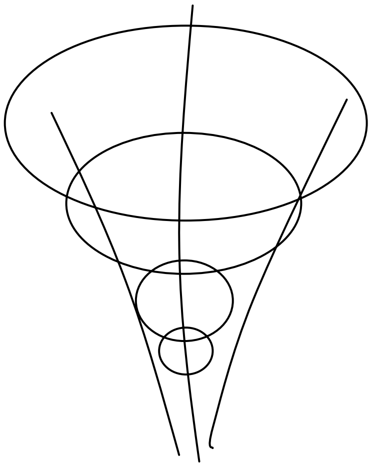
•In deep water…

$$\begin{matrix} \cosh k(z_0 + h) \\ \sinh k(z_0 + h) \end{matrix} \quad \Rightarrow \quad \frac{e^{k(z_0 + h)}}{2} \quad \text{as} \quad kh \to \infty$$

This means…..

$a \approx b$

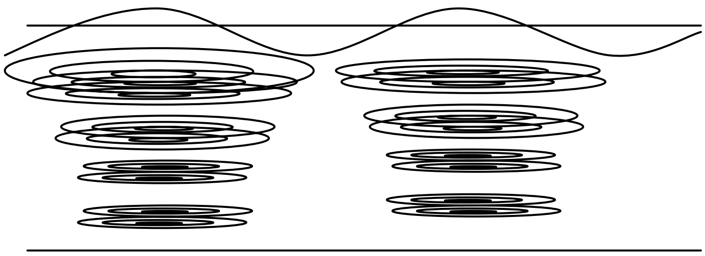
Therefore, the orbit becomes circle, decaying exponentially



decaying e^{kz}

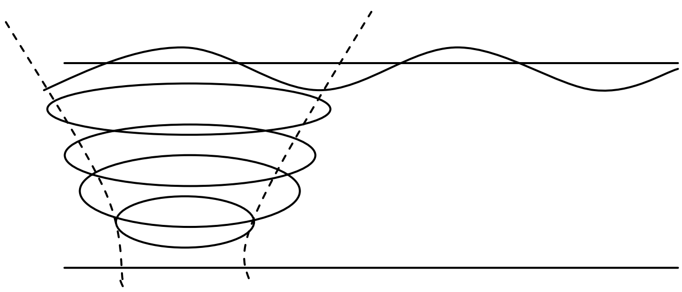
•In shallow water…

$$\cosh k(z_0 + h) \to 1$$
$$\sinh k(z_0 + h) \to 0$$



Almost flat…

•In finite depth…. Elliptic orbit



Elliptic orbit

5.2.5. Dispersion relation

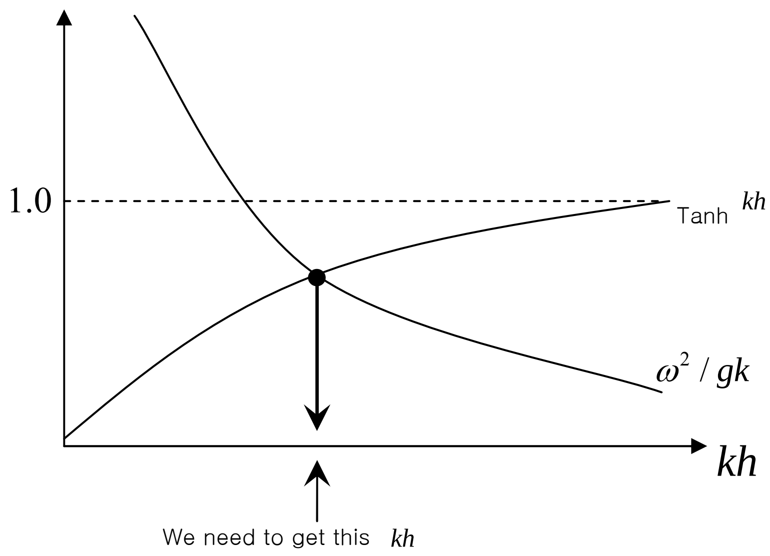
• $\omega^2 = gk \tanh kh$

$\omega^2 = gk$ in deep water
$\omega^2 = gk^2 h$ in shallow water

- $\omega \uparrow \Rightarrow k \uparrow$
  $T \uparrow \Rightarrow \lambda \uparrow$
  $\Rightarrow$Longer waves are faster…

•When k is known, straight forward to compute ω
•When ω is known, maybe complicated to get k

In general…



•Approximation of k

(i) kh>3 $\Rightarrow$ $\lambda < 2h$: Deep water

$$\omega^2 = gh \Rightarrow k = \frac{\omega^2}{g}$$

(ii) kh<<1 $\Rightarrow$ typically, $\lambda > 20h$

$$\omega^2 \cong gk^2 h \Rightarrow k \cong \frac{\omega}{\sqrt{gh}}$$

(iii) Otherwise, Put $C = \frac{\omega^2 h}{g}$

A. If c>2, $kh \approx C(1 + 2e^{-2C} - 12e^{-C} + ...)$

B. If c<2, $kh \approx \sqrt{C}(1+0.169C+0.031C^2+......)$

5.2.5. Wave speed : phase velocity

•$V_P = \frac{\lambda}{T} = \frac{\omega}{k}$

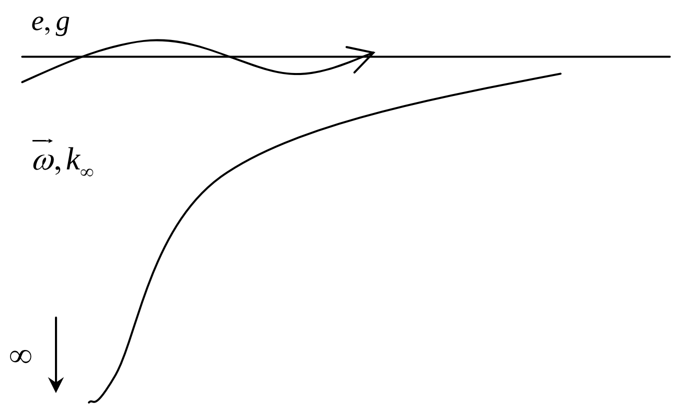
Using dispersion relation $\omega^2 = gk \tanh kh$

$$V_P = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kh}$$

• $V_P = \sqrt{\frac{\omega}{k}}$ in deep water

$\sqrt{gh}$ in shallow water <- not a function of k. i.e. ω

•When ω is constant….



$\omega^2 = gk_\infty = gk \tanh kh$ or

$$\frac{k_\infty}{k} = \tanh \ kh$$

Notice that

$\frac{k_\infty}{k} < 1$

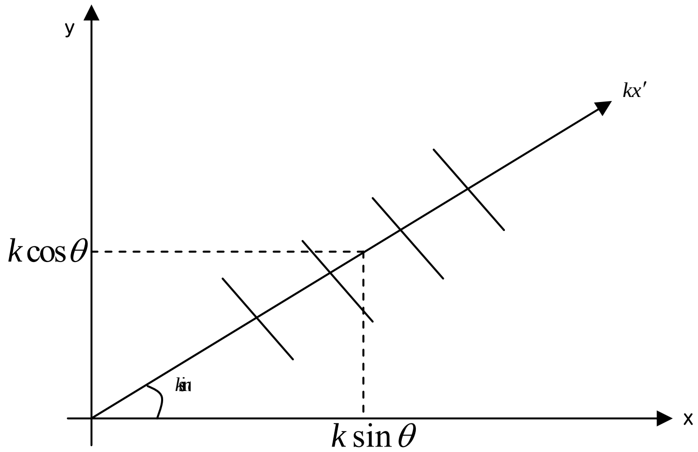
$\Rightarrow \frac{V_P}{V_{P,\infty}} = \frac{\omega/k}{\omega/k_\infty} = \frac{k_\infty}{k}\left(= \frac{\lambda}{\lambda_\infty}\right) = \tanh kh$

•Wave speed in deep water is faster than that in finite depth

•Wavelength in deep water > wave length in finite depth

5.3 3-D Plane waves

5.3.1. 3-D Plane Progressive Waves (PPW)

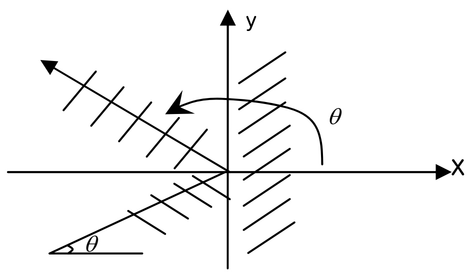


$$\phi = \frac{gA}{\omega} K(z)\sin(kx' - \omega t)$$
$$= \frac{gA}{\omega} K(z)\sin(kx\cos\theta + ky\sin\theta - \omega t)$$

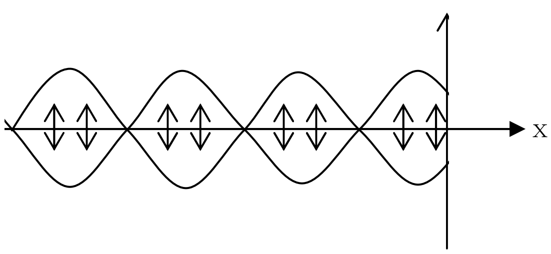
$$k_x = k\cos\theta$$
$$k_y = k\sin\theta$$

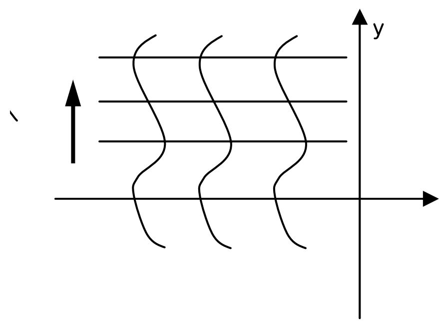
5.3.2 3-D Standing waves (Oblique Standing Waves)

$$\eta_1 = A\cos(kx\cos\theta + ky\sin\theta - \omega t)$$
$$\eta_2 = A\cos(kx\cos(\pi - \theta) + ky\sin(\pi - \theta) - \omega t)$$



$$\eta_1 + \eta_2 = 2A\cos(kx\cos\theta) \times \cos(ky\sin\theta - \omega t)$$
$$\text{Standing} \times \text{Progressive}$$





5.3.3 General Form of Superposition…

2D : $\eta = \int_{-\infty}^{\infty} A(\omega)\cos(kx - \omega t)\,d\omega$

3D : $\eta = \int_0^{2\pi}\int_{-\infty}^{\infty} A(\omega)\cos(kx\cos\theta + ky\sin\theta - \omega t)\,d\omega$

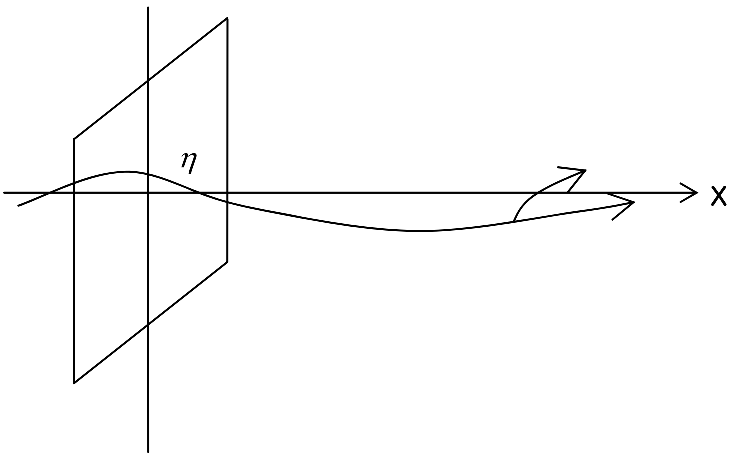
In discrete forms…

$$\eta = \sum A_i \cos(k_i x - \omega_i t)$$

$$\eta = \sum_i \sum_j A_{ij} \cos(k_j x\cos\theta_i + k_j y\sin\theta_i - \omega_j t)$$

5.4 Wave energy & Group Velocity

5.4.1 Sectional Wave Energy



•Kinetic energy

Kinetic Energy $= \int_{-h}^{\eta} \frac{1}{2}|\nabla\phi|^2\,dz$

$= \int_{-h}^{0} \frac{1}{2}|\nabla\phi|^2\,dz + \int_{0}^{\eta} \frac{1}{2}|\nabla\phi|^2\,dz$

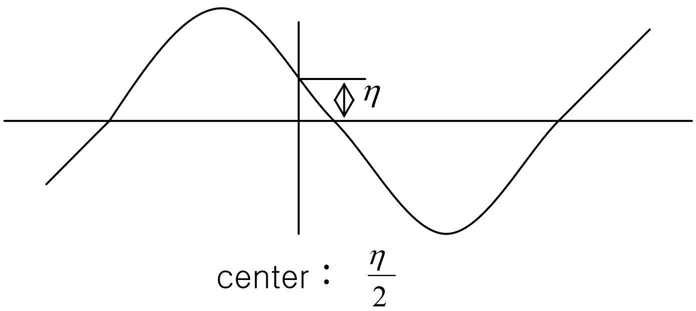
$O(\varepsilon^2)$ $\quad$ $O(\varepsilon^3)$

$\cong \frac{1}{4}\rho g A^2$

•Potential Energy

P.E. $= (\rho g \eta)\left(\frac{\eta}{2}\right)$

$= \frac{1}{2}\rho g \eta^2$



5.4.2 Mean Energy Density

$$\overline{E} \equiv \frac{1}{\lambda T}\int_0^{\lambda}\int_0^{T}(K.E.+P.E.)dtdx = \frac{1}{2}\rho g A^2$$ -> for both deep & finite depth

Thus, Wave energy $\propto A^2$

5.4.3 Energy flux across a vertical plane

Rate of work done by wave flow passing a vertical plane
= Energy flux across the plane

$$\overline{P} = F \bullet v$$

For our case…

$$\overline{P} = \int_{-h}^{\eta} P \bullet u_n \, dz$$

Where P : pressure
u_n : Normal velocity (= u)

$$\overline{P} = \int_{-h}^{\eta}\left(-\rho\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial x}\right)dz$$

$$\cong \int_{-h}^{0}\left(-\rho\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial x}\right)dz$$ for linear problem

$$= \frac{1}{2}\rho g A^2 \bullet \left(\frac{1}{2}+\frac{kh}{\sinh 2kh}\right)\frac{\omega}{k}$$

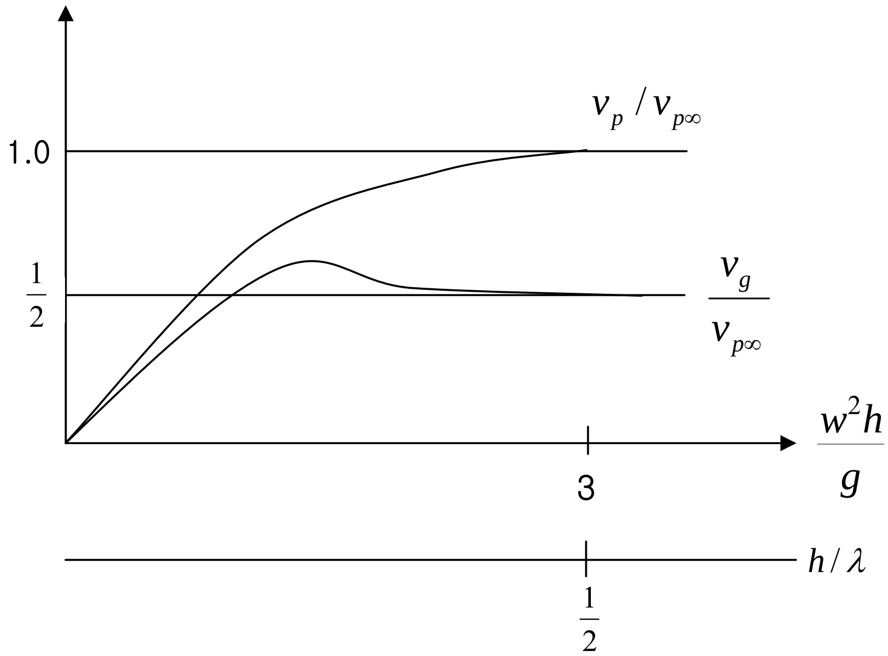
5.4.4 Group velocity

$$\overline{P} = \overline{E}V_g$$

Where $V_g = \left(\frac{1}{2}+\frac{kh}{\sinh\ 2kh}\right)V_p$: Vp is phase velocity

$kh \to \infty$ $\qquad$ $kh \to 0$

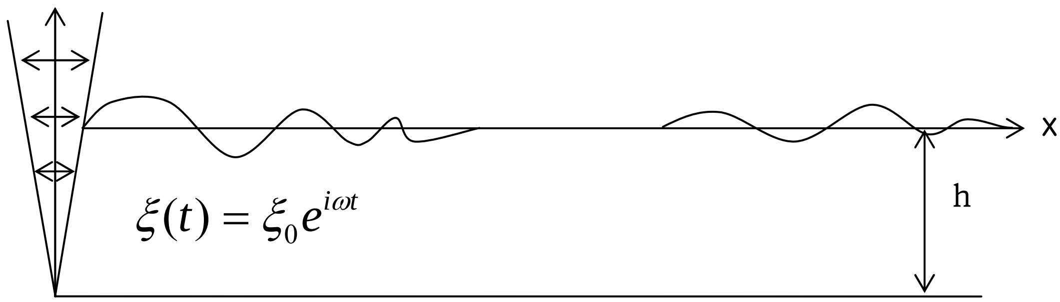
$V_g = \frac{1}{2} V_p$ $\qquad$ $V_g \approx V_p$



•The other way go derive V_g

$$\frac{d\omega}{dk} = \frac{\omega}{k}\left(\frac{1}{2} + \frac{kh}{\tanh kh}\right)$$

Chapter 6. Wave-Maker problem



6.1 Boundary Value Problem

At far,

$$\phi \to \mathrm{Re}\{\varphi e^{i\omega t}\} \quad : \text{Free wave}$$

2D or 3D plane waves

At near,

$$\phi = \mathrm{Re}\{\varphi e^{i\omega t}\} + \mathrm{Re}\{\psi e^{i\omega t}\}$$

① Laplace equation

$$\nabla^2 \phi 0 \Rightarrow \nabla^2 \varphi = 0, \nabla^2 \psi = 0$$

② F.S.B.C. (Free Surface Boundary Condition)

$$\phi_{tt} + g\phi_z \quad on \quad z = 0$$

$$\psi_z - \frac{\omega^2}{g}\psi = 0 \quad on \quad z = 0$$

③ Bottom Boundary Condition

$$\psi_z = 0 \quad on \quad z = -h$$

④ Body Boundary Condition

$$\frac{\partial \psi}{\partial x} V_n \quad on \quad x = 0$$

⑤ Radiation condition

As $x \to \infty, \psi = 0$

Why? φ already satisfies

6.2 Velocity potentials of local waves

$$① \Rightarrow \frac{X_{xx}}{X} = -\frac{Z_{zz}}{Z} = -k^2 \quad \text{(similar to 2-D plane waves)}$$

$$X_{xx} + k^2 X = 0$$

$$⑤ \Rightarrow X = Ae^{\sigma x} + Be^{-\sigma x}$$

$$\Rightarrow X = Be^{-\sigma x}$$

$$Z = Ce^{i\sigma z} + De^{-i\sigma z}$$

$$③ \frac{\partial Z}{\partial z} = i\sigma\left\{Ce^{-i\sigma h} - De^{i\sigma h}\right\} = 0$$

$$\Rightarrow C = De^{2i\sigma h}$$

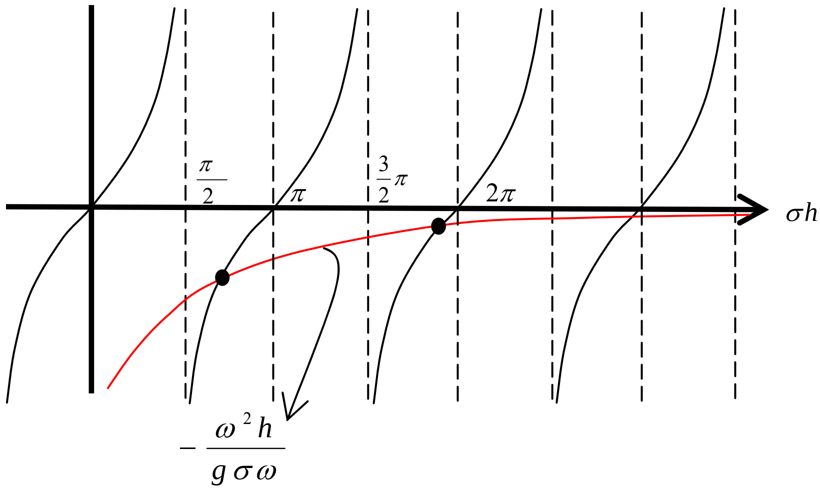
$$\Rightarrow \psi = 2BDe^{i\sigma h} \cos\{\sigma(z+h)\} e^{-\sigma x}$$

$$② \, e^{-\sigma x}(-\sigma)\sin\{\sigma(z+h)\} - \frac{\omega^2}{g} e^{-\sigma x} \cos\{\sigma(z+h)\} = 0 \qquad \text{on } z=0$$

$$\sigma \tanh \sigma h = -\frac{\omega^2}{g}$$

•Dispersion of Local waves

•there is infinite number of modes



•Velocity Potential : general form

$$\psi = \sum_{n=1}^{\infty} \tilde{\psi}_n e^{-\sigma_n x} \cos\{\sigma_n (z+h)\} \equiv \sum_{n=1}^{\infty} \tilde{\psi}_n e^{-i\omega t}$$

Note that

$$\int_{-h}^{0} \psi_n(z) \bullet \psi_m(z) dz \quad = 0 \quad m \neq n$$

$$\neq 0 \quad m = n$$

6.3. Amplitude of φ

Bottom Boundary Condition

$$\frac{\partial \phi}{\partial x} = \mathrm{Re}\{i\omega \xi_0 e^{i\omega t}\}$$

$$\frac{\partial}{\partial x}\left[\{\varphi e^{i\omega t}\} + \left\{\sum_{n=1}^{\infty} \psi_n e^{-i\omega t}\right\}\right]$$

$$= \left\{ik\varphi + \sum_{n=1}^{\infty} (-\sigma_n)\psi_n\right\} e^{-i\omega t}$$

$$\Rightarrow \int_{-h}^{0} \left\{-ik\varphi + \sum_{n=1}^{\infty} (-\sigma_n)\psi_n\right\} \varphi dz$$

$$= \int_{-h}^{0} i\omega \xi_0 \varphi dz$$

From orthogonality

$$-k\int_{-h}^{0}\varphi^{2}dz=\omega\int_{-h}^{0}\xi_{0}\varphi dz$$

Then, substituting φ, we can get

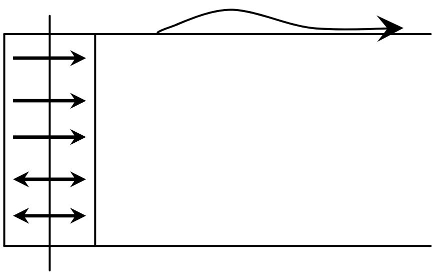
$$A=-\frac{\omega}{k}\int_{-h}^{0}\hat{\varphi}\xi_{0}(z)dz$$

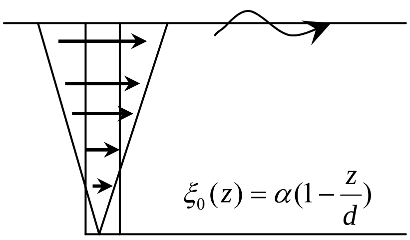
where

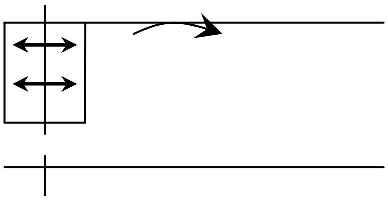
$$\hat{\varphi}=\frac{\sqrt{2}\cosh k(z+h)}{\sqrt{h+\frac{g}{\omega^{2}}\sinh^{2}(kh)}}$$

When $\xi_0(z)$ is known, we can get A using above…

e.g.

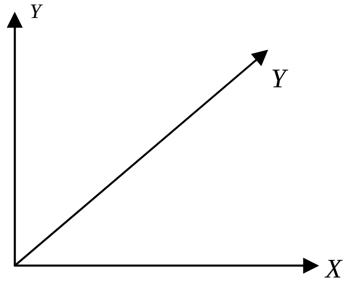


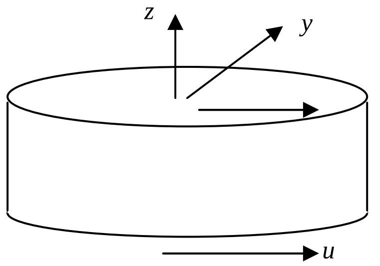




Chapter 7. Ship waves

7.1 Moving frame





$$X = x + Ut \Rightarrow \begin{array}{l} x = X - Ut \\ y = Y \\ z = Z \end{array}$$

$$\left.\frac{\partial}{\partial t}\right|_{X,Y,Z} = \left.\frac{\partial}{\partial t}\right|_{x,y,z} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} = \left.\frac{\partial}{\partial t}\right|_{x,y,z} - U\frac{\partial}{\partial x}$$

•Linear F.S.B.C.

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad \text{on z=0 (*)}$$

Notice that this is valid in XYZ Frame in xyz frame. (?)

$$\frac{\partial^2}{\partial t^2}\phi = \left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 \phi$$

$$\therefore (*) \Rightarrow \frac{\partial^2 \phi}{\partial t^2} - 2U\frac{\partial^2 \phi}{\partial x \partial t} + U^2\frac{\partial^2 \phi}{\partial x^2} + g\frac{\partial \phi}{\partial z} = 0$$

•Steady flow $\frac{\partial}{\partial t}() = 0$

Steady F.S.B.C. with moving speed U

$$U^2 \frac{\partial^2 \phi}{\partial x^2} + g \frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z=0$$

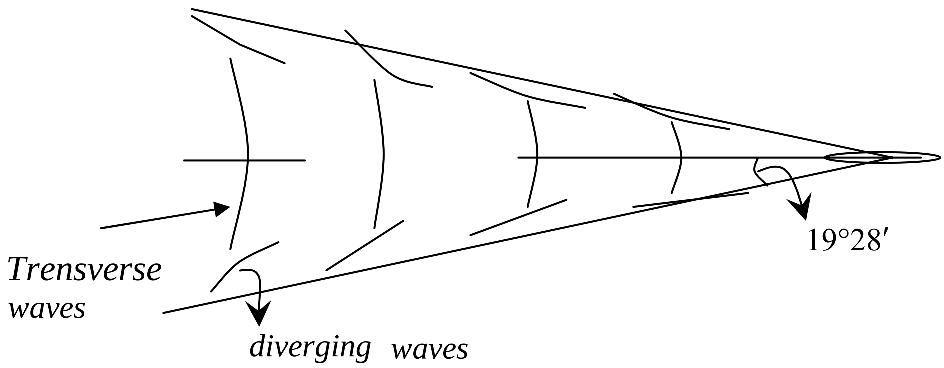
•Wave elevation

$$\eta = -\frac{1}{g} \frac{\partial \phi}{\partial t}\bigg|_{xyz} = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t} - U \frac{\partial \phi}{\partial x} \right)$$

If steady,

$$\eta = \frac{U}{g} \frac{\partial \phi}{\partial x} \qquad \text{(insert P.7-1)}$$

7.2 Kelvin Waves



• $\frac{\partial^2 \phi}{\partial x^2} + \frac{g}{U^2} \frac{\partial \phi}{\partial z} = 0 \quad on \quad z = 0$

$\frac{\partial^2 \phi}{\partial x^2} + \frac{gL}{U^2} \frac{1}{L} \frac{\partial \phi}{\partial z} = 0 \quad L: characteristic\ length. \quad i.e. \quad ship\ length$

$\phi_{xx} + \frac{1}{Fr^2} \frac{1}{L} \frac{\partial \phi}{\partial z} = 0 \quad Fr: Froude\ Number$

(i) Fr $\to 0$ $\qquad \phi_z \to 0$

(ii) Fr $\to \infty$ $\qquad \phi_{xx} \to 0 \Rightarrow \phi = 0$

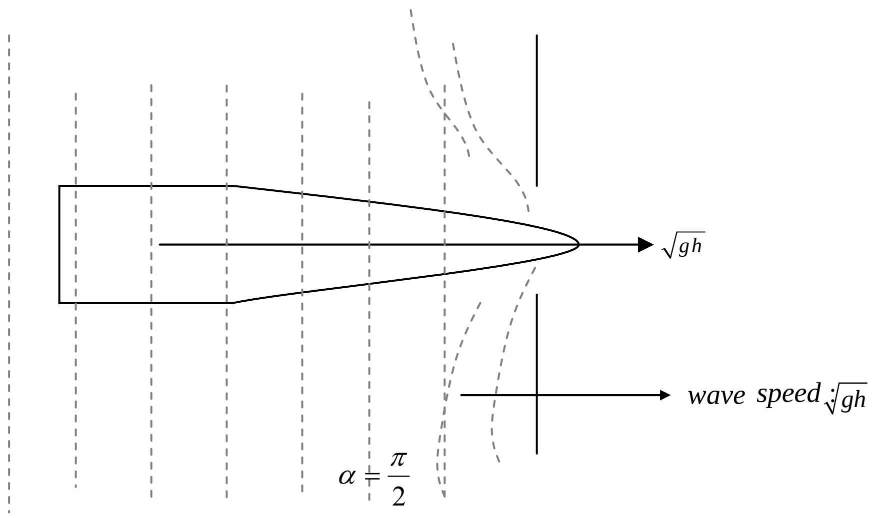
(iii)

※Fr is a key parameter in wave resistance problem.

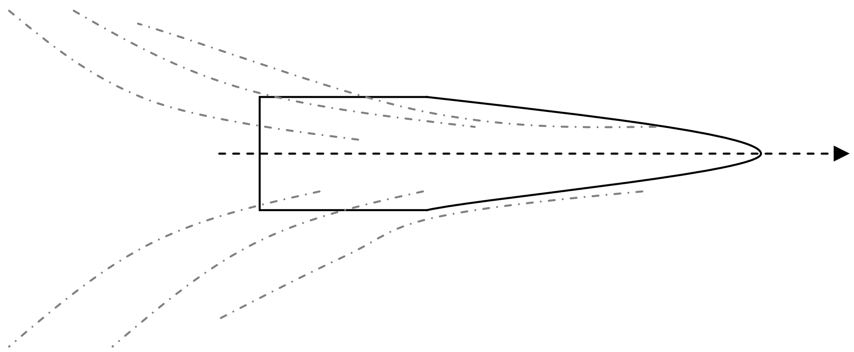
•In shallow depth, (i.e. $\lambda >> h$), Fn = $\dfrac{V}{\sqrt{gh}}$: depth Froude Number, is a key parameter

(e.g.) if $Fn = \dfrac{V}{\sqrt{gh}} = 1$, i.e.

$\sqrt{gh} = U$



If $Fn = \dfrac{V}{\sqrt{gh}} > 1$



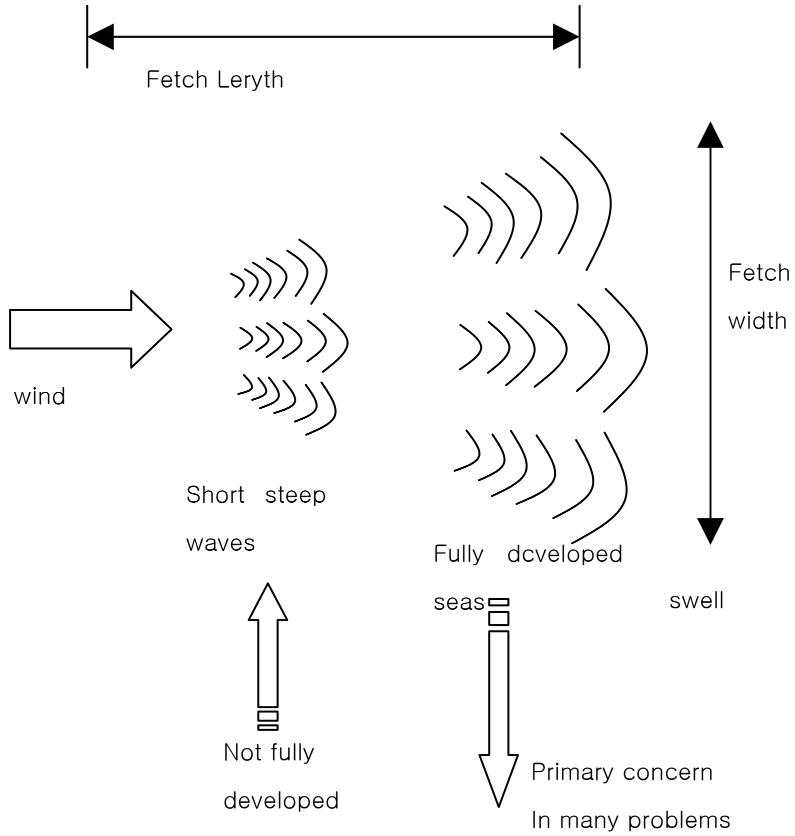
Only diverging waves

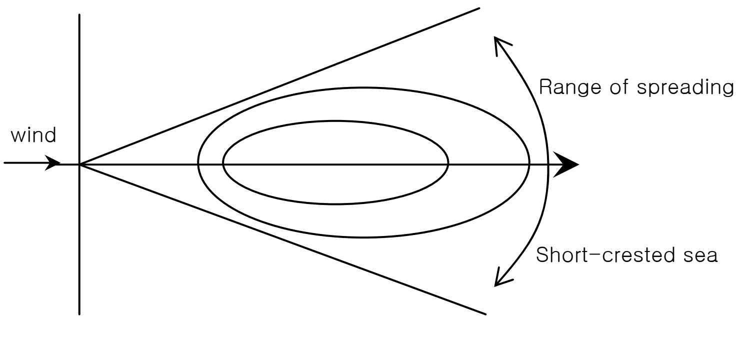
• $Fn = 1.0$: *critical depth Froude #*

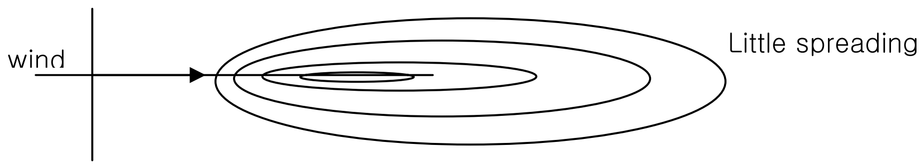
• $\alpha \approx \sin^{-1}\left(\dfrac{1}{Fn}\right)$

Chapter 8. Wave Spectra

8.1 Random wave generation







long-crested sea

8.2 Stochastic Process

•Definition

(1) Stationary

A stochastic Process x(t) is stationary if its density function is independent in time, i.e. f(x,t) = f(x)

(2) Homogeneity

A stochastic process x(t) is homogeneous if its density function is independent on the spatial location.

(3) Ergodicity

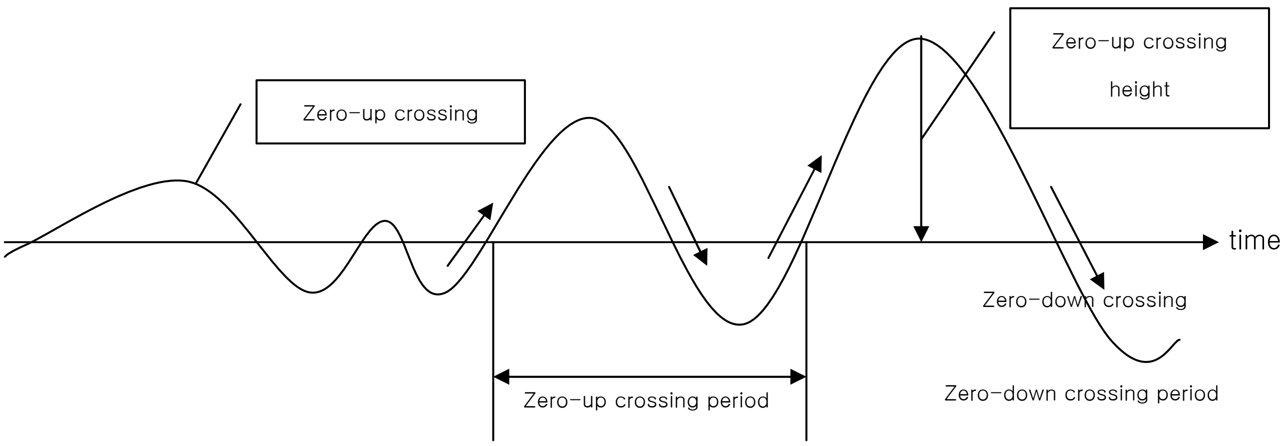
A stochastic process x(t) is ergodic if

$$\bar{x} = E[x] = \frac{1}{2T}\lim_{T\to\infty}\int_{-T}^{T} x(t)dt$$

And

$$\sigma^2 = E\left[(x-\bar{x})^2\right] = \frac{1}{2T}\lim_{T\to\infty}\int_{-T}^{T}(x-\bar{x})^2 dt$$

In most practical applications, a stochastic process which is stationary and homogeneous is ergodic.



• Superposition of multiple waves

- We will consider the random waves as a summation of multiple wave components
- $\eta = \int_0^\infty \int_0^\infty \eta(\omega,\theta)d\omega\, d\theta$ for short-crested waves

– $\eta = \int_0^\infty \eta(\omega, \theta) d\omega$ for long-crested waves

• Consider a random variable x.

-Central Limit theorem

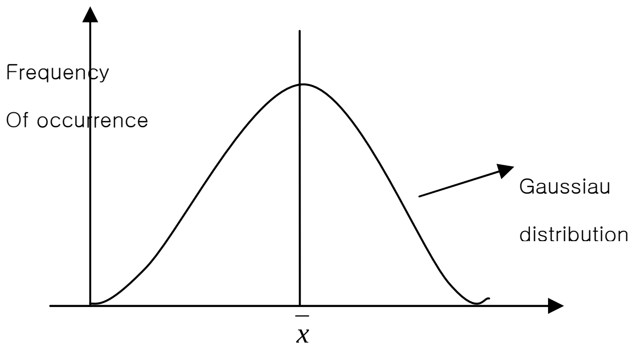
If a random variable x can be expressed as the sum of a large number of independent random variable x_j

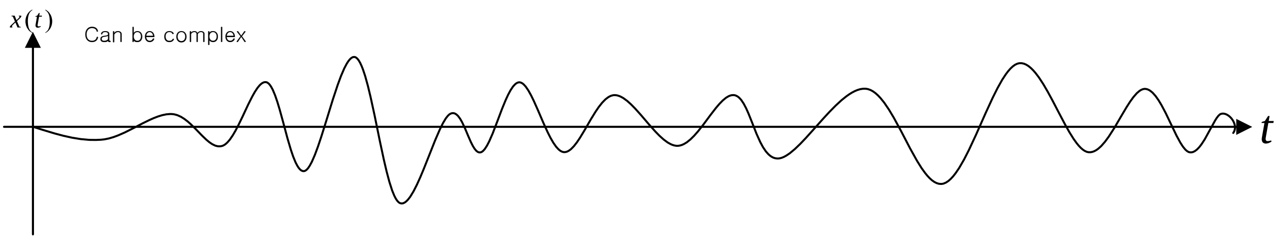
$$x = x_1 + x_2 + x_3 + x_4 + ... + x_n \qquad \text{(n: large)}$$

Then the density f(x) of x is the Gaussian function, s.t.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\bar{x})^2 / 2\sigma^2}$$

Where $\bar{x}, \sigma^2$ are mean and variation





Mean : $\bar{x} = E[x] = \frac{1}{2T} \lim_{T\to\infty} \int_{-T}^{T} x(t) dt$

Variance: $\sigma^2 = E\left[(x-\bar{x})^2\right] = \frac{1}{2T} \lim_{T\to\infty} \int_{-T}^{T} (x-\bar{x})^2 dt$

We say this system is "ergodic"

8.3 Stochastic Description of Random Waves

• $\overline{\eta} = 0$

$$\eta = \sum_k \eta_k(x, y, t) = \sum_k A_k \cos(\omega_k t + \theta_k) \; ; \; \theta_k \text{ is phase}$$

※If $x = x_1 + x_2 + ... + x_n$

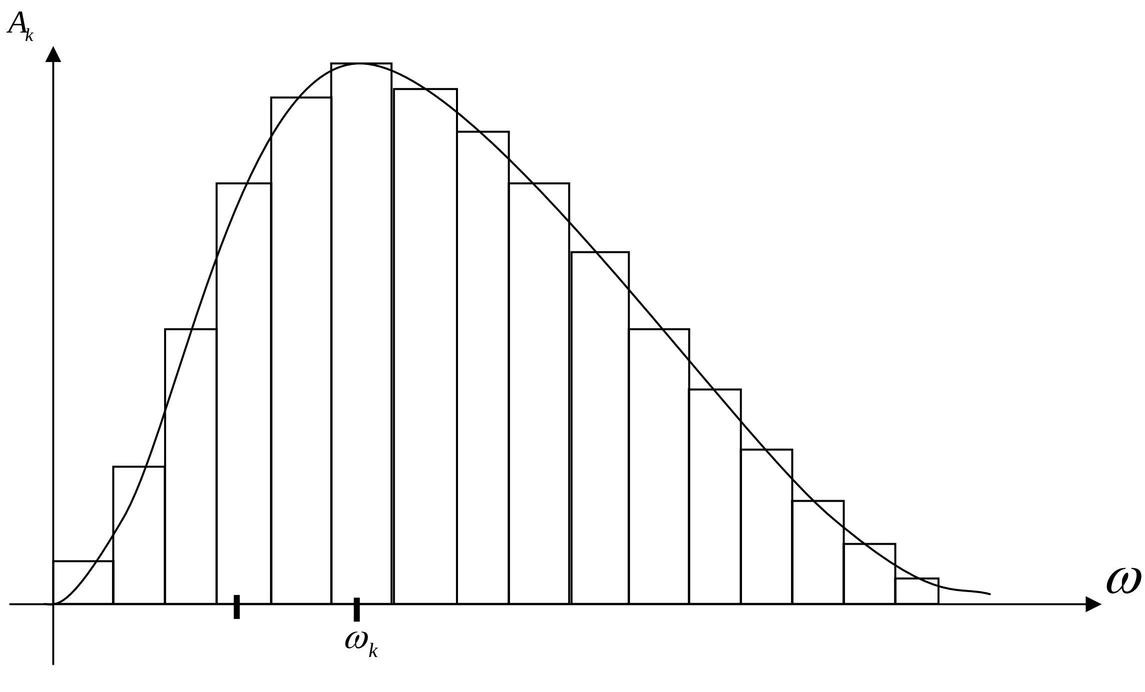
$x_1, x_2, ..., x_n$: *independent*

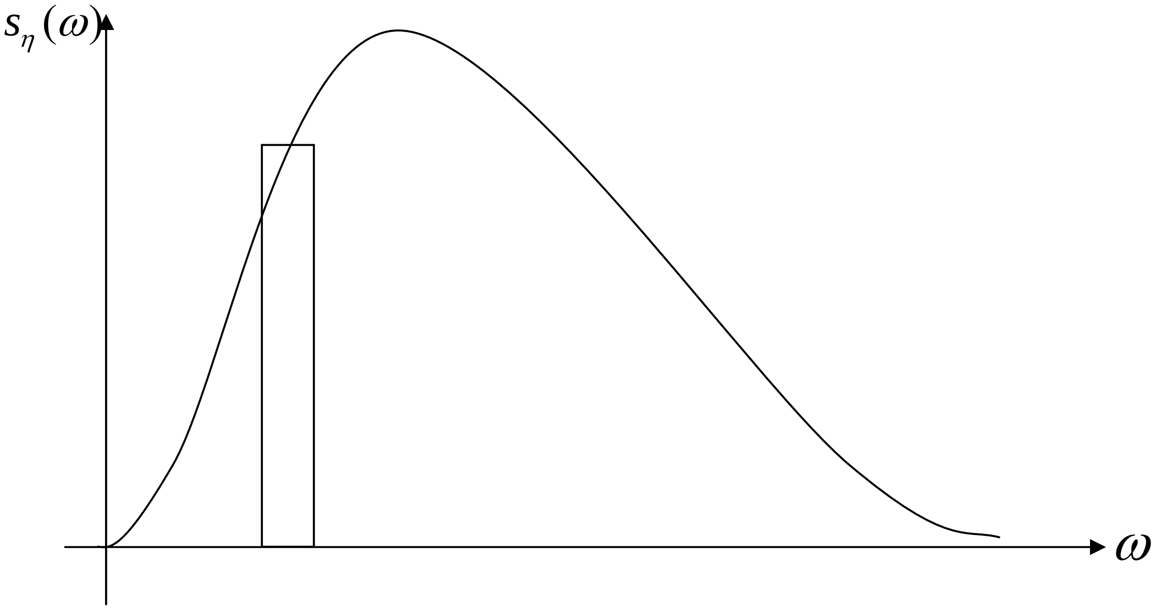
$$\overline{x} = \sum_k^N \frac{x_k}{N}, \sigma^2 = \sum_k^N (x_k - \overline{x})^2$$

$$\eta_k = A_k \cos(\omega_k t + \theta_k)$$

$$E[\eta_k{}^2] = \frac{1}{2} A_k{}^2$$

$$\Rightarrow \sigma^2 = \sum_k \frac{1}{2} A_k{}^2$$





$S_\eta(\omega)\Delta\omega_k = \frac{1}{2}A_k{}^2$

$S_\eta(\omega)$: Poser spectrum of A_k

$\Rightarrow$ energy specturum

8.4 Wave Spectra

(1) Pierson-Moskowitz spectrum

$$S_\eta(\omega) = \frac{8.1\times10^{-3} g^2}{\omega^5} e^{-0.74(g/V\omega)^4}$$

(2) Bretschneider Spectrum

$$S_\eta(\omega) = \frac{A}{\omega^5} e^{-B/\omega^4}$$

A, B : constant

•ITTC Spectrum

A = $8.1\times10^{-3} g^2$

B = $\frac{3.11}{{H_{1/3}}^2}$, $H_{1/3}$: significant wave height

•ISSC spectrum

A = $\frac{173 {H_{1/3}}^2}{T_1^4}$

B = $\frac{691}{T_1^4}$

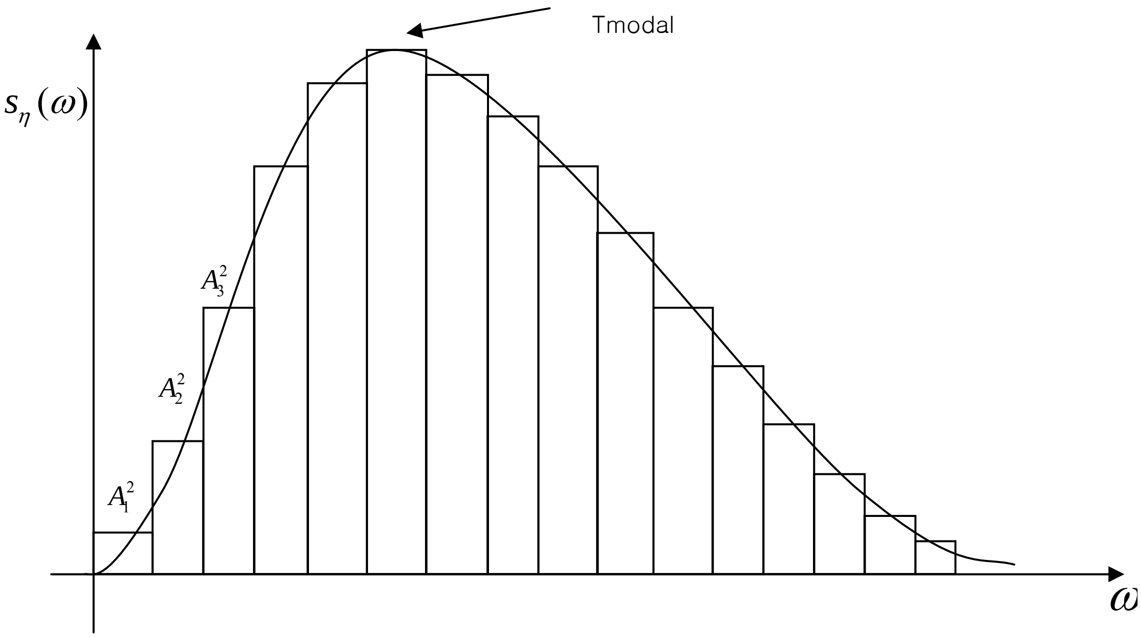
T_1 : mean wave period = $2\pi \frac{m_0}{m_1}$

• $m_k = \int_0^\infty \omega^k S_\eta(\omega) d\omega$

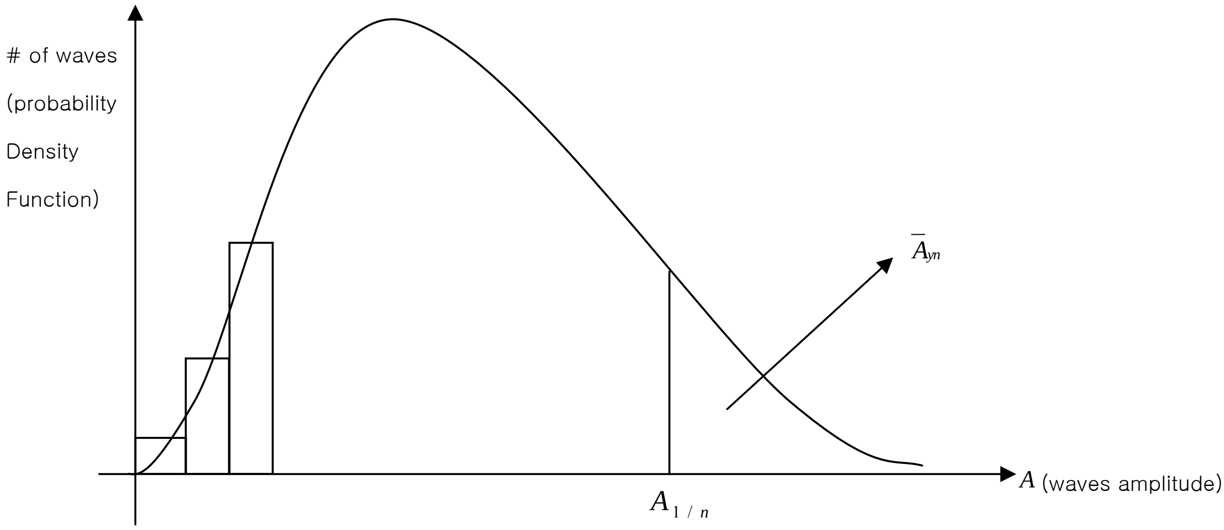
• $H_{1/3} = 4\sqrt{m_0}$

•ITTC spectrum = ISSC spectrum

$$S_\eta(\omega) = \frac{0.11}{2\pi} {H_{1/3}}^2 T_1 \left(\frac{\omega T_1}{2\pi}\right)^{-5} e^{-0.44\left(\frac{\omega T_1}{2\pi}\right)^{-4}}$$: the other form



Energy= $\sum \frac{1}{2}\rho g A_{k^2}$

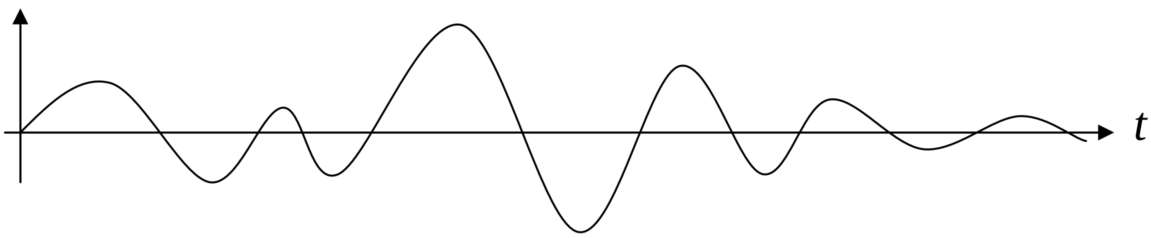


$\bar{A}_{1/n}$: $1/n-th$ highest wave amplitude

$\bar{A}_{1/3}$: significant wave amplitude

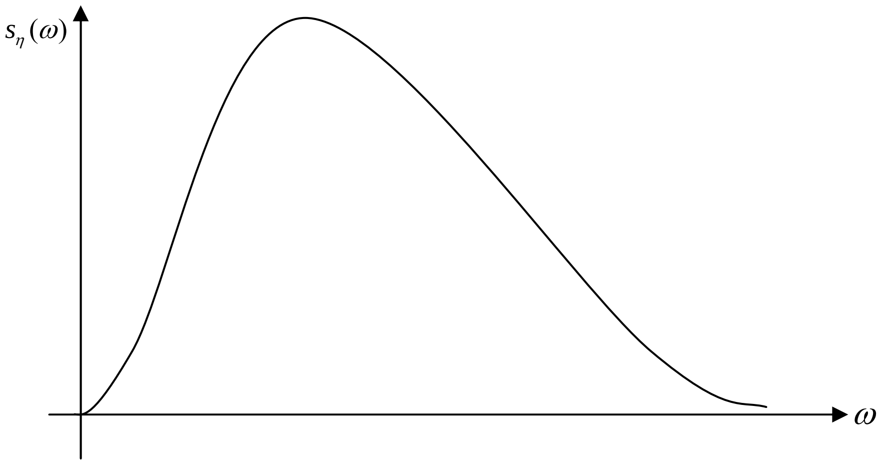
⦿Summary

- $\eta(t) = \sum A_k \cos(\omega_k t + \theta_k)$



$\eta(t)$ shows Gauss Distribution : Central Limit Theorem

- $\sigma^2 \;=\; E[\eta^2] = \sum \frac{1}{2} A_k(\omega_k)$
-



Wave spectrum

$$\sigma^2 = \int_0^\infty S(\omega)d\omega \approx \sum S(\omega_k)\Delta\omega$$

$$\Rightarrow \sigma^2 \Rightarrow \frac{1}{2}{A_k}^2 = S_\eta(\omega_k)\Delta\omega$$

Or $A_k = \sqrt{2S_\eta(\omega_k)\Delta\omega}$

•Wave spectra

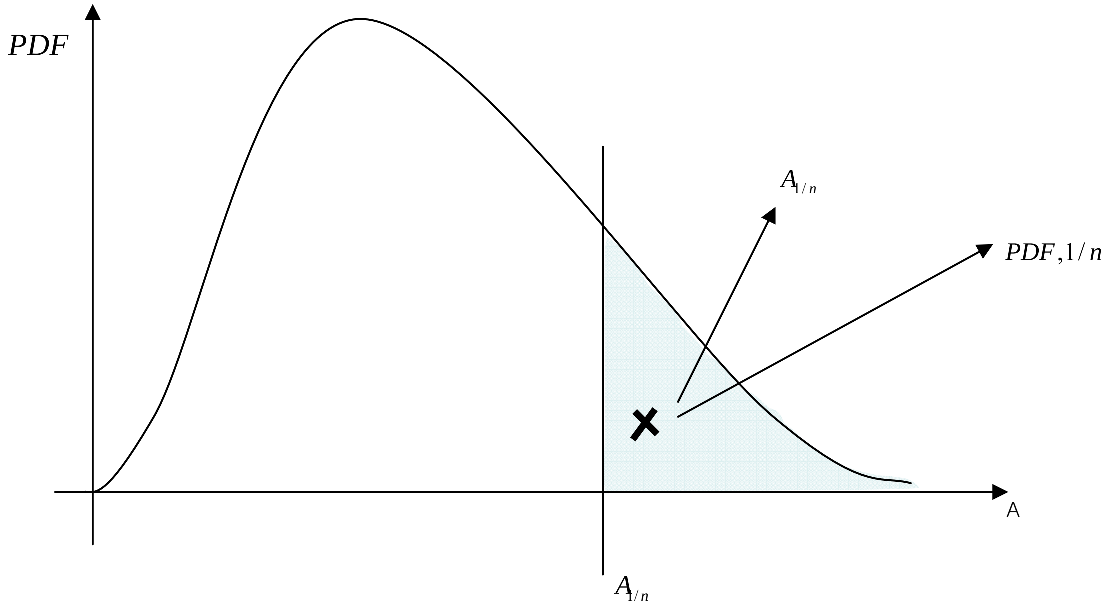
(1) Pierson-Moskowitz Spectrum

(2) Bretschneider spectrum

$$S_\eta(\omega) = \frac{A}{\omega^5} e^{-B/\omega^4}$$

A&B are functions of $H_{1/3}$ & T_1

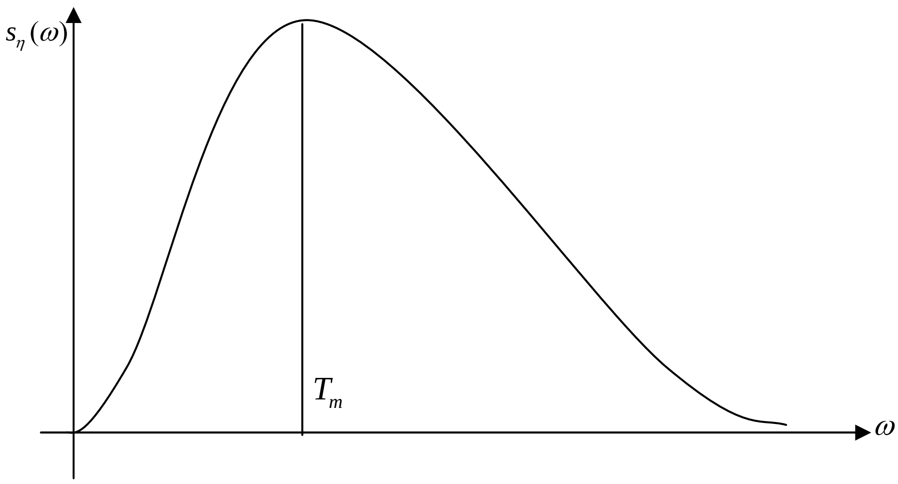
• $H_{1/n}$



$\overline{A}_{1/n}$: $1/n-th$ highest wave amplitude

$\overline{A}_{1/3}$: significant wave amplitude

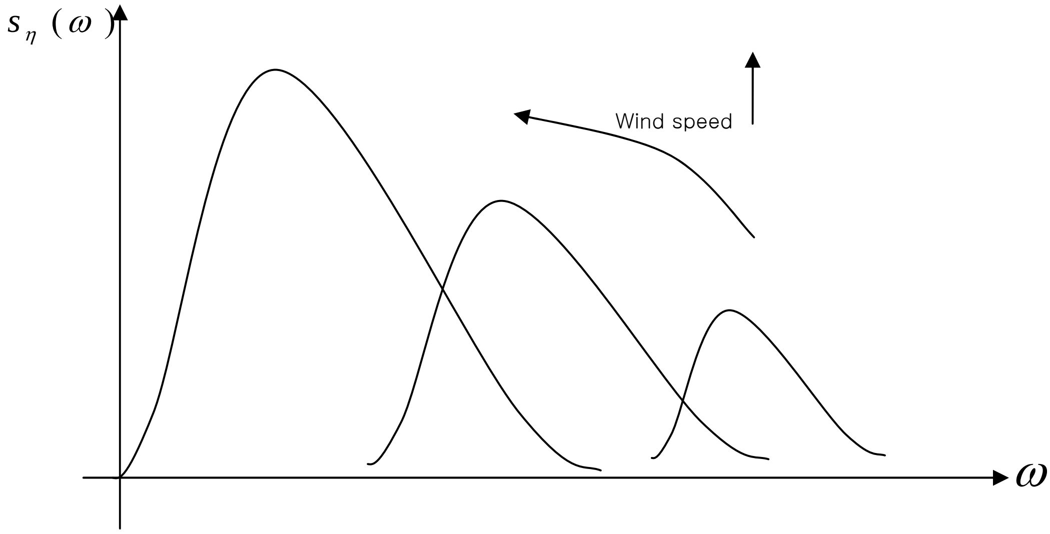
$H_{1/3} = 2\,\overline{A}_{1/3}$



- $m_k = \int_0^\infty \omega^k S_\eta(\omega)\,d\omega$
- $H_{1/3} = 4\sqrt{m_0}$
- $T_1 = 2\pi \dfrac{m_0}{m_1}$

$$T_m = 1.408 T_2 \quad \text{where} \quad T_2 = 2\pi\sqrt{\frac{m_0}{m_2}}$$

①



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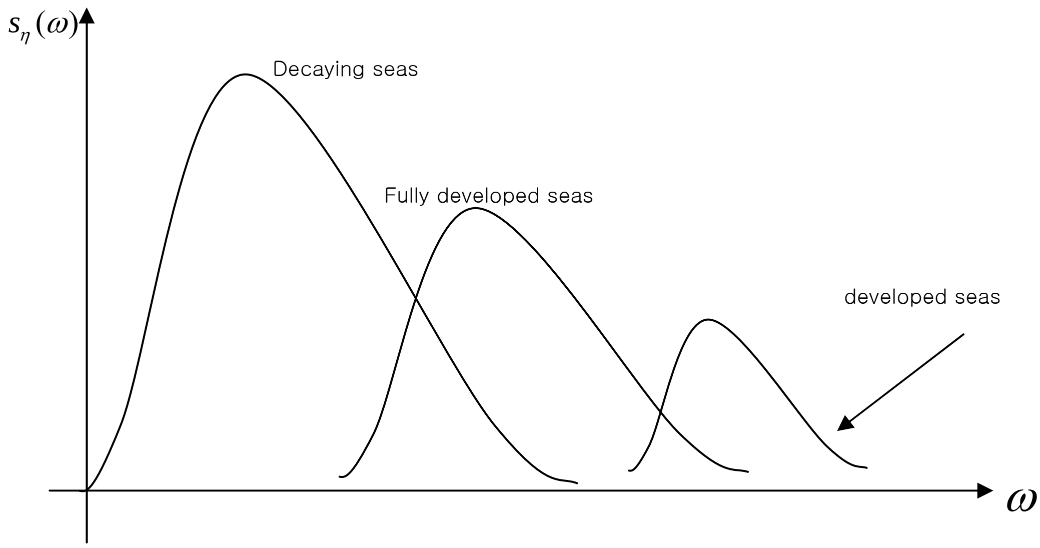

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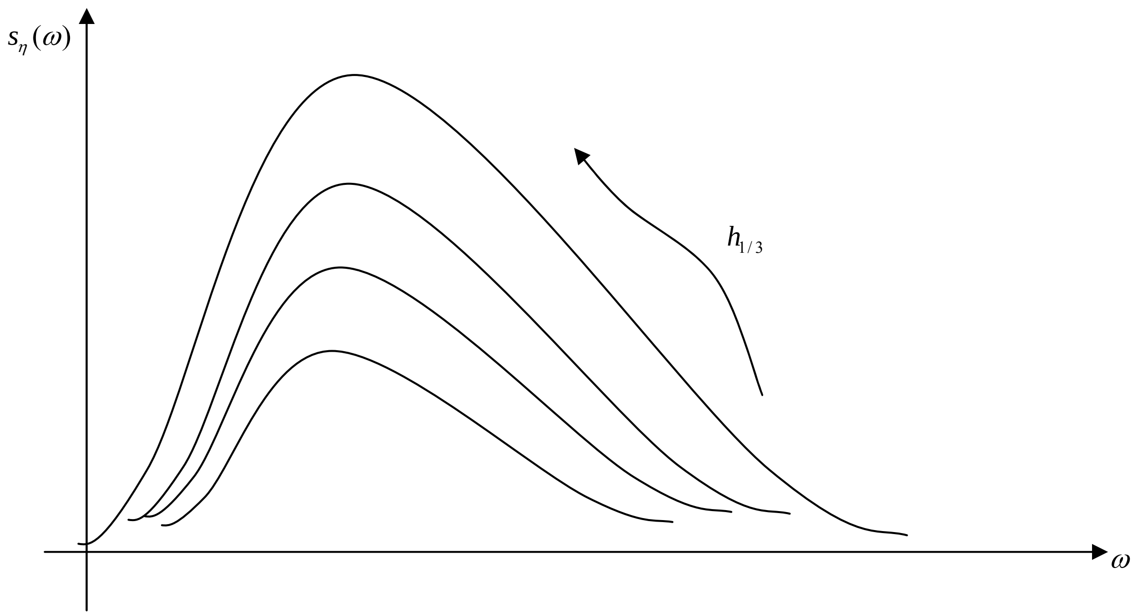
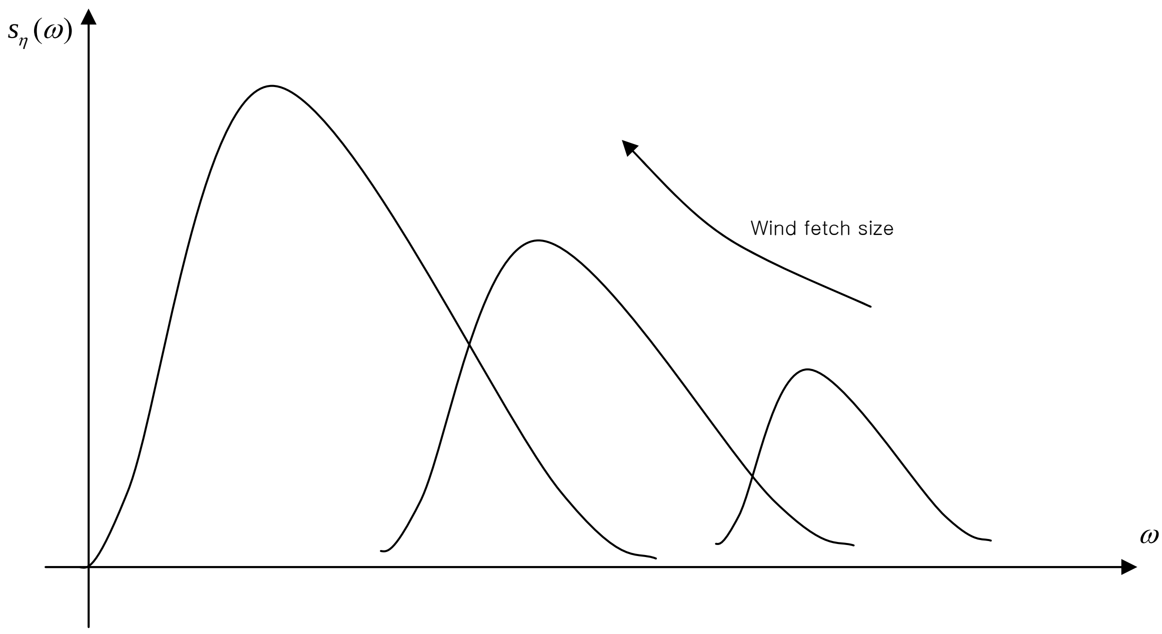
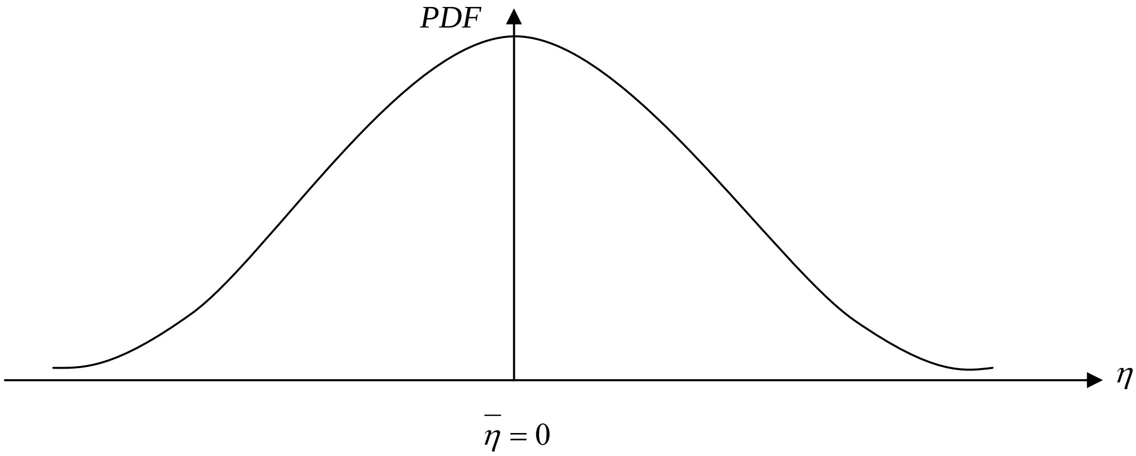


②



8.5 Statistics of wave peaks

• $\eta(t)$: Gaussian distribution (by central limit theorem)



$$f_\eta \text{(PDF of } \eta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\eta^2}{2\sigma^2}}$$

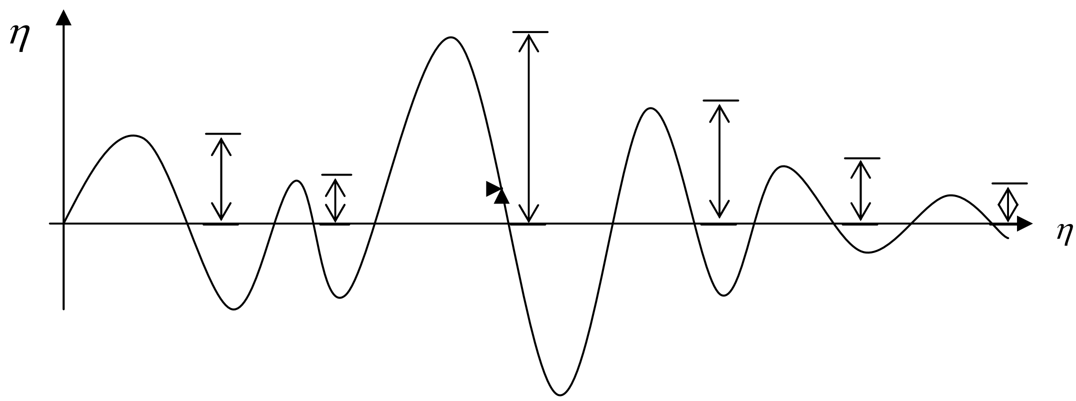
We know that

$$\sigma^2 = \sum \frac{1}{2}\eta_k{}^2 = \int_0^\infty S_\eta(\omega) d\omega$$

•define $m_k = \int_0^\infty \omega^k S_\eta(\omega) d\omega$

$m_0 = \sigma^2$: area of $S_\eta(\omega)$

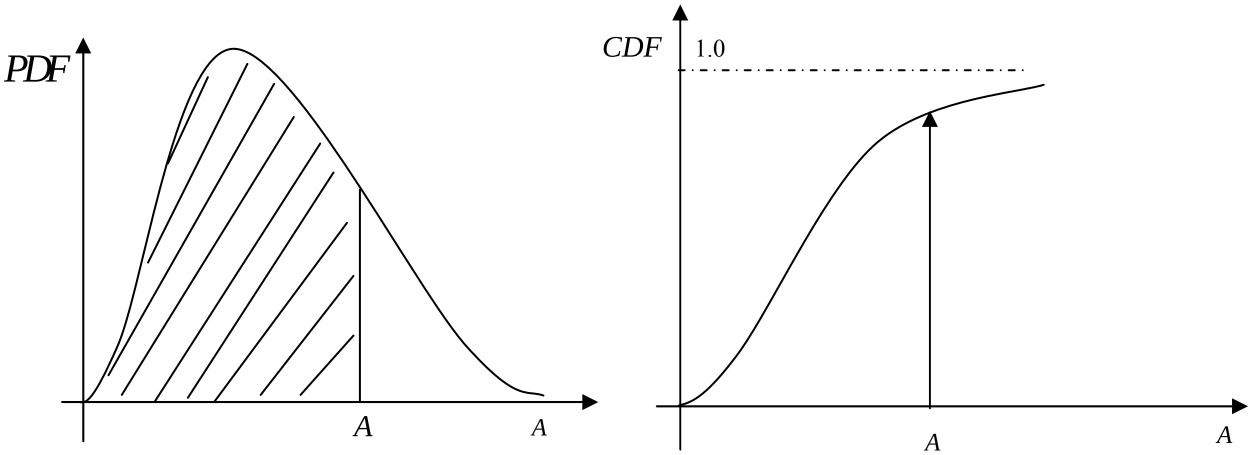
•Probability of Peaks



PDF of Peaks of $\eta(t)$: Gaussian, Narrow-banded Distribution.
⇒Rayleigh Distribution

PDF of A $\equiv f_H(A) = \frac{A}{m_0} e^{-\frac{A^2}{2m_0}}$ where h : wave height

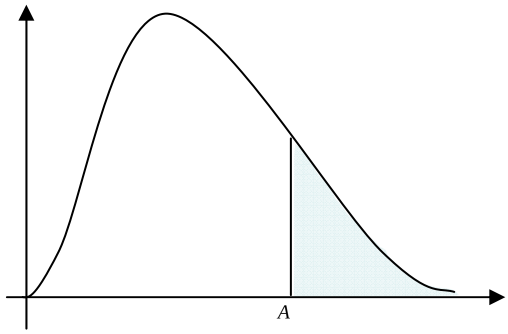
•Cummulative probability function(CPF)

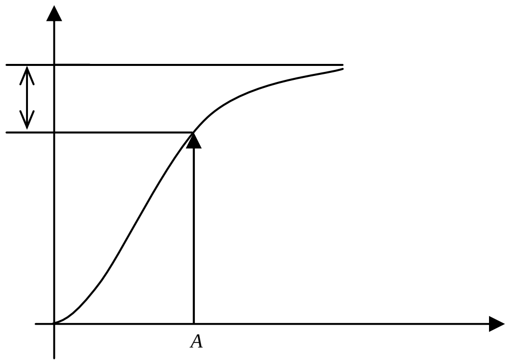


$$\mathrm{P(A)} = \int_0^A \frac{x}{m_0} e^{-\frac{x^2}{2m_0}} dx = -e^{-\frac{x^2}{2m_0}} \Bigg|_0^A = 1 - e^{-\frac{A^2}{2m_0}}$$

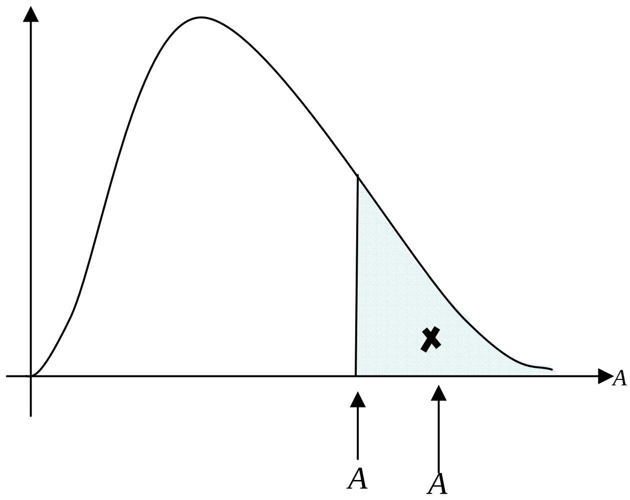
1 - P(A) : Peak > A

$$= e^{-\frac{A^2}{2m_0}}$$



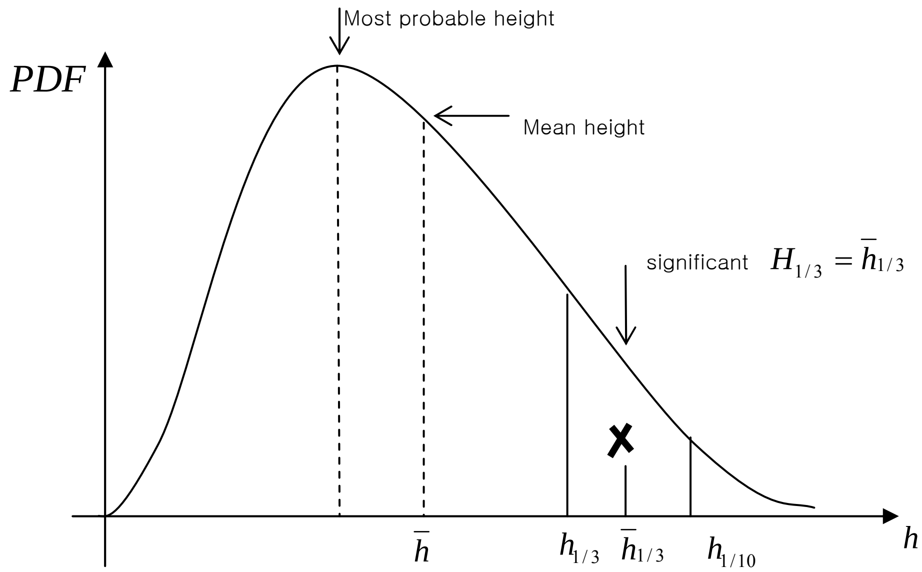


•1/n-th highest & averaged 1/n-th highest wave



$$H_{1/3} = significant\ wave\ height = 2\overline{A}_{1/3}$$

•



$$\bar{h} / H_{1/3} = 0.64$$

$$\bar{h}_{1/10} / H_{1/3} = 1.29$$

$$\bar{h}_{1/\infty} / H_{1/3} = 1.68$$

$$m_k = \int_0^\infty \omega^k S_\eta(\omega) d\omega$$

$$H_{1/3} = 4\sqrt{m_0}$$

$$\bar{h}_{1/10} = 5.1\sqrt{m_0}$$

$$\bar{h} = 2.5\sqrt{m_0}$$

T_1 = defined in Bretschneider Spectrum = $2\pi \dfrac{m_0}{m_1}$

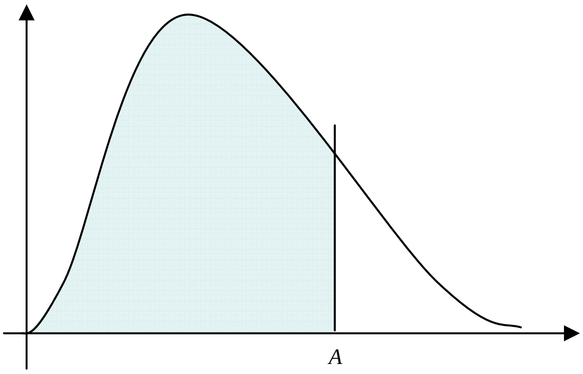
$T_2 = 2\pi\sqrt{\dfrac{m_0}{m_2}}$: average period between “zero upcrossing”

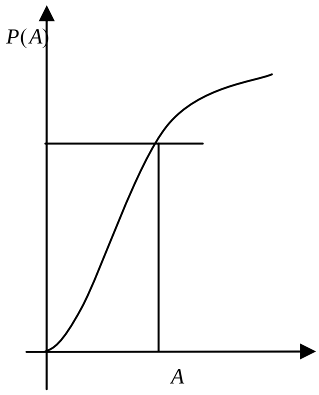
$T_{peak} = 1.408 T_2$: Most probable -> corresponding period = T_{peak}

8.6 Prediction of wave amplitude

•CPF of Rayleigh Distribution

$$P(A) = \int_0^A \frac{x}{m_0} e^{-\frac{x^2}{2m_0}} dx = 1 - e^{-\frac{A^2}{2m_0}}$$





•1-P(A) : Probability of exceedance

e.g. 1% -> P(A) = 99%

•1-P(A) = Q(A)

Q(A) = $e^{-\frac{A^2}{2m_0}}$

$$\ln Q(A) = -\frac{A^2}{2m_0}$$

$$A = \sqrt{-2m_0 \ln Q(A)}$$

e.g. Q(A) = 0.01 -> 1% $\Rightarrow \quad A = \sqrt{9.2103 m_0}$

Q(A) = 0.0001 -> 0.1% $\Rightarrow$ $A = \sqrt{13.8155 m_0}$

•Probability of 1/N occurance

$$A = \sqrt{-2m_0 \ln \frac{1}{N}}$$

$$= \sqrt{-2m_0 \ln N}$$

•Application to wave amplitude prediction

“Return Period” $\Rightarrow$ The time between successive occurrence

e.g. 10 year return period

•means ; expect the occurrence of the same event after 10 year later.

$\Rightarrow$For ocean engineers (and naval architects)

“100 year return period” of ocean wave is a primary concern

•M year return period

N(Number of wave occurrence) = $\dfrac{time\ of\ year}{T_{mean}}$

If T_{mean} : second

$$N = \frac{M^{Year} \times 365^{Days/year} \times 24^{Hours/Day} \times 3600^{Second/Hour}}{T_{mean}}$$

e.g. M = 100

$$N = \frac{3.1536 \times 10^7}{T_{mean}}$$

•Procedure of wave amplitude prediction

$\Rightarrow$ For a specific M and $S_\eta(\omega)$

(1) Compute $m_k = \int_0^\infty \omega^k S_\eta(\omega) d\omega$

m_0, m_1, m_2 is needed.

e.g. $T_{peak} = 1.408 T_2$ or $T_2 = 2\pi\sqrt{\frac{m_0}{m_2}}$ where $T_2 = T_{mean}$

(2) $N = \frac{3.1536\times10^7}{T_{mean}}$

(3) $A = \sqrt{2m_0 \ln N}$

m_0 = 2.25, M = 100years & $T_2 = T_{mean}$ = 10 second

N = 3.1536×10^8

$A = \sqrt{2\times2.25\times\ln(3.1536\times10^8)} = 9.3841m$

$\Rightarrow$There is a possibility that the largest wave in 100 years has the amplitude of 9.384m

8.7 Short-term Prediction & Long-term Prediction

(1) Short-term Prediction
For "a" specific spectrum, we can apply the above concept
$\Rightarrow$Short-term prediction

(2) Long-term prediction

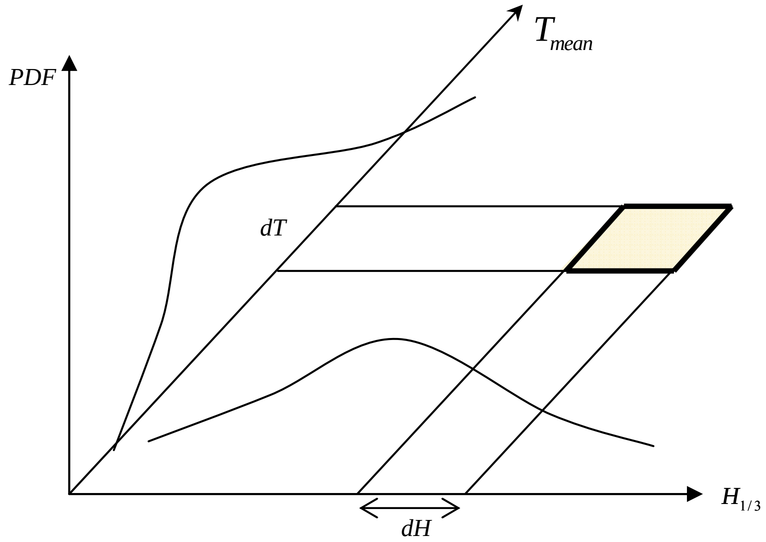
•Consider a set(table) of $S_\eta(\omega)$.

(3) For instance, for the Bretschneider spectrum, we can define a set of $S_\eta(\omega)$ s.t.

	$T_1^{(1)}$	$T_1^{(2)}$	$T_1^{(3)}$	$\cdots$	$T_1^{(I)}$
$H_{1/3}^{(1)}$					
$H_{1/3}^{(2)}$					
$H_{1/3}^{(3)}$					
$\vdots$				$\vdots$	
$H_{1/3}^{(J)}$					

LONG-TERM PREDICTION

•



We need to consider all the range of $H_{1/3}$ and T_{mean}.

•Define

$$\frac{\text{Expected Time of } \left(H_{1/3} - \frac{\Delta H}{2} < \text{H} < H_{1/3} + \frac{dH}{2} \right) \cap \left(T_{mean} - \frac{dT}{2} < \text{T} < T_{mean} + \frac{dT}{2} \right)}{\text{Total time}}$$

$$\equiv p(H_{1/3}, T_{mean}) dHdT$$

•Expected 'number' of waves for $p(H_{1/3}, T_{mean}) dHdT$

$$= \frac{\text{Total time}}{T_{mean}} p(H_{1/3}, T_{mean}) dHdT$$

•Expected 'number' of waves for $p(H_{1/3}, T_{mean})$ and $H > H_0$

$$= \frac{\text{Total time}}{T_{mean}} p(H_{1/3}, T_{mean}) dHdT \times p(H > H_0, T_{mean})$$

•For all $H_{1/3}$ and Tmean, the probability of $H > H_0$ is

$$Q = \frac{\text{Expected total number of waves for } H > H_0}{\text{Expected total number of waves}}$$

Or

$$Q = \frac{\int_0^\infty \int_0^\infty \frac{T_{total}}{T_{mean}} p(H_{1/3,} T_{mean}) p(H > H_{0,} T_{mean}) dH dT}{\int_0^\infty \int_0^\infty \frac{T_{total}}{T_{mean}} p(H_{1/3,} T_{mean}) dH dT}$$

•For Rayleigh Distribution

$$p(H > H_{0,} T_{mean}) = e^{-\frac{A_0^2}{2m_0}} = e^{-\frac{H_0^2}{gm_0}} \quad (2A_0 = H_0)$$

$$\text{(or} \quad = e^{-\frac{2H_0^2}{H_{1/3}^2}} \quad (\because H_{1/3} = 4\sqrt{m_0}))$$

•If the occurrences of $H_{1/3}$ and Tmean are independent.

$$p(H = H_{1/3,} T = T_{mean}) = p(H_{1/3}) p(T_{mean})$$

$$Q = \frac{\int_0^\infty \frac{T_{total}}{T_{mean}} p(T_{mean}) dT \int_0^\infty p(H_{1/3}) e^{-\frac{H_0^2}{8m_0}} dT}{\int_0^\infty \frac{T_{total}}{T_{mean}} p(T_{mean}) dT \underbrace{\int_0^\infty p(H_{1/3}) dT}_{=1.0}}$$

$$\Rightarrow \quad Q = \int_0^\infty p(H_{1/3}) e^{-\frac{H_0^2}{8m_0}} dH$$

$$= \int_0^\infty p(H_{1/3}) e^{-2(\frac{H_0}{H_{1/3}})^2} dH$$

•In a discrete case (e.g.table)

$$Q \approx \sum p(H_{1/3}) e^{-2(\frac{H_0}{H_{1/3}})^2} dH$$

Or

$$Q = \sum_i e^{-2(\frac{H_0}{H_{1/3i}})^2} p_i$$

Where $p_i = p(H_{1/3i}) dH$

•Numerical Implementation

	T_{mean1}	T_{mean2}	T_{mean3}	$\cdots$	$\cdots$	
$H_{1/3,1}$						$N_{H,1}$
$H_{1/3,2}$						$N_{H,2}$
$\vdots$						$\vdots$
$\sum$	$N_{T,1}$	$N_{T,2}$	$N_{T,3}$	$\cdots$		N_{Total}

$$P_i = \frac{N_{H,i}}{N_{Total}}$$

$$T_{mean} \text{ in total time } \approx \sum T_{mean,j,} P_{T,j} \equiv \overline{T}_{mean}$$

$$\text{Where } P_{T,j} = P(T_{mean,j})\Delta T$$

$$= \frac{\mathrm{N_{T,j}}}{\mathrm{N_{Total}}}$$

•Design wave height of M-year return period.

$$Q = \frac{\overline{T}_{mean}}{3.1536 \times 10^7 \times M} = \sum e^{-2(\frac{H_0}{H_{1/3,i}})^2} p_i$$

$$= \sum e^{-2(\frac{H_0}{H_{1/3,i}})^2} \frac{N_{H,i}}{N_{Total}}$$

•For a given M, we need an iteration to get a corresponding H_0.

•We can plot H_0 as a function of Q or $1 - Q = P$

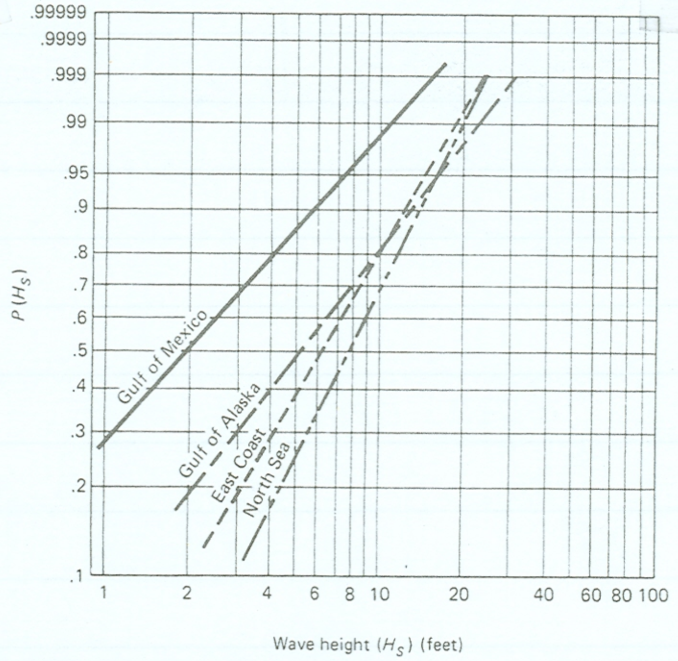


Fig. 3.33. Long-term probability distributions for several offshore areas. (After Riggs.[44])

P_{ij} = Probability of T_1 & $H_{1/3}$

•Probability of exceedance = $1-\sum_i^I \sum_j^J P(A)P_{ij}$ (?)

8.8 Other Probability Functions

(1) Weibull (Gumbel III tupe)

$$PDF = abx^{b-1}e^{-ax^b}$$

$$CPF = 1-e^{-ax^b} \qquad x \geq 0$$
$$= 0 \qquad x \prec 0$$

(2) Fretchet

(3) Gumbel type I

(4) Gamma Function …

Chapter 9. Hydrodynamic Force on Offshore Structures and Marine Vehicles

9.1. Force on a structure under wave & current action

(1) Froude-Kryloff Force ; Pure incident-wave component. Force integrated on body surface without any interaction between the body & wave.

$$\vec{F}_{F.K} = \int_{S.B.} Pincident\ wave\ ds$$

e.g. for linear waves

$$\vec{F}_{F.K} = \int_{S.B.} \underline{-P\frac{\partial \phi}{\partial t}}\ ds$$

Why?

Bernolli e.g. $\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + gz + \frac{p}{\varphi} = C(t)$

$$C(t) = gz + \frac{P_\infty}{\rho}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi = -\frac{p - p_\infty}{\rho}$$

Linearization & put

$p_\infty = 0$(∵ no disturbance in calm. There is only hydro static)

$$\Rightarrow Peinear = -p\frac{\partial \phi}{\partial t}$$

(※ Linear F.S.B.C is different form this. Why?

That's the condition on $z = 0$ which should be imposed on $z = \eta$ in exact case)

(2) Diffraction Forces ; Force due to the existence of body. The body is assumed to be at rest .(i.e. no motion.) when there is no body motion,
Force due to wave = F.K.force + Diffraction force.

(3) Radiation Forces ; Force due to moving body in calm water. In potential theory, this is mostly due to wave generation

(4) Drag Forces ; Force due to viscosity. Frictional drag, form drags are in this category. Drag force $\propto$ velocity^2

(5) Lift Forces ; Force due to non-symetrical separation or vortices on the body. Then lift force acts transversely to the velocity.

(6) Other Forces ; misc. force. e.g. nonlinear force mixed of above, higher-order forces.

•We usually group "Froude-Kryloff force + radiation force" referred them as the fluid inertia forces.

Why? Thos are related to acceleration of fluid.

•"In the absence of current", we define the Keulegan-Carpenter number s.t.

$$K.C = \frac{TV_m}{d}$$

Where

d ; body length (diameter in many cases)

T ; wave period

V_m; maximum fluid velocity

※ in linear wave theory

$$V_m = \omega A \;,\; T = \frac{2\pi}{\omega}$$

$$K.C = \frac{2\pi \not{\omega} A}{\not{\omega} d} = \frac{2\pi A}{d}$$

•Physical meaning of K.C

↓ Possible Maximum

$$K.C \approx \frac{\text{Advancing Distance of Fluid Motion in a period}}{\text{body size}}$$

•<u>small</u> ← $K.C$ → <u>big</u>

for fixed body.

Inertia is important

Viscous drag is dominant

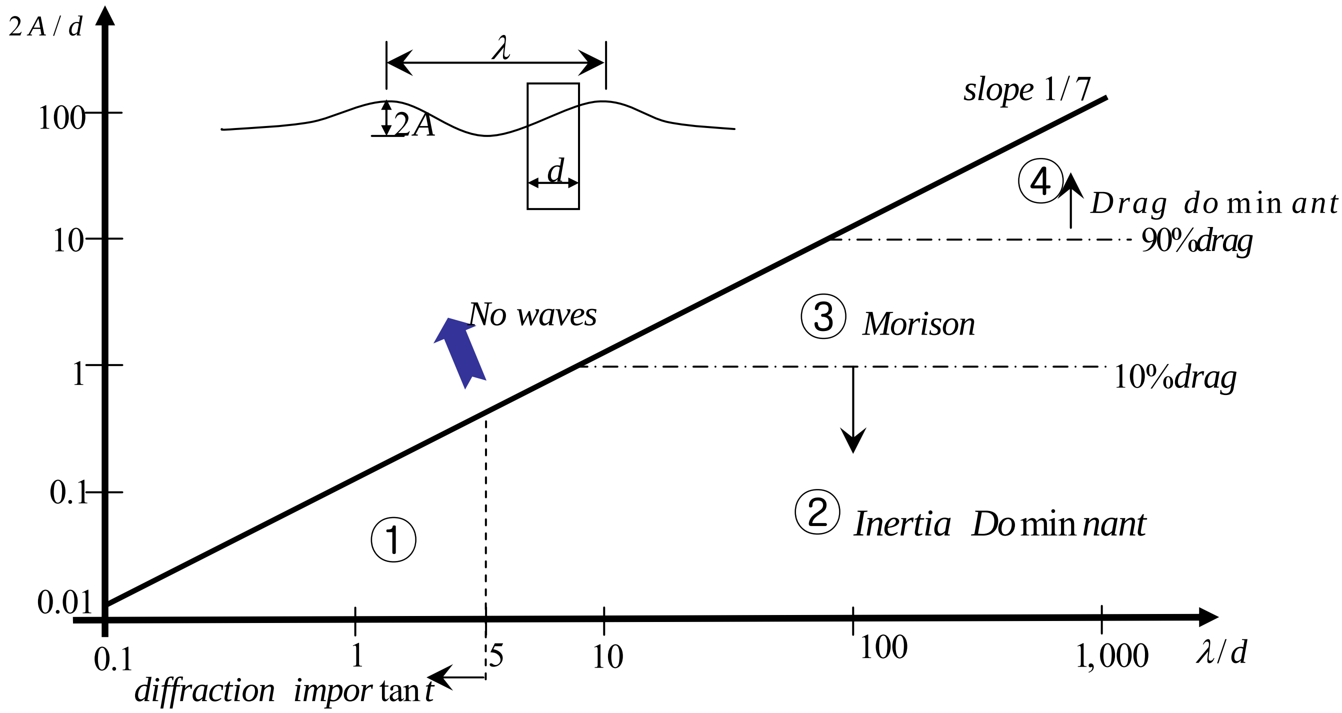


Figure4 ; Load Regimes

•Typically

① $\lambda / d \prec 5$ (or $\lambda \prec 5d$) ; Ignorable drag inertia and diffraction forces are important.

② $\lambda / d \succ 5$ and $2A/d \prec 1$ (or $\lambda \succ 5d$ & $2A \prec d$) ;

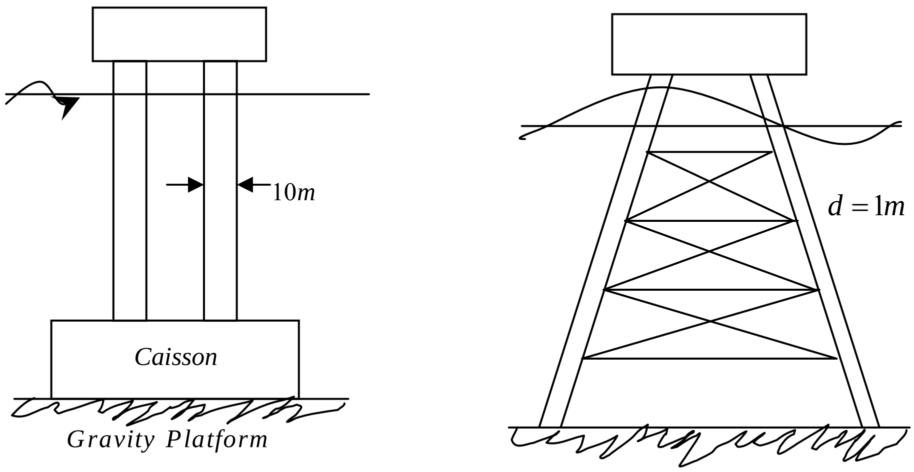
($2A$: wave height) insignificant drag and diffraction Inertia forces are important.

③ $1 \prec 2A/d \prec 10$; Both the drag and inertia forces are important.

(Morrison equation is useful.)

④ $2A/d \succ 10$ (or $2A \succ 10d$) ; The Drag forces are important.

•Example ; $\lambda = 400m,\ A = 15m$ for design wave.



(i) Caisson ; $2A/d = 0.3,\quad \lambda/d = 4 \quad \Rightarrow$ ① case

(ii) Legs ; $2A/d = 3,\quad \lambda/d = 40 \quad \Rightarrow$ ③ case

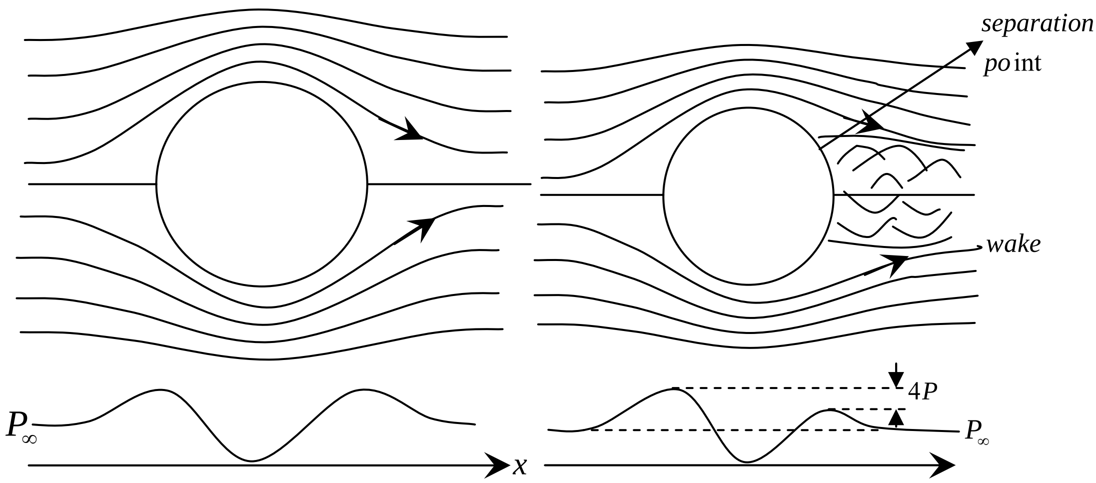
(iii) Jacket ; $2A/d = 30,\quad \lambda/d = 400 \quad \Rightarrow$ ④ case

9.2. Drag Force

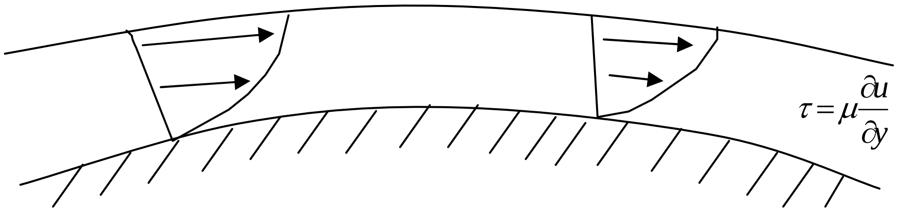
•Viscous Drag

① Pressure Drag ("from drag" in ship resistance)

② Frictional Drag



pressure no separation　　　　with separation



•Viscous Drag= function of R_n

R_n = Raynolds number = $\dfrac{VL}{\upsilon}$

•Typically we called

① subcritical : $R_n \prec 10^5$

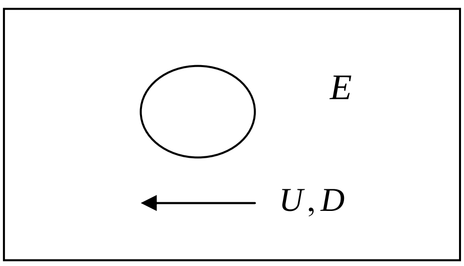
② overcritical : $R_n \succ 2 \times 10^6$

③ critical : $10^5 \prec R_n \prec 2 \times 10^6$

9.3. Morrison Equation

•Total Energy due to a moving body
= Energy of Irrotational Fluid Motion
+ Energy of (Wake) Viscous (Effort) Motion
+ Energy of Body

• (Force on the Body) • Velocity
= Rate of Energy Change



control volume

$$\frac{dE}{et} = D.U \quad \text{(D: Drag on the body)}$$

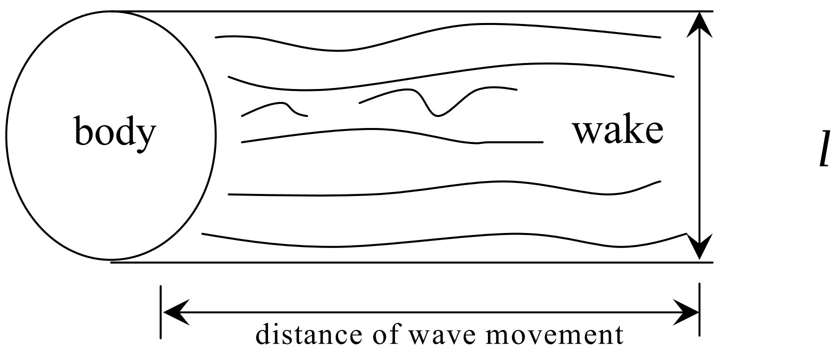
•Kinematic Energy Due to irrotational flow

$$K.E = \frac{1}{2} ma\left| u \right|^2 \qquad \text{(ma: added mass)}$$

$$\frac{d}{dt}(K.E) = ma\frac{du}{dt} \bullet u$$

(※ potential energy is not a primary concern as long as buoyancy does not change.)

•Viscous flow



$$\left| u \right| \bullet time$$

$$E_v = \frac{1}{2}\{\rho|u| \cdot t \cdot l\}u^2 C_D$$

$* C_D$: coefficient

$$\frac{dE_v}{dt} = \frac{1}{2}\rho u^2 |u| l C_D$$

•Body Motion:

$$E_{B.D.} = \frac{1}{2} mu^2$$

$$\frac{dE_{B.D.}}{dt} = mu\frac{du}{dt}$$

•Force :

$$D = \frac{1}{U}\{\frac{1}{2}\rho u^2 |u| l C_D + (m+ma)u\frac{du}{dt}\}$$

$$\frac{1}{2}\rho A u |u| C_D + (m+ma)\frac{du}{dt}$$

•Morrison EQ

$$D = \frac{1}{2}\rho C_D l u |u| + \rho C_M \forall \frac{du}{dt}$$

C_D : drag coefficient

C_M : (virtual) mass coefficient

• $C_D . C_M$ = function of

(1) $\square_n (C_D)$

(2) K.C.

(3) Shape

(4) Separation point (C_D)

(5) Roughness (C_D)

$$\bullet C_M = \begin{cases} (m+ma)/\rho\forall & \text{: moring body} \\ (\rho\forall + ma)/\rho\forall & \text{: moring fluid} \end{cases}$$

•For a 2-D circular cylinder,

$$D = \frac{1}{2}\rho C_D d u |u| + \rho C_M \frac{\pi}{4} d^2 \frac{du}{dt}$$

* d :diameter

•Application for force computation

(1) Select an appropriate Wave Theory.
Predict velocity of flow.

(2) Select the set of Appropriate $C_M . C_D$

(3) Apply Morrison E_q

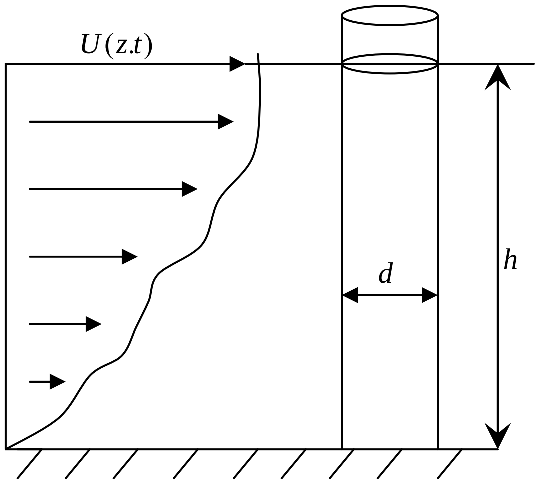
•For linear waves (cylinder)

$C_D = 1 \sim 1.4,\ \ C_M = 2.0$

For nonlinear waves

$C_D = 0.8 \sim 1.0,\ \ C_M = 2.0$

9.4. Force on Cylinder

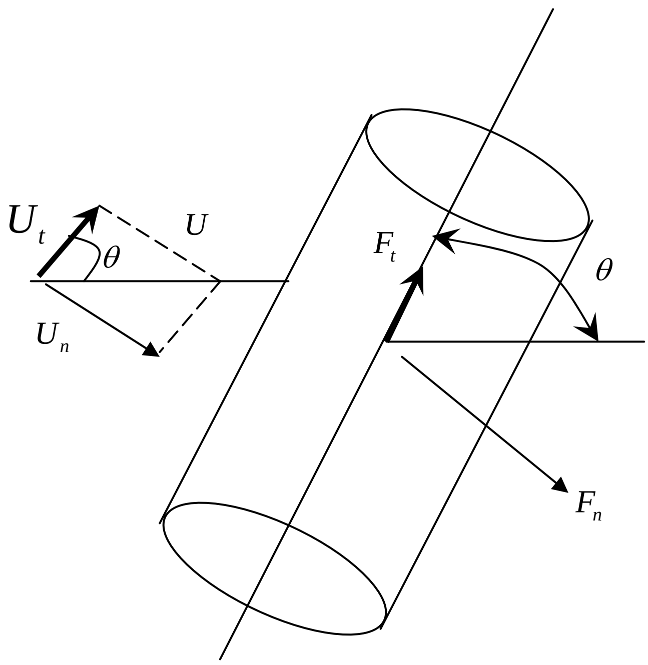


• $F(t) = \int_{-h}^{0} dF dz$

• $M(t) = \int_{0}^{h} zF dz$: moment w.r.t. sea floor

• $dF = C_M \rho \dfrac{\pi d^2}{4} \dfrac{du}{dt} + \dfrac{1}{2} C_D \rho d u |u|$

•Forces on inclined cylinders.



$U_n = U \sin\theta$

$U_t = U \cos\theta$

$$dF = C_M \rho \frac{\pi d^2}{4} U_n + C_D \frac{1}{2} \rho d u_n |u_n|$$

$$dF = \frac{1}{2} \rho C_f \pi d |u_t| u_t$$

$* C_f$: frictional coeff.

• $F_n = \int_0^h dF_n dz$

$F_n = \int_0^h dF_t dz$

※ Strip method; We assume that the interactions between sections are ignorable.

9.5. Forces on Cylinders in Waves

(1) Inertia force

$$F_I = \int_{-h}^{0} -\rho C_M \frac{\pi d^2}{4} A\omega^2 \frac{\cosh\ k(z+h)}{\sin\ kh} \sin \omega t dz$$

$$=- - \rho \frac{\pi d^2}{4} \omega^2 A \frac{1}{k} \sin(\omega t)$$

$$= - F_{I.O} \sin(\omega t)$$

where $F_{I.O} = \rho \frac{\pi d^2}{4} \frac{\omega^2 A}{k}$

(2) Drag Force

$$F_D = \int_{-h}^{0} \frac{1}{2}\rho C_D d(\omega A)^2 \frac{\cosh^2 \ k(z+h)}{\sinh^2 \ kh} \cos(\omega t)\left|\cos(\omega t)\right| dz$$

$$= \frac{1}{2}\rho C_D d(\omega A)^2 \frac{\{s\,\text{inh}(2kh)+2kh\}}{4k \ \sinh^2(kh)} \cos(\omega t)\left|\cos(\omega t)\right|$$

$$= \mathrm{F_D} \cos\omega t \left|\cos\omega t\right|$$

※

$$u = \frac{\partial \phi_2}{\partial x} = \frac{gA}{\omega} k \frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\omega t) = \omega A \frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\omega t)$$

$$\frac{du}{dt} = \frac{gA}{\omega}\cdot\omega\cdot\frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t) = \omega^2 A \frac{\cosh k(z+h)}{\cosh kh}\sin(kx-\omega t)$$

※ We will consider u & $\frac{du}{dt} at \cdot x = 0$

•Ratio of maximum forces

$$\frac{F_{D.O}}{F_{I.O}} = \frac{C_D}{\pi C_M}\frac{A}{d}\{\frac{\sinh(2kh)}{2}+kh\}\frac{1}{\sinh^2 kh}$$

$\Rightarrow$ Depend on $\frac{C_D}{C_M}$, $\frac{A}{d}$

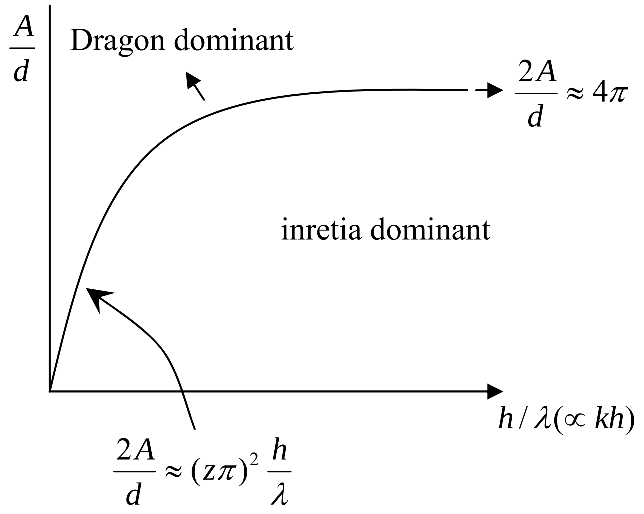
$\Rightarrow$ When $\frac{A}{d}$ is small, $F_{D.O} \succ F_{I.O}$

•Typically C_M =2, $C_D = 1$

When $\frac{F_{D.O}}{F_{I.O}} = 1.0$?

(i) $kh \to \infty$; $\frac{A}{d} \approx 2\pi$ or $\frac{2A}{d} \approx 4\pi$

(ii) $kh \to 0$; $\frac{2A}{d} \approx (2\pi)^2 \frac{h}{\lambda}$



A / d
Dragon dominant
2A / d ≈ 4π
inretia dominant
h / λ(∝ kh)
2A / d ≈ (zπ)^2 h / λ