Electromagnetics

<Chap. 1~ Chap. 6> Static electric & magnetic fields

Course Intro

(1st class of **week 1**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Course Intro | Contents for 1st class of week 1

1. Syllabus

2. A Review of *"Introduction to electromagnetism with practice"* (기초전자기학 및 연습; 430.202B)

    - Mathematical basis: vector calculus and important theorem (Ch. 2)
    - Electrostatics (Static Electric Field) (Ch. 3)
    - Magnetostatics (Static Magnetic Field) (Ch. 6)

Course Intro | Syllabus (1/2)

Lecture: Electromagnetics (전자기학; 430.203A-002)
* *Prerequisite*: 기초전자기학 및 연습 (430.202B), 공학수학 1, 2 (or any equivalent courses)
* Course will be offered in English.

Staff:
- ***Instructor:*** Jaesang Lee (email: jsanlgee@snu.ac.kr)
  - Office: 301-906 (Office hour: Mon/Wed/Fri 2:00-3:00 pm)
- ***Course TA:*** 양광모 (kwangmo95@snu.ac.kr)
- 전자기학 학습도우미: 최선진 (csj7481@snu.ac.kr)

Textbook: D. K. Cheng, "*Field and Wave Electromagnetics*", 2nd Ed. Addison-Wesley, 1989.

Homework:
- Total 7 sets
- A problem set (HW) will be given approximately ***every two weeks at the end of Thursday class***.
- HW deadline: one week after assignment
  - Submit it to TA at the end of the class
  - Drop it in the HW submission box at my office (301-906) on Thursday until 6 pm
- No late homework will be accepted unless special occasion.

Exam: ***Two*** midterm, ***One*** Final Exams

Grading Policy:
Attendance (10 %)
Homework (15 %)
Midterm I (20 %)
Midterm II (25 %)
Final (30 %)

Course Intro | Syllabus (2/2)

Description

기초전자기학 및 연습

- **static** electric fields (Ch. 3), **static** magnetic fields (Ch. 6) / Vector calculus (Ch. 2)

전자기학

Principle objective: Understand interaction between charges and currents at a distance based on EM model

- Time-varying electric & magnetic fields and their coupling → **Maxwell's equations** (Ch. 7)
- A particular solution to Maxwell's equation: **plane electromagnetic waves** (Ch. 8)
- How electromagnetic waves propagate in various media (i.e. ***transmission lines*** and ***waveguides***) (Ch. 9 and 10)
- Wireless and long-distance propagation of electromagnetic waves (***Antennas***) (Ch. 11)

Schedule

Week	Topic	Reading	HW / Exam
1	Introduction / Review of static EM fields	Ch. 1~6	
2	Maxwell's Equations	Ch. 7	HW1
3	Plane Electromagnetic Waves I	Ch. 8	
4	Plane Electromagnetic Waves II	Ch. 8	HW2
5	Plane Electromagnetic Waves III	Ch. 8	**Midterm I**
6	Waveguides I	Ch. 10	
7	Waveguides II	Ch. 10	HW3
8	Waveguides III / Intro of Transmission Lines	Ch. 9~10	
9	Transmission Lines I	Ch. 9	HW4
10	Transmission Lines II	Ch. 9	**Midterm II**
11	Transmission Lines III	Ch. 9	
12	Transmission Lines IV	Ch. 9	HW5
13	Antennas I	Ch. 11	
14	Antennas II	Ch. 11	HW6
15	Antennas III	Ch. 11	**Final**

Course Intro | Electromagnetics

What is electromagnetics?

Study of electric & magnetic phenomena caused by static or moving electric charges

- Source of Electric fields - positive & negative electric charges
- Source of Magnetic fields - moving charges (i.e. currents)

Field: Spatial distribution of a quantity

- a function of space coordinates: (x, y, z) or (r, ϕ, z) or (r, θ, ϕ)
- may or may not be a function of time

What did we learn previously?

Electrostatics
When electrons are at **rest**

Static Electric Field

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \end{cases}$$

Magnetostatics
When electrons move at a **constant** speed

Static Magnetic Field

$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \end{cases}$$

Only functions of space: $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{H}(x, y, z)$
Independently defined!

What will we learn?

Electromagnetics
When movement of electrons changes with **time**

Time-varying, Coupled Electric Field & Magnetic Field
= Electromagnetic Field

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \end{cases} \qquad \begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t} \end{cases}$$

Functions of both space and time: $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{H}(x, y, z, t)$
Coupled!

Course Intro | Must-know vector calculus

Gradient

= maximum space rate of change of the *scalar field*

normal direction

$$\nabla V \triangleq \boldsymbol{a}_{\max(dV/dl)} \max\left(\frac{dV}{dl}\right) = \boldsymbol{a}_{\boldsymbol{n}} \frac{dV}{dn}$$

$$\triangleq \boldsymbol{a}_x \frac{\partial V}{\partial x} + \boldsymbol{a}_y \frac{\partial V}{\partial y} + \boldsymbol{a}_z \frac{\partial V}{\partial z}$$

$$dV = (\nabla V) \cdot d\boldsymbol{l}$$

Vector

Divergence

= The net **outward flux** of *vector field* per unit volume as the volume about the point tends to zero

$$\nabla \cdot \boldsymbol{E} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \boldsymbol{E} \cdot d\boldsymbol{s}}{\Delta v}$$

$$\triangleq \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Scalar

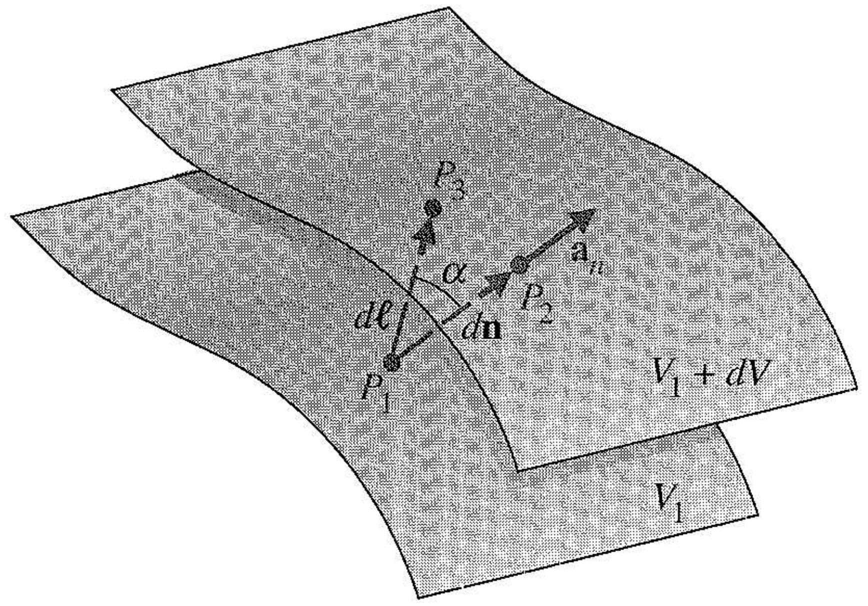
Curl

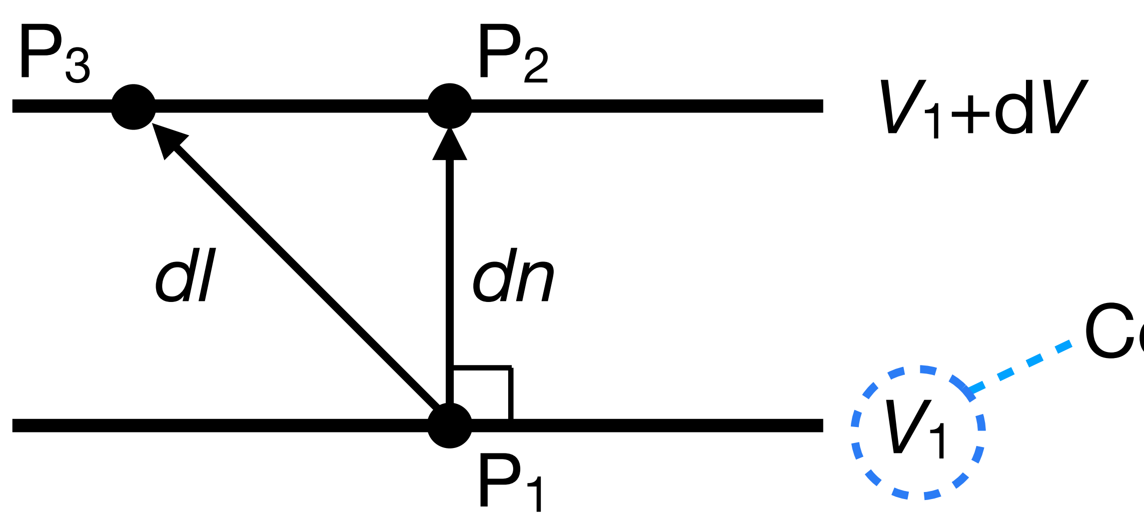
= Maximum net **circulation** of *vector field* per unit area as the area tends to zero

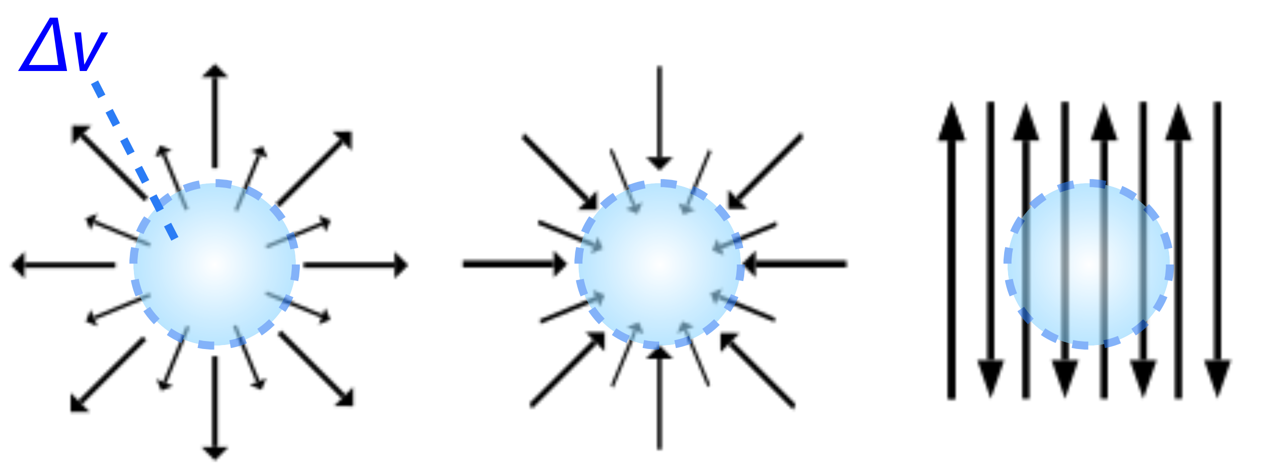
$$\nabla \times \boldsymbol{A} \triangleq \boldsymbol{a}_{\boldsymbol{n}} \lim_{\Delta s \to 0} \frac{\oint_C \boldsymbol{A} \cdot d\boldsymbol{l}}{\Delta s}$$

$$\triangleq \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

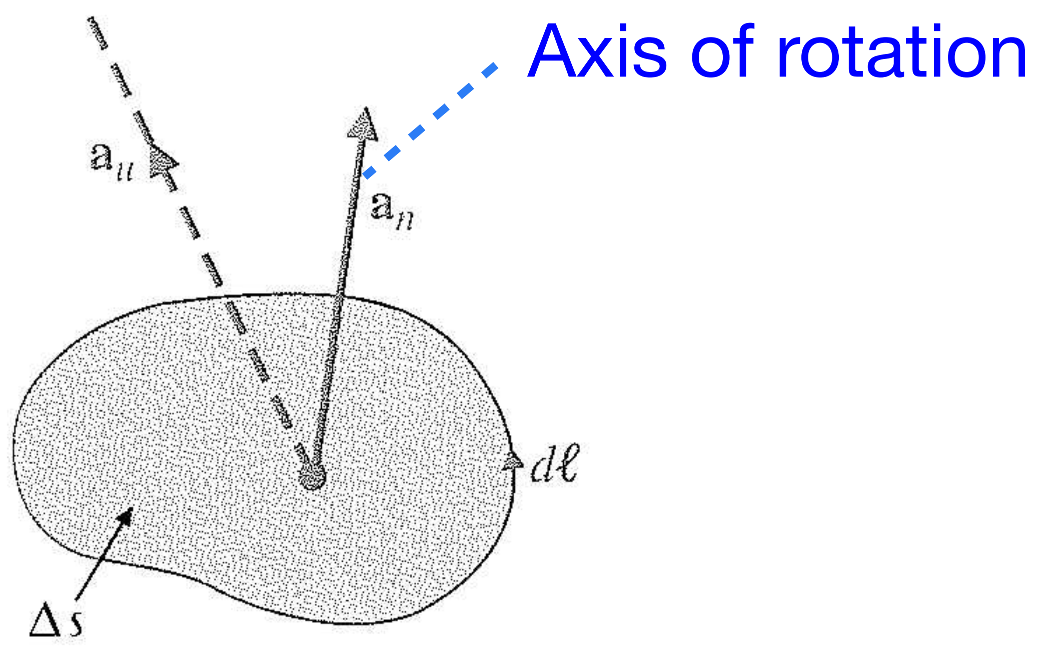
Vector







$\nabla \cdot \boldsymbol{E} > 0$ $\nabla \cdot \boldsymbol{E} < 0$ $\nabla \cdot \boldsymbol{E} = 0$



Course Intro | Mathematical basis for electromagnetics

Helmholtz's theorem

- Any vector field can be decomposed into irrotational (curl-free) and solenoidal (divergence-free) vector fields.
- A vector field can be completely determined if both its divergence and curl are specified everywhere.

Null Identity I

$$\boldsymbol{F} = \boldsymbol{F}_i + \boldsymbol{F}_s \text{ where } \begin{cases} \nabla \times \boldsymbol{F}_i = 0 & \rightarrow \boxed{\nabla \times (-\nabla V) = 0} \rightarrow \boldsymbol{F}_i = -\nabla V \\ \nabla \cdot \boldsymbol{F}_s = 0 & \rightarrow \boxed{\nabla \cdot (\nabla \times \boldsymbol{A}) = 0} \rightarrow \boldsymbol{F}_s = \nabla \times \boldsymbol{A} \end{cases}$$

$\boldsymbol{F}_i = -\nabla V$: Irrotational vector (by **flow** source)

$\boldsymbol{F}_s = \nabla \times \boldsymbol{A}$: Solenoidal vector (by **vortex** source)

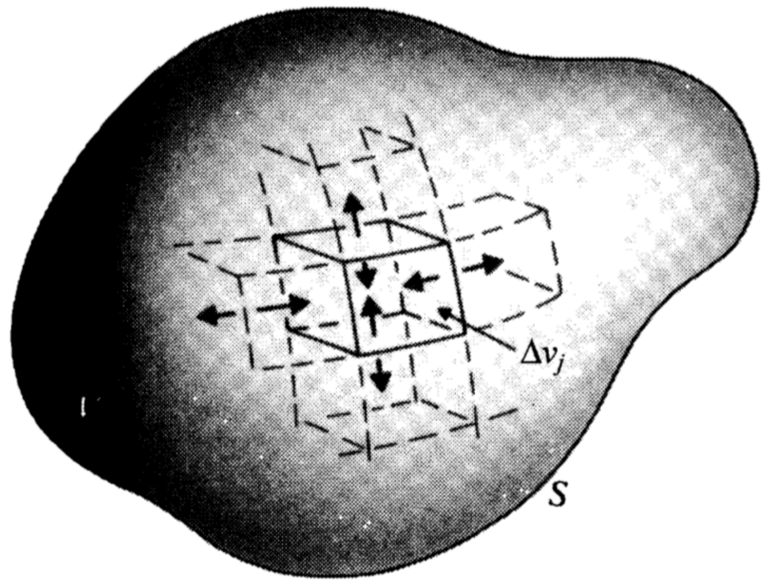
Null Identity II

Stokes theorem

$$\int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{E} \cdot d\boldsymbol{l}$$

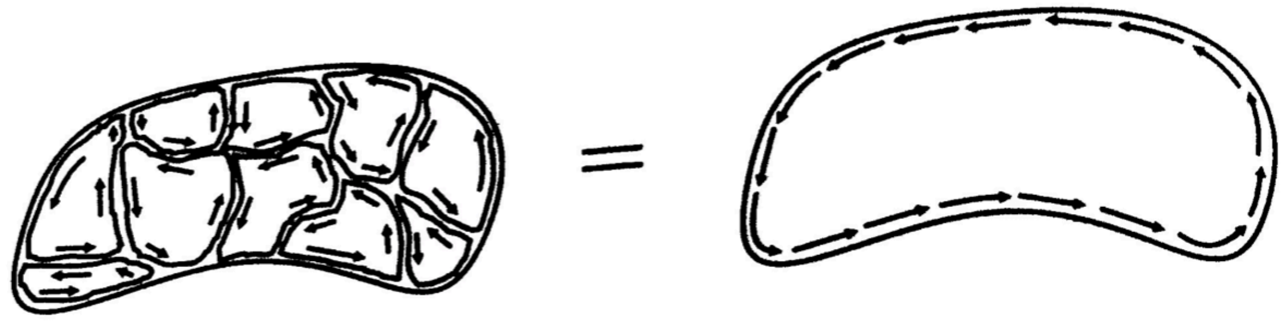
Divergence theorem

$$\int_V \nabla \cdot \boldsymbol{D}\, dv = \oint_S \boldsymbol{D} \cdot d\boldsymbol{s}$$



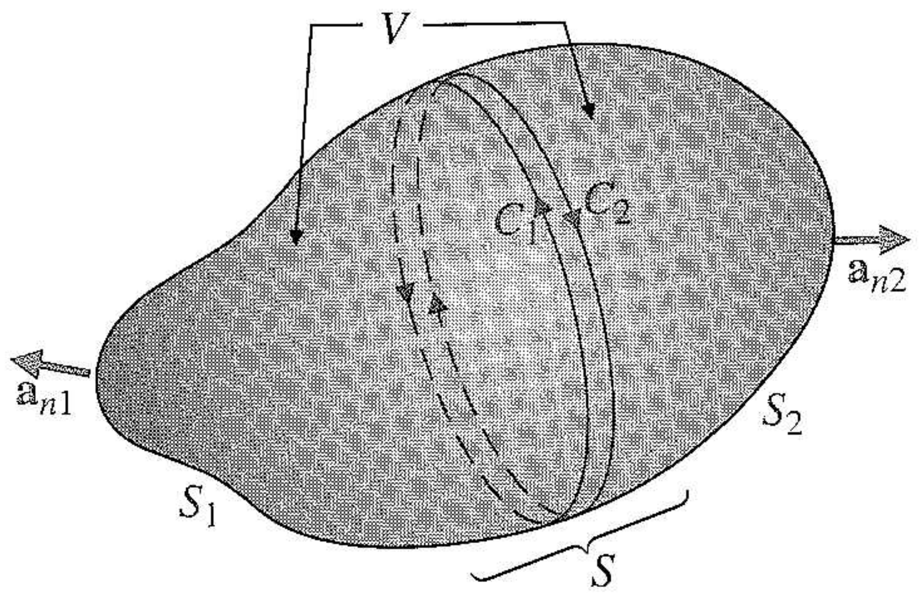
Proof of Null Identity I

$$\int_S [\nabla \times (-\nabla V)] \cdot d\boldsymbol{s} = -\oint_C \nabla V \cdot d\boldsymbol{l} = \oint_C dV = 0$$



Proof of Null Identity II

$$\begin{aligned} \int_V \nabla \cdot (\nabla \times \boldsymbol{A})\, dv &= \oint_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} \\ &= \oint_{S_1} (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{a}_{n1}\, ds + \oint_{S_1} (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{a}_{n2}\, ds \\ &= \oint_{C_1} \boldsymbol{A} \cdot d\boldsymbol{l} + \oint_{C_2} \boldsymbol{A} \cdot d\boldsymbol{l} = 0 \end{aligned}$$



Course Intro | 4 fundamental field quantities and 4 postulates

Field quantities

	Name	Symbol	Unit
Electric	Electric field intensity	$\boldsymbol{E}$	V/m
	Electric flux density	$\boldsymbol{D}$	C/m^2
Magnetic	Magnetic field intensity	$\boldsymbol{H}$	A/m
	Magnetic flux density	$\boldsymbol{B}$	T

Needed for evaluating the fields in the medium

Fundamental postulates

Source: free charge

Electrostatics
$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \end{cases}$$
where in the medium, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

Polarization vector of "induced" electric dipoles

Magnetostatics
$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \end{cases}$$
where in the medium, $\boldsymbol{H} = \dfrac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$

Magnetization vector of "induced" magnetic dipoles

Source: current density

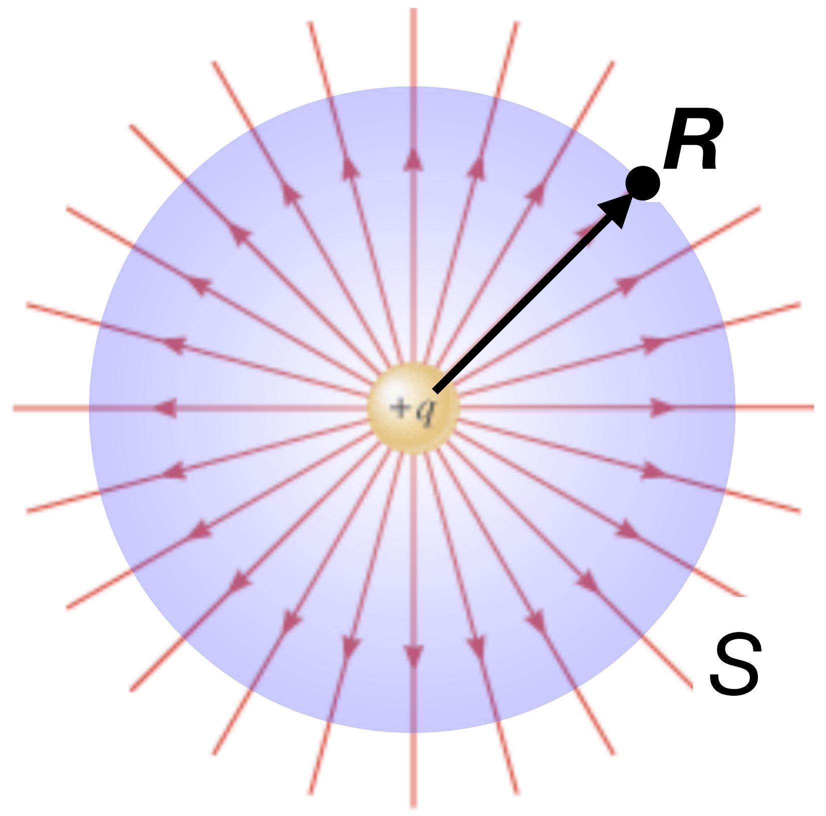
Course Intro | Coulomb's law and E-field

E-field intensity $\boldsymbol{E}$ at $\boldsymbol{R}$ due to a positive charge q at ***origin***

$$\boldsymbol{E} = \boldsymbol{a_R} E_R = \boldsymbol{a_R} \frac{q}{4\pi\varepsilon_0 R^2} \text{ (V/m)}$$

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{q}{\varepsilon_0}$$: fundamental postulate in free space

$$\text{(l.h.s)} \oint_S (\boldsymbol{a_R} E_R) \cdot (\boldsymbol{a_R} ds) = E_R \oint_S ds = 4\pi R^2 E_R$$



E-field intensity $\boldsymbol{E}$ at $\boldsymbol{R}$ due to a positive charge q at $\boldsymbol{R'}$

$$\boldsymbol{E}(\boldsymbol{R}) = \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{R} - \boldsymbol{R}'}{|\boldsymbol{R} - \boldsymbol{R}'|^3} = \boldsymbol{a_q} \frac{q}{4\pi\varepsilon_0} \frac{1}{|\boldsymbol{R} - \boldsymbol{R}'|^2} \quad \text{where} \quad \boldsymbol{a_q} = \frac{\boldsymbol{R} - \boldsymbol{R}'}{|\boldsymbol{R} - \boldsymbol{R}'|}$$

Coulomb's law

$$\boldsymbol{F} = q_2 \boldsymbol{E}_1(\boldsymbol{R_2}) = \frac{q_2 q_1}{4\pi\varepsilon_0} \frac{\boldsymbol{R}_2 - \boldsymbol{R}_1}{|\boldsymbol{R}_2 - \boldsymbol{R}_1|^3} = \boldsymbol{a_{21}} k_e \frac{q_2 q_1}{|\boldsymbol{R}_2 - \boldsymbol{R}_1|^2}$$

:Attractive or repulsive force acting between two point charges

E-field due to a **group** of n discrete charges

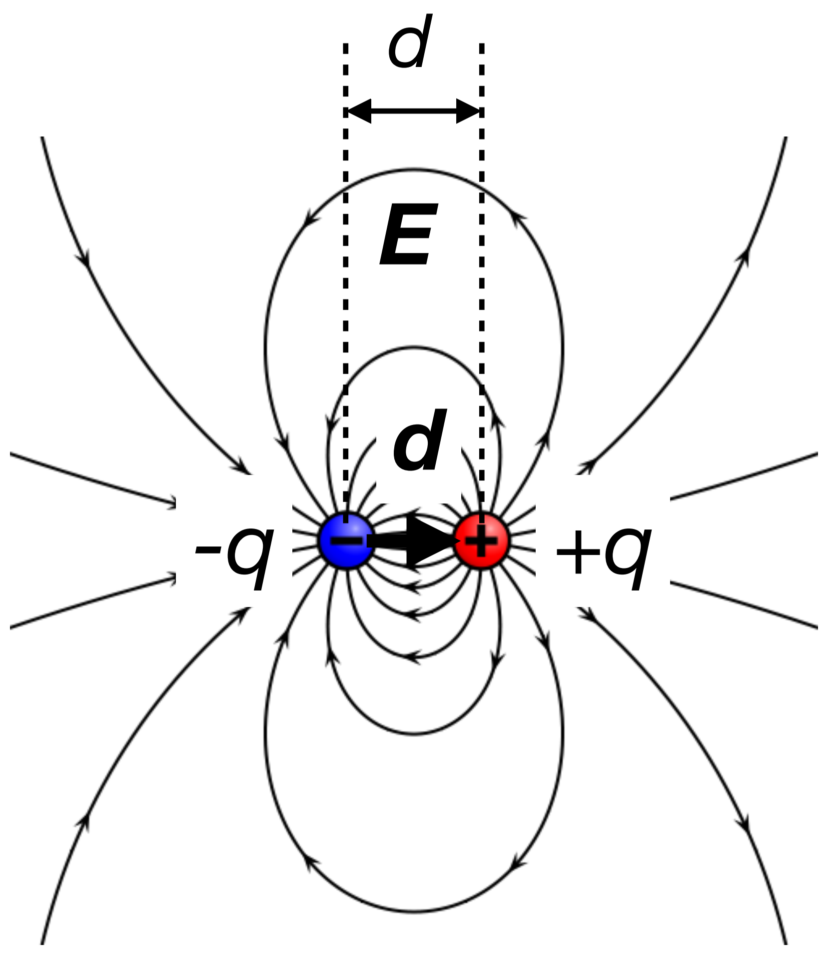
$$\boldsymbol{E}(\boldsymbol{R}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} q_k \frac{\boldsymbol{R} - \boldsymbol{R}_k'}{|\boldsymbol{R} - \boldsymbol{R}_k'|^3}$$

E-field due to an **electric dipole** (Ch. 3-3.1)

$$\boldsymbol{E}(\boldsymbol{R}) \cong \frac{1}{4\pi\varepsilon_0 R^3} \left[3 \frac{\boldsymbol{R} \cdot \boldsymbol{p}}{R^2} \boldsymbol{R} - \boldsymbol{p} \right]$$

where $\boldsymbol{p} = q\boldsymbol{d}$

Electric Dipole Moment



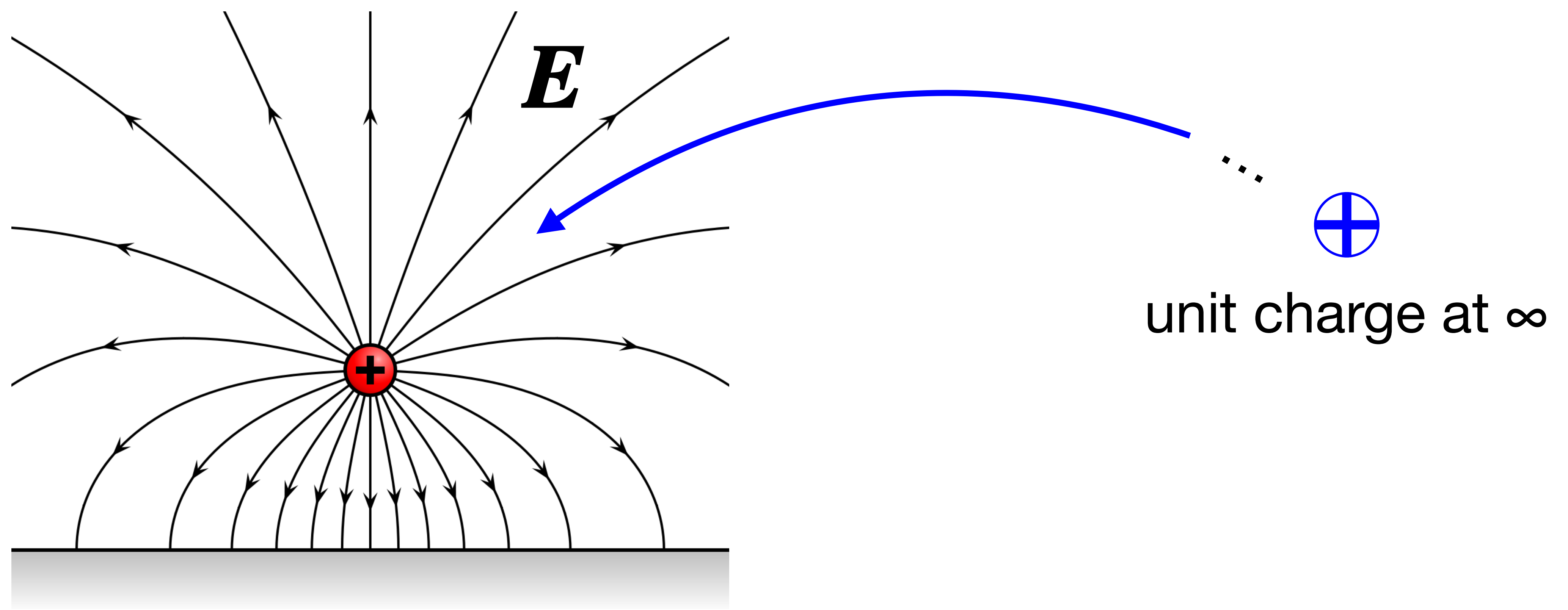
Course Intro | Electric Potential

Definition

Work required to move a **unit positive charge** from the reference point (usually, **infinity**) to a specific point **against** the E-field

$$\frac{W}{q}\ (\mathrm{J/C}) = -\int_\infty^R \boldsymbol{E}\cdot d\boldsymbol{l} = \int_\infty^R \nabla V\cdot d\boldsymbol{l} = \int_\infty^R dV = V_R\ (\mathrm{V})$$

Since $\nabla\times\boldsymbol{E}=0$, $\boxed{\boldsymbol{E}=-\nabla V}$ according to ***Null Identity I***



V due to a point charge

$$V = -\int_\infty^R \boldsymbol{E}\cdot d\boldsymbol{l}$$
$$= \int_\infty^P \left(\boldsymbol{a_R}\frac{q}{4\pi\varepsilon_0 R^2}\right)\cdot(\boldsymbol{a_R}dR)$$
$$= \frac{q}{4\pi\varepsilon_0 R}$$

V due to n discrete charges

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{|\boldsymbol{R}-\boldsymbol{R}'_k|}$$

V due to electric dipole

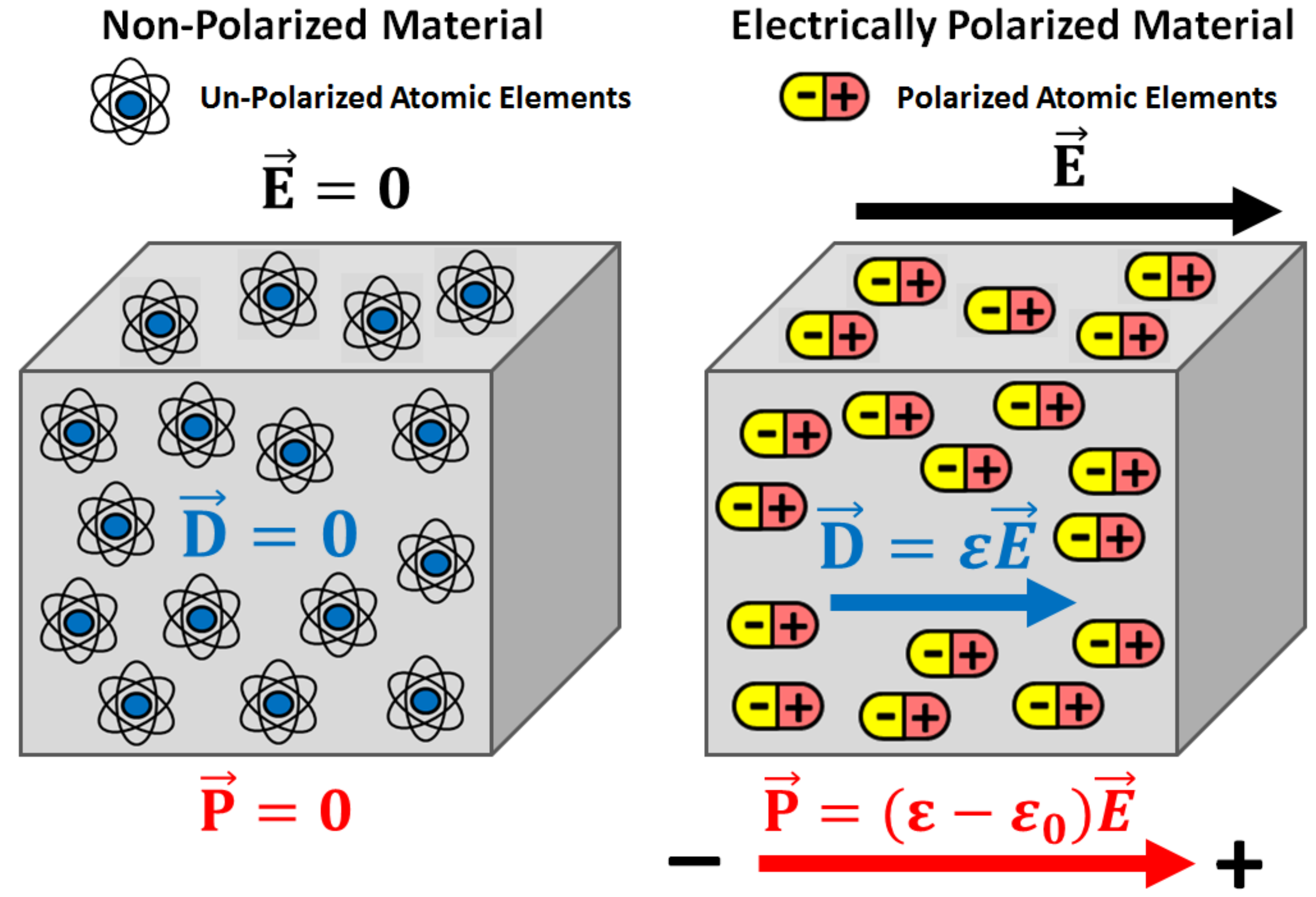
$$V = \frac{\boldsymbol{p}\cdot\boldsymbol{a_R}}{4\pi\varepsilon_0 R^2}$$

V due to continuous distribution

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho}{R}dv' \quad \text{(volume)}$$
$$V = \frac{1}{4\pi\varepsilon_0}\int_{S'}\frac{\rho_S}{R}ds' \quad \text{(surface)}$$
$$V = \frac{1}{4\pi\varepsilon_0}\int_{L'}\frac{\rho_L}{R}dl' \quad \text{(line)}$$

Electric Flux Density, *D* (1/2)

Dielectric



- **Electrical insulator** that can be **electrically polarized** by an **applied E-field**
  - Electric charges are still bound, but **slightly displaced** from their equilibrium positions
- Such displacements **polarize a dielectric material** and create **electric dipole**
- Induced electric dipoles modify E-field both inside and outside the dielectric

Polarization vector

$$\boldsymbol{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{p}_k}{\Delta v}$$

Polarization density vector

: volume density of permanent or induced electric dipole moments in a dielectric

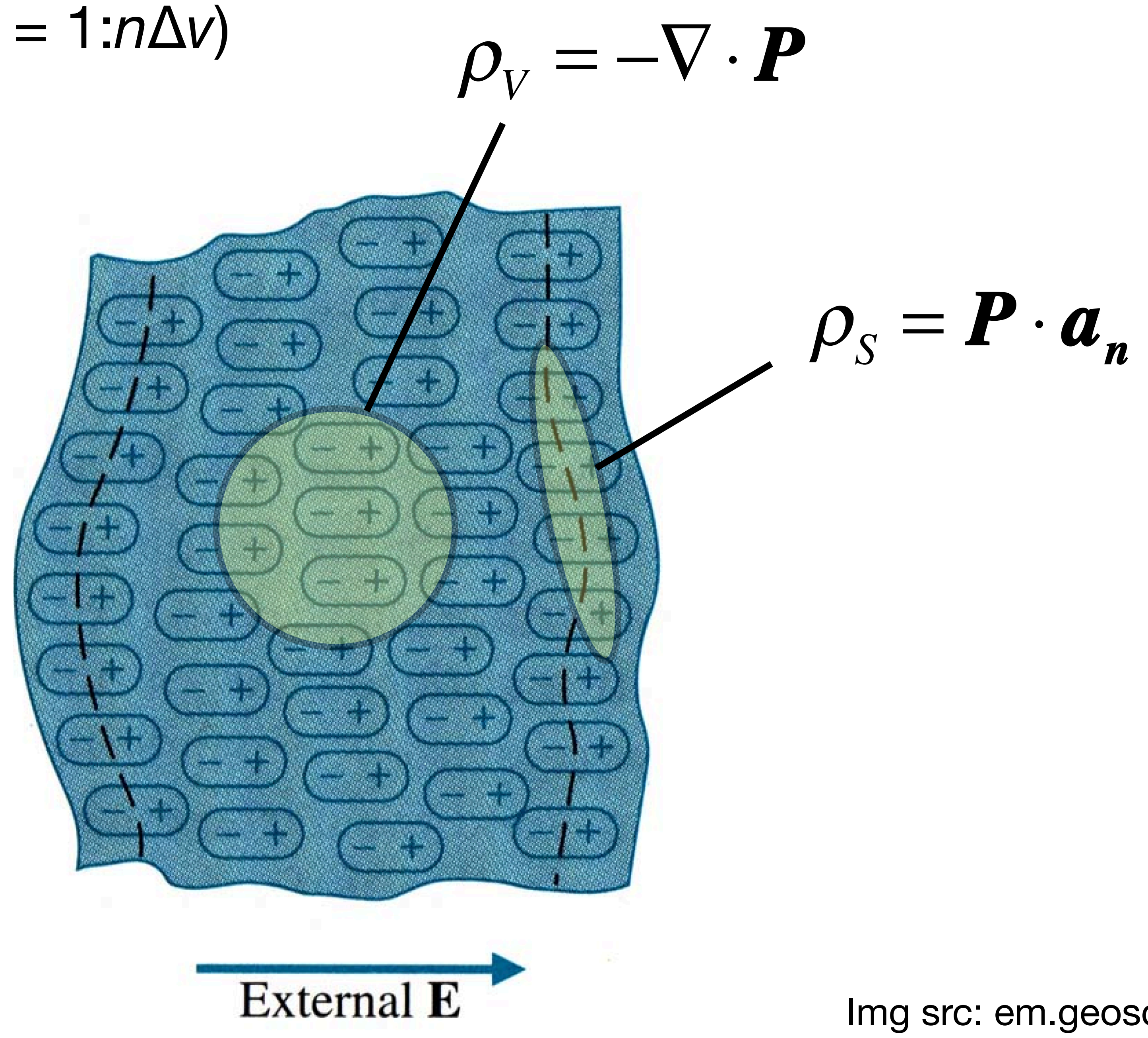
P indicates macroscopic effects of all the induced dipoles, $\boldsymbol{p}_k = q\boldsymbol{d}_k$ (k = 1:$n\Delta v$)

V due to polarized dielectric

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\boldsymbol{P}\cdot\boldsymbol{a}_R}{R^2}dv' \quad \left(\because V = \frac{\boldsymbol{p}\cdot\boldsymbol{a}_R}{4\pi\varepsilon_0 R^2}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint_{S'} \frac{\boldsymbol{P}\cdot\boldsymbol{a}'_n}{R}ds'\right] + \frac{1}{4\pi\varepsilon_0}\left[\int_{V} \frac{-\nabla'\cdot\boldsymbol{P}}{R}dv'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint_{S'} \frac{\rho_S}{R}ds'\right] + \frac{1}{4\pi\varepsilon_0}\left[\int_{V} \frac{\rho_V}{R}dv'\right]$$ (refer to 3-7.1 for derivation)



Img src: em.geosci.xyz

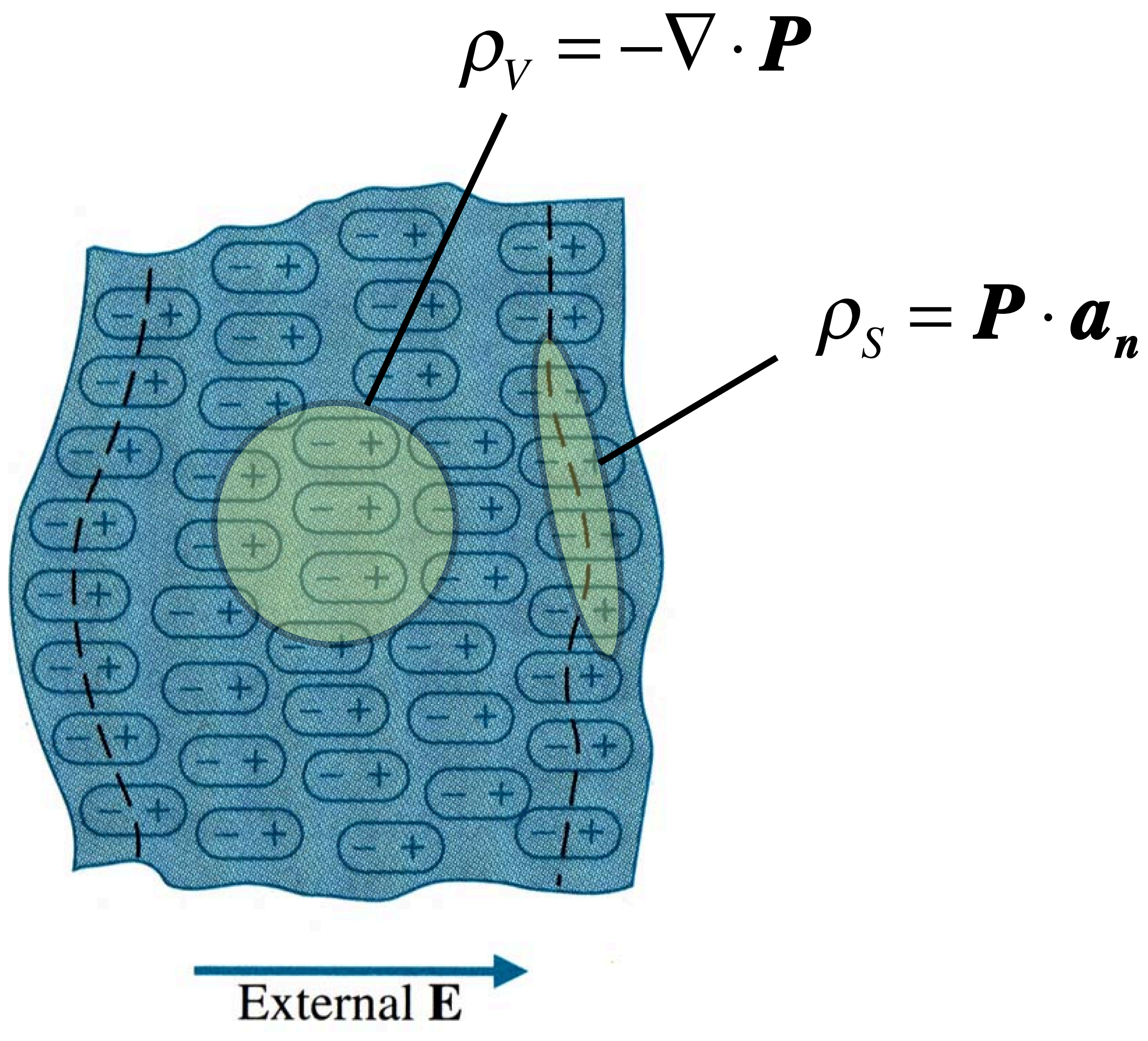
Electric Flux Density, *D* (2/2)

Divergence of *E* "in the dielectric"

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}(\rho + \rho_V)$$

E: electric field intensity "in the dielectric"
ρ: free charge density
ρ_V: polarized charge density

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho, \quad (\because \rho_V = -\nabla \boldsymbol{P})$$



Electric Flux Density, *D*

$$\boldsymbol{D} \triangleq \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \qquad (\mathrm{C/m^2})$$

Divergence postulate "in any medium"

$$\nabla \cdot \boldsymbol{D} = \rho \qquad (\mathrm{C/m^3})$$

Permittivity

For **linear and isotropic** medium,

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$$ where χ_e is electric susceptibility

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon_0 \varepsilon_r \boldsymbol{E} = \varepsilon \boldsymbol{E}$$

(i.e. Material specific)

Constitutive relation

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$\varepsilon = \varepsilon_r \varepsilon_0$: Absolute permittivity (F/m)

$\varepsilon_r = 1 + \chi_e$: Relative permittivity (dimensionless)
= Dielectric constant of the medium

Two fundamental postulates (repeated)

Differential form		Integral form		
$\nabla \cdot \boldsymbol{D} = \rho$	$\xrightarrow{\text{Divergence theorem}}$	$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q$	Gauss's Law	Total **outward** flux of ***D*** density over **any closed surface S** in any medium equals to the total charge Q enclosed in that surface.
$\nabla \times \boldsymbol{E} = 0$	$\xrightarrow{\text{Stoke's theorem}}$	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$	Kirchhoff's voltage law	Scalar line integral of ***E*** (=voltage) vanishes around any closed path = Kirchhoff's voltage law

Divergence theorem

$$\left[\int_V \nabla \cdot \boldsymbol{D}\, dv = \oint_S \boldsymbol{D} \cdot d\boldsymbol{s}\right] = \left[\int_V \rho\, dv = Q\right]$$

Stokes theorem

$$\left[\int_S \nabla \times \boldsymbol{E}\, d\boldsymbol{s} = \oint_C \boldsymbol{E} \cdot d\boldsymbol{l}\right] = 0$$

Course Intro | Boundary Conditions

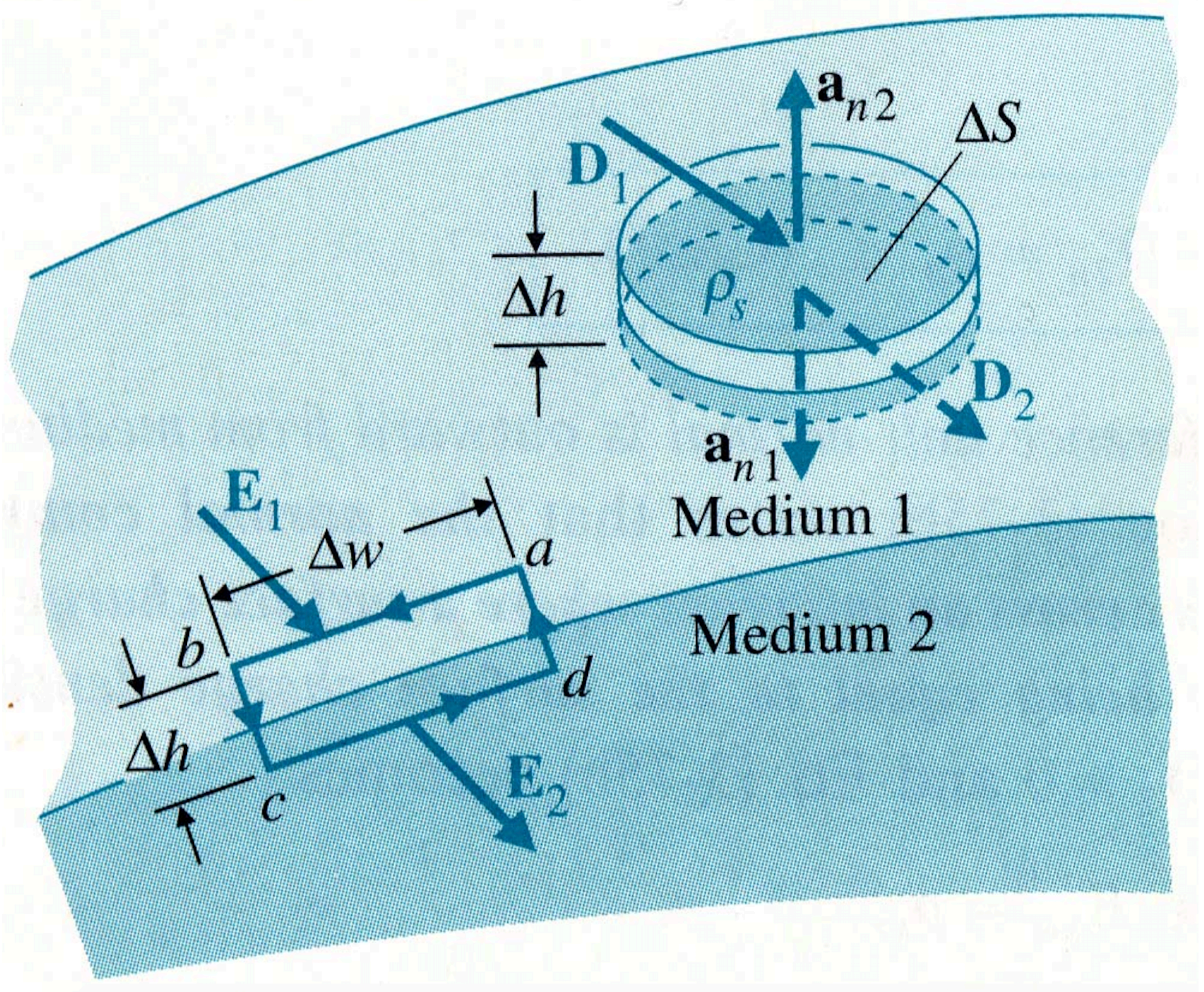
Boundary condition (B.C.)

Tangential component

$$E_{1t} = E_{2t} \quad \text{(V/m)} \qquad \left(\because \oint_{\text{abc da}} \boldsymbol{E}\cdot d\boldsymbol{l} = \boldsymbol{E_1}\cdot\Delta\boldsymbol{w} + \boldsymbol{E_2}\cdot(-\Delta\boldsymbol{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0\right)$$

Normal component

$$\boldsymbol{a_{n2}}\cdot(\boldsymbol{D_1} - \boldsymbol{D_2}) = \rho_s \qquad \left(\because \oint_S \boldsymbol{D}\cdot d\boldsymbol{s} = (\boldsymbol{D_1}\cdot\boldsymbol{a_{n2}} + \boldsymbol{D_2}\cdot\boldsymbol{a_{n1}})\Delta S = \rho_s\Delta S\right)$$



Conductors in a static *E*

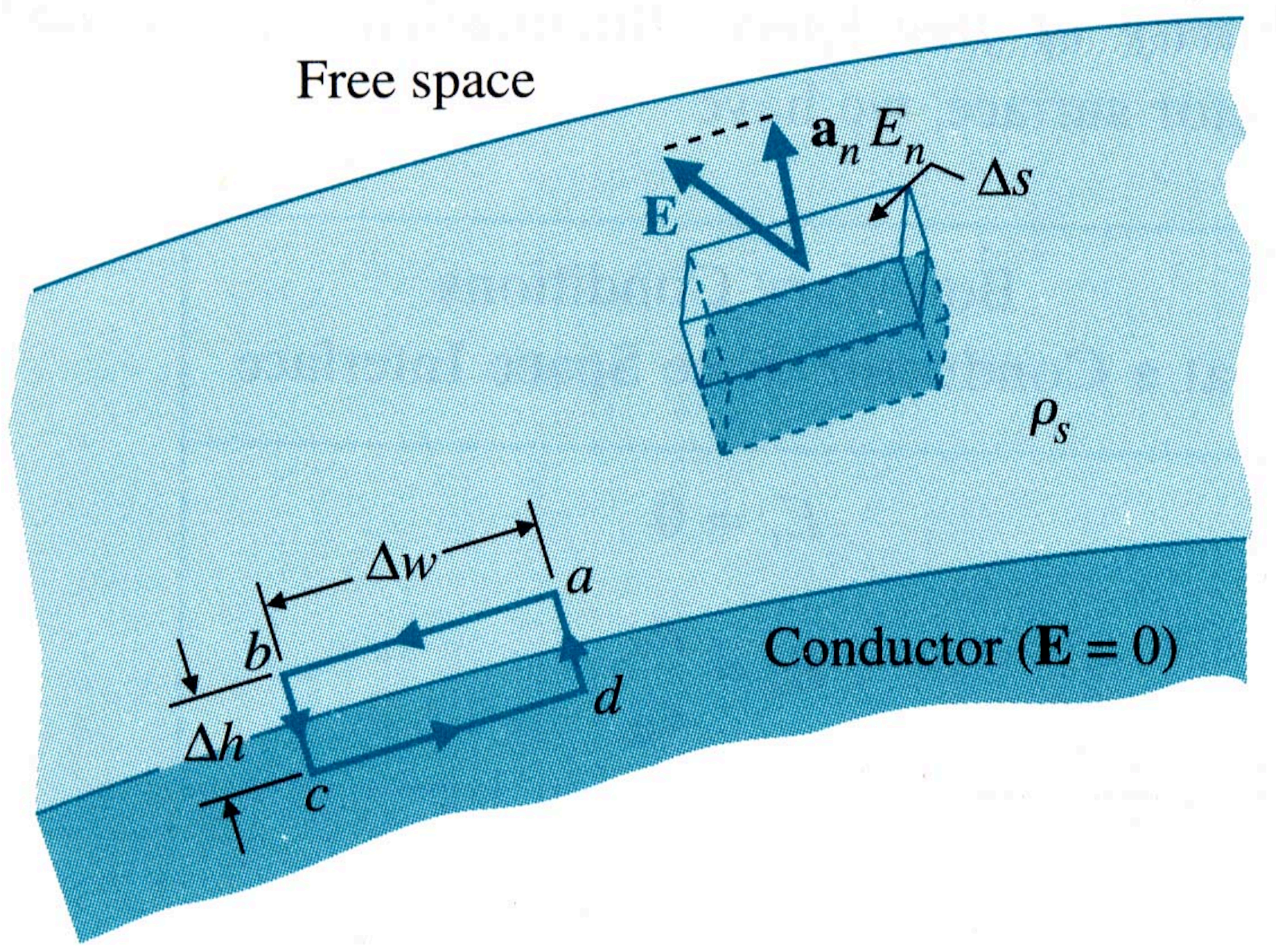
$$\text{Inside} \begin{cases} \rho = 0 \\ \boldsymbol{E} = 0 \end{cases}$$

B.C. at a conductor / free space interface

$$\begin{cases} E_t = 0 \\ E_n = \dfrac{\rho_s}{\varepsilon_0} \end{cases}$$

- E-field on the surface everywhere normal to the surface
- Conductor surface = equipotential surface

$$\left(\because \oint_S \boldsymbol{E}\cdot d\boldsymbol{s} = \frac{\rho_s\Delta S}{\varepsilon_0} \rightarrow E_n\Delta S = \frac{\rho_s\Delta S}{\varepsilon_0}\right)$$



Course Intro | Fundamental law of physics

Conservation of electric charge

Sum of positive and negative charges in a closed (isolated) system NEVER changes

Equation of Continuity

a net current flows out of (into) the volume = a net charge in the volume decreases (increases)

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \quad \left(A / m^3\right)$$

For steady-state currents (i.e. $\partial\rho/\partial t = 0$)

$$\nabla \cdot \boldsymbol{J} = 0 \qquad \sum_j I_j = 0$$

Kirchhoff's current law

sum of all the currents leaving out of & entering into a junction in a circuit is zero

Where current is change of charge vs. time:

$$I = \frac{dq}{dt} \quad [A]$$

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{s}$$: Total current I flowing through S = Flux of $\boldsymbol{J}$ vector through S

Where $\boldsymbol{J}$ (A/m^2) is the volume current density:

a measure of current flowing through a unit area normal to the direction of the current

Electromagnetics

<Chap. 1~ Chap. 6> Static electric & magnetic fields

Course Intro

(2nd class of **week 1**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Course Intro | Contents for 2nd class of week 1

1. Review of last class
   - Vector-valued functions and mathematical theorem (Ch. 2)
   - Electrostatics (Ch. 3)

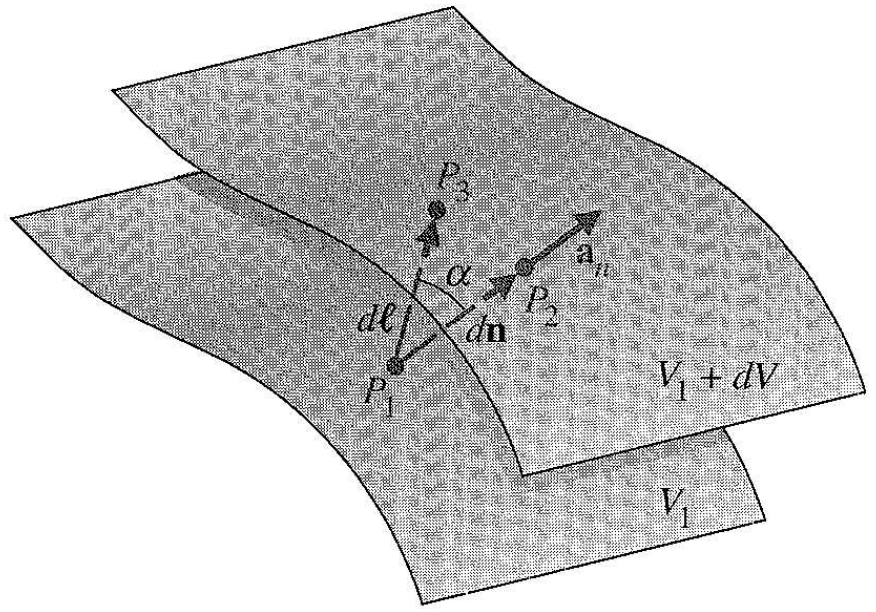
2. (Cont'd) Review of *"Introduction to electromagnetism with practice"* (기초전자기학 및 연습; 430.202B)
   - Steady electric currents (Ch. 5)
   - Magnetostatics (Static Magnetic Field) (Ch. 6)

Course Intro | Review of the last class (1/2)

Vector-valued functions

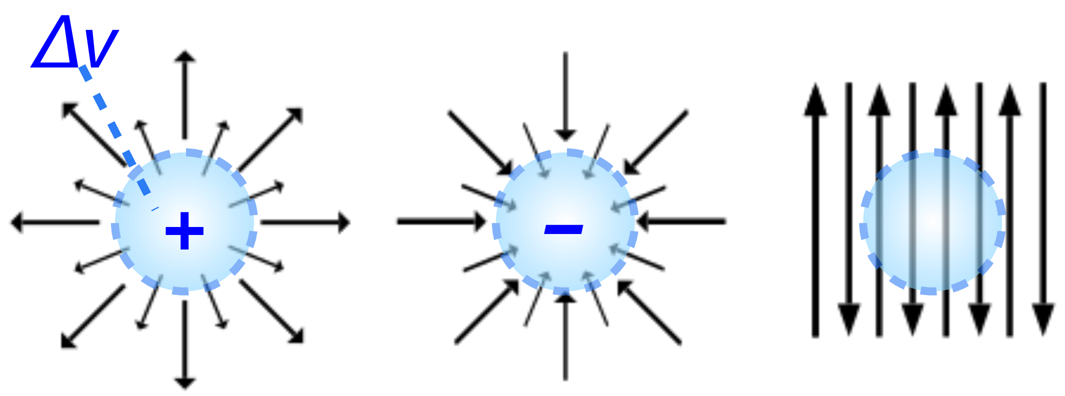
Gradient

$$\nabla V \triangleq \boldsymbol{a}_{\max(dV/dl)} \max\left(\frac{dV}{dl}\right) = \boldsymbol{a}_{\boldsymbol{n}} \frac{dV}{dn}$$



Divergence

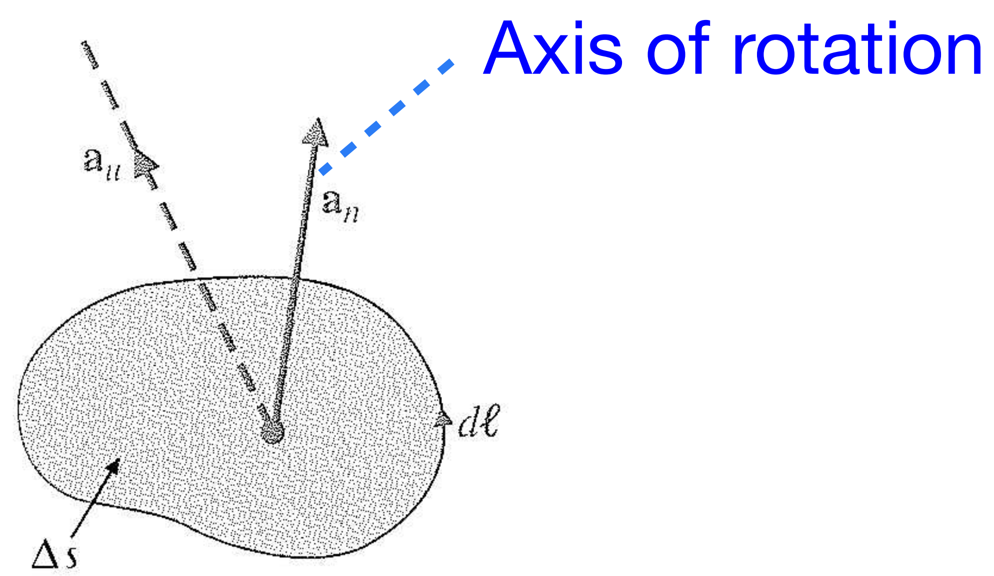
$$\nabla \cdot \boldsymbol{E} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \boldsymbol{E} \cdot d\boldsymbol{s}}{\Delta v}$$



$\nabla \cdot \boldsymbol{E} > 0$ $\quad\nabla \cdot \boldsymbol{E} < 0$ $\quad\nabla \cdot \boldsymbol{E} = 0$

Curl

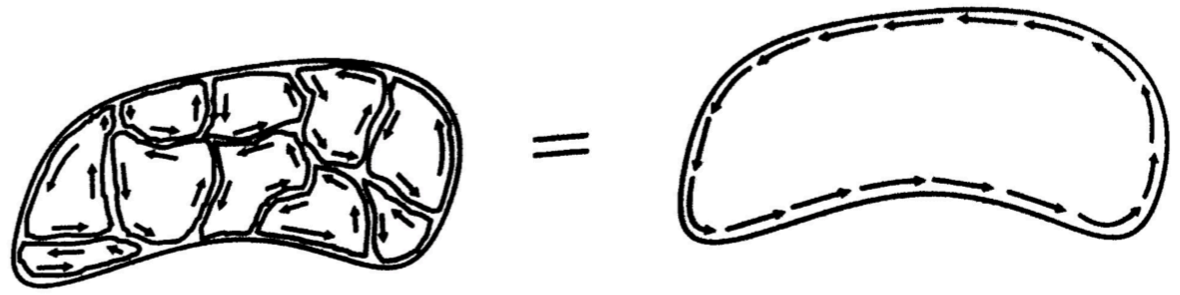
$$\nabla \times \boldsymbol{A} \triangleq \boldsymbol{a}_{\boldsymbol{n}} \lim_{\Delta s \to 0} \frac{\oint_C \boldsymbol{A} \cdot d\boldsymbol{l}}{\Delta s}$$



Mathematical theorem

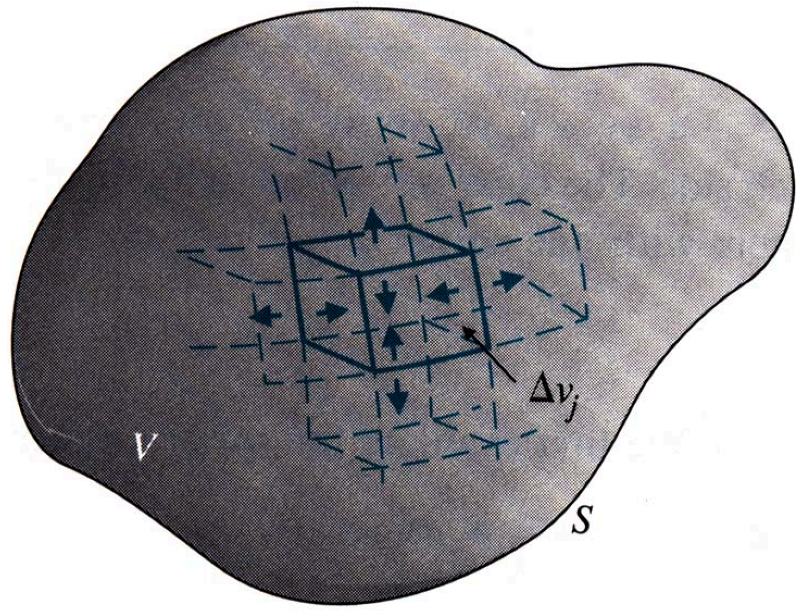
Stokes theorem

$$\int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{E} \cdot d\boldsymbol{l}$$



Divergence theorem

$$\int_V \nabla \cdot \boldsymbol{D}\, dv = \oint_S \boldsymbol{D} \cdot d\boldsymbol{s}$$



Helmholtz theorem

$$\boldsymbol{F} = \boldsymbol{F}_i + \boldsymbol{F}_s = -\nabla V + \nabla \times \boldsymbol{A}$$

where $\nabla \times \boldsymbol{F}_i = \nabla \times (-\nabla V) = 0$: Null Identity I

where $\nabla \cdot \boldsymbol{F}_s = \nabla \cdot (\nabla \times \boldsymbol{A}) = 0$: Null Identity II

Course Intro | Review of the last class (2/2)

Electrostatics
Static electric fields

Source:
Stationary charges

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \end{cases}$$

Magnetostatics
Static magnetic fields

Source:
Steady-state current

$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \end{cases}$$

Only functions of space: $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{H}(x, y, z)$
Independently defined!

Differential form → **Integral form**

$$\nabla \cdot \boldsymbol{D} = \rho \xrightarrow{\text{Divergence theorem}} \oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q$$

$$\nabla \times \boldsymbol{E} = 0 \xrightarrow{\text{Stoke's theorem}} \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \oint_C dV = 0$$

E-field intensity *E* at *R* due to a positive charge *q* at *R'*

$$\boldsymbol{E}(\boldsymbol{R}) = \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{R} - \boldsymbol{R}'}{|\boldsymbol{R} - \boldsymbol{R}'|^3} = \boldsymbol{a}_q \frac{q}{4\pi\varepsilon_0} \frac{1}{|\boldsymbol{R} - \boldsymbol{R}'|^2}$$

Coulomb's law

$$\boldsymbol{F} = q_2 \boldsymbol{E}_1(\boldsymbol{R}_2) = \frac{q_2 q_1}{4\pi\varepsilon_0} \frac{\boldsymbol{R}_2 - \boldsymbol{R}_1}{|\boldsymbol{R}_2 - \boldsymbol{R}_1|^3} = \boldsymbol{a}_{21} k_e \frac{q_2 q_1}{|\boldsymbol{R}_2 - \boldsymbol{R}_1|^2}$$

Electric potential

$$\frac{W}{q} = -\int_\infty^R \boldsymbol{E} \cdot d\boldsymbol{l} = \int_\infty^R \nabla V \cdot d\boldsymbol{l} = \int_\infty^R dV = V_R$$

$$\boldsymbol{E} = -\nabla V \ \ (\because \nabla \times \boldsymbol{E} = 0)$$

Electric Flux Density

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon_0 \varepsilon_r \boldsymbol{E} = \varepsilon \boldsymbol{E}$$

If medium is linear and isotropic

 Electric current (Essential as a source of magnetic field!)

Derivation

- Amount of charges passing through $\Delta\boldsymbol{s}$

$$\Delta Q = qN\left(\boldsymbol{u}\Delta t \cdot \Delta\boldsymbol{s}\right), \quad \text{unit: } \left(C \cdot \frac{1}{m^3} \cdot \frac{m}{s} \cdot m^2 = C\right)$$

where N is the number of charges per unit volume, $\boldsymbol{u}\Delta t$ is a distance vector that charge carriers moved

- Since electric current = time rate of change of charge,

$$\Delta I = \frac{\Delta Q}{\Delta t} = qN\boldsymbol{u} \cdot \Delta\boldsymbol{s}$$
$$= \boldsymbol{J} \cdot \Delta\boldsymbol{s} \quad (\text{A or C/s})$$

where

Volume current density

$$\boldsymbol{J} = qN\boldsymbol{u} = \rho\boldsymbol{u} \quad (\text{A/m}^2)$$

volume charge density

∴Total current ***I*** flowing through a surface ***S***:

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{s} \quad (\text{A})$$

Course Intro | Ohm's law

Conduction current

- Result of **drift** motion of **many groups of charge carriers** affected by **E-field**

$$\boldsymbol{J} = \sum_i q_i N_i \boldsymbol{u_i} = \rho \boldsymbol{u} \quad (\mathrm{A/m^2})$$

- where ***u*** is average drift velocity,

$$\boldsymbol{u} = -\mu_e \boldsymbol{E} \quad (\mathrm{m/s})$$ where μ_e is **electron mobility** ($\mathrm{m^2/V \cdot s}$)

Ohm's law

$$\boldsymbol{J} = \rho_e \boldsymbol{u} = -\rho_e \mu_e \boldsymbol{E} = \sigma \boldsymbol{E} \quad (\mathrm{A/m^2})$$

where $\sigma = -\rho_e \mu_e$ is **conductivity**

(a measure of how well the medium conducts electrons)

* For semiconductors,

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

mobility

Carrier concentration

Mateirals	Electron mobility ($\mathrm{cm^2/V \cdot s}$)
Silicon	1,360
GaAs	8,000
GaN	1,500
Organic semiconductors	$10^{-8} \sim 10^{-3}$

Thin organic device (e.g. OLED) is not an option, but **a must** to avoid using >1,000V operating voltage!

- Ohmic media

Isotropic material satisfying the relationship, $\boldsymbol{J} = \sigma \boldsymbol{E}$

Equation of continuity

Conservation of electric charge

- Sum of positive and negative charges in a closed (isolated) system NEVER changes

Equation of Continuity

- a net current flows out of the volume = a decreased time rate of net charge in the volume

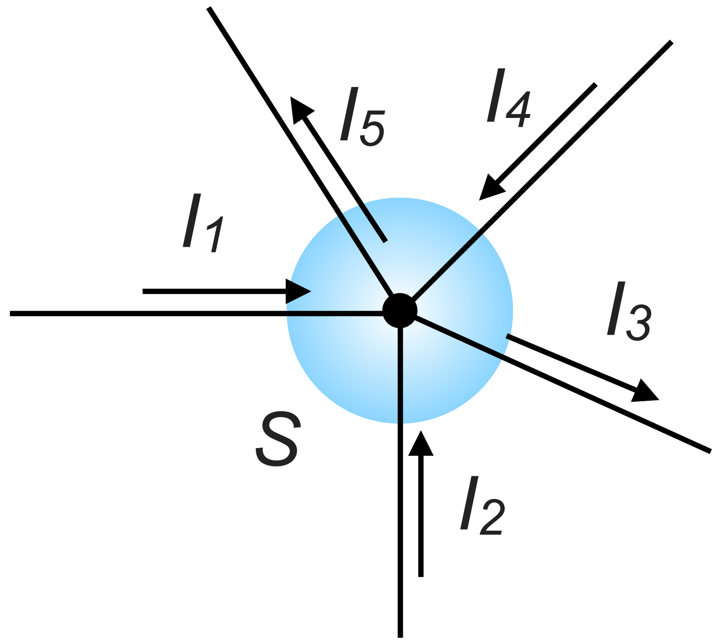
$$\boxed{\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \quad \left(A/m^3\right)} \qquad \left(\because I = \left[\oint_S \boldsymbol{J} \cdot ds = \int_V \underline{\nabla \cdot \boldsymbol{J}}\, dv\right] = \left[-\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho\, dv = \int_V \underline{\frac{\partial \rho}{\partial t}}\, dv\right]\right)$$

- For steady-state currents (i.e. ∂ρ/∂t = 0)

$$\nabla \cdot \boldsymbol{J} = 0 \longrightarrow \int_V \nabla \cdot \boldsymbol{J}\, dv = \oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = 0 \longrightarrow \sum_j I_j = 0$$

Divergence theorem

Kirchhoff's current law
sum of all the currents leaving out of & entering into a junction in a circuit is zero



Course Intro | Joule’s law

Power dissipation (loss)

- Under **E-field** in the conductor, electrons **drift** and **collide** with atoms on lattice sites → **lose kinetic energy** from E-field into **thermal vibration (i.e. heat)**

- Power loss = Power delivered to a charge *q* by ***E***-field

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \lim_{\Delta t \to 0} \frac{q\boldsymbol{E} \cdot \Delta \boldsymbol{l}}{\Delta t} = q\boldsymbol{E} \cdot \boldsymbol{u}$$

where Δ*w* is work “**done by E-field**” moving a charge *q* a distance Δ***l***

c.f.) For electric potential, $\frac{W}{q}\ (\mathrm{J/C}) = -\int_{\infty}^{R} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{\infty}^{R} \nabla V \cdot d\boldsymbol{l} = \int_{\infty}^{R} dV = V_R\ (\mathrm{V})$

- Power delivered to **many** charges in *dv* by ***E***-field

Power of i-th group of charges

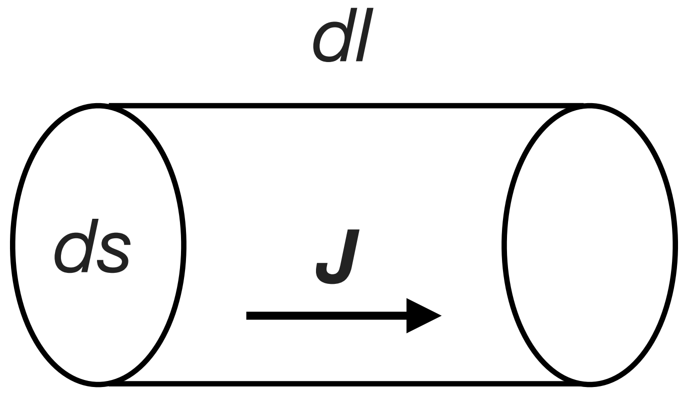
$$dP = \sum_i P_i = \sum_i q_i (N_i dv) \boldsymbol{E} \cdot \boldsymbol{u}_i = \boldsymbol{E} \cdot \left(\sum_i q_i N_i \boldsymbol{u}_i \right) dv = \boldsymbol{E} \cdot \boldsymbol{J} dv \longrightarrow \frac{dP}{dv} = \boldsymbol{E} \cdot \boldsymbol{J}\ \ (\mathrm{W/m^3})$$

: Power density

- Total power delivered to many charges = total power dissipated as heat for a given volume V

$$\boxed{\therefore P = \int_V \boldsymbol{E} \cdot \boldsymbol{J}\, dv\ \ (\mathrm{W})}$$

: Joule’s law



$$P = \int_V \boldsymbol{E} \cdot \boldsymbol{J}\, dv = \int_L E\, dl \int_S J\, ds = VI = I^2 R\ \ (\mathrm{W})$$

Magnetostatics (1/2)

Magnetic field

- Source: moving charge (=electric current)
- When a test charge *q* ***moves*** in ***a magnetic field***, it experiences the ***magnetic force*** as

$$F_m = q\boldsymbol{u} \times \boldsymbol{B} \quad \text{(N)}$$

where ***u*** is the velocity vector (m/s) and ***B*** is the magnetic flux density (T or Wb/m^2)

Lorentz's force equation

$$\boldsymbol{F} = \boldsymbol{F}_e + \boldsymbol{F}_m = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \quad \text{(N)}$$

: Electromagnetic force on *q*

Another fundamental postulate of electromagnetic model

(Cannot be derived by other postulates!)

Static magnetic field

- Source: steady-state currents
- Two fundamental postulates for *static magnetic field* in any medium,

$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \end{cases}$$

Source: steady-state current

where in the medium, $\boldsymbol{H} = \dfrac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$

Magnetization vector of "induced" magnetic dipoles

c.f.) $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

Polarization vector of "induced" electric dipoles

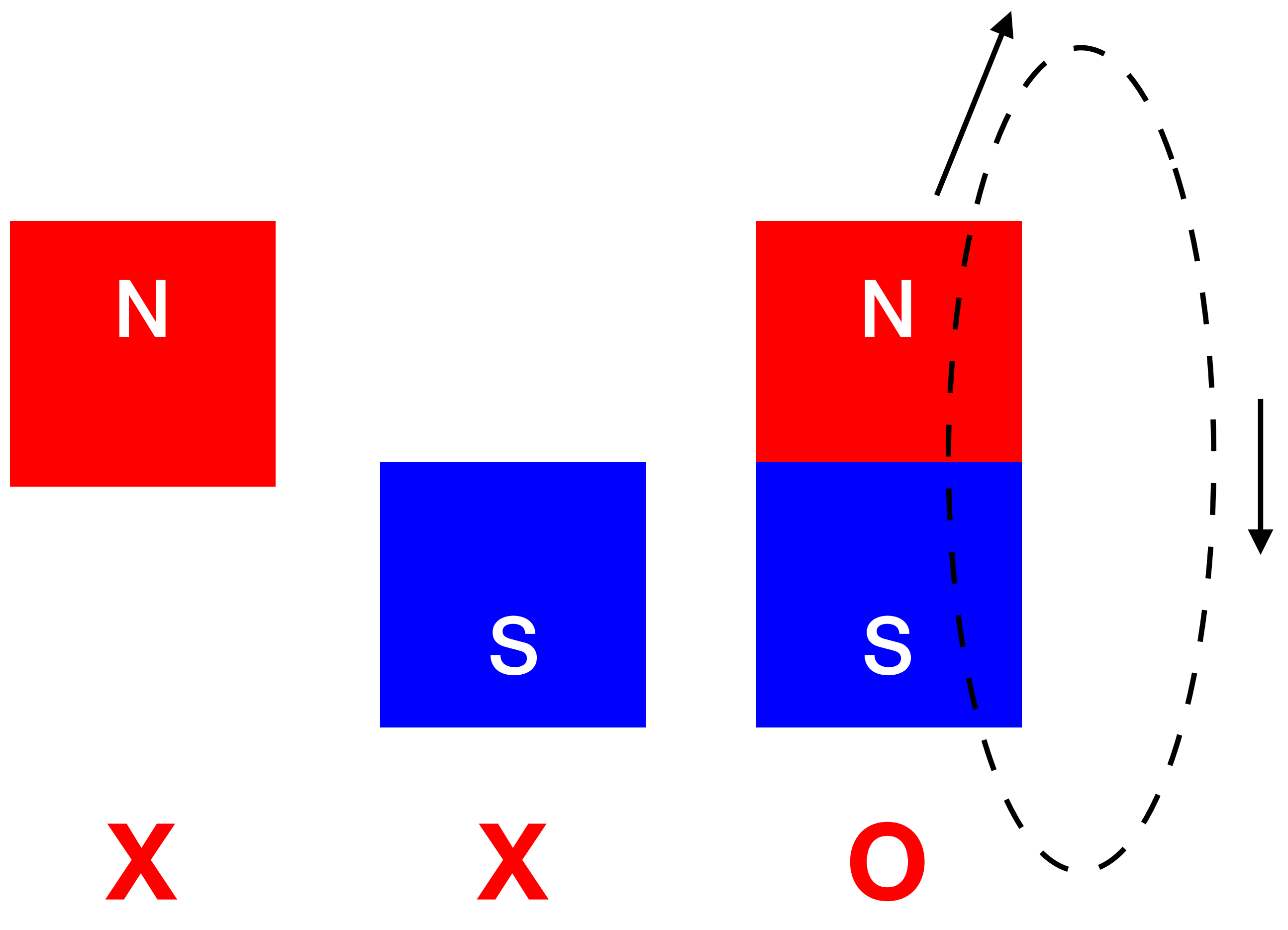
Course Intro | Magnetostatics (2/2)

Static magnetic field

- Divergence postulate of ***B***

Divergence theorem

$$\nabla \cdot \boldsymbol{B} = 0 \longrightarrow \boxed{\int_V \nabla \cdot \boldsymbol{B}\, dv = \oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0 \quad \textit{The law of Conservation of Magnetic Flux}}$$



  - There are **NO** magnetic charges (or **monopoles**) (i.e. they are always **paired**!)
  - Magnetic field is not a flow source, but a **solenoidal** source
  - Magnetic flux always closes upon itself

- Curl postulate of ***H***

Null Identity II

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \longrightarrow \nabla \cdot (\nabla \times \boldsymbol{H}) = 0 = \nabla \cdot \boldsymbol{J} \text{ (For steady-state current) } \left(\because \nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \right)$$

Stoke's theorem

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \longrightarrow \left[\int_S (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{H} \cdot d\boldsymbol{l} \right] = \left[\int_S \boldsymbol{J} \cdot d\boldsymbol{s} = I \right]$$

Ampere's circuital law

$$\boxed{\therefore \oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I \quad (\mathrm{A})}$$

Only useful when there is symmetrical geometry.
(*i.e.* when ***B*** is constant over the closed path *C*.)

Course Intro | "Vector" Magnetic Potential

Magnetic potential

- Divergence-free postulate of ***B***

$$\nabla \cdot \boldsymbol{B} = 0 \xrightarrow[\text{Null Identity I}]{} \nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$

$\therefore \boldsymbol{B} = \nabla \times \boldsymbol{A}$ (T) where ***A*** is *vector magnetic potential*

c.f.) $\boldsymbol{E} = -\nabla V$ (V/m) $\xleftarrow[\text{Null Identity II}]{}$ $\nabla \times \boldsymbol{E} = 0$

Vector Poisson's equation

- Starting from a curl postulate,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \longrightarrow \nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{J}$$

If we choose $\nabla \cdot \boldsymbol{A} = 0$ for simplicity,

$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$: Vector Poisson's equation

Laplacian of *A*

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

or

$$\boxed{\nabla^2 \boldsymbol{A}} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times \nabla \times \boldsymbol{A}$$

$$= \boldsymbol{a_x} \nabla^2 A_x + \boldsymbol{a_y} \nabla^2 A_y + \boldsymbol{a_z} \nabla^2 A_z$$

Lorentz Gauge

$$\nabla \cdot \boldsymbol{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

Coulomb's Gauge

$$\nabla \cdot \boldsymbol{A} = 0$$

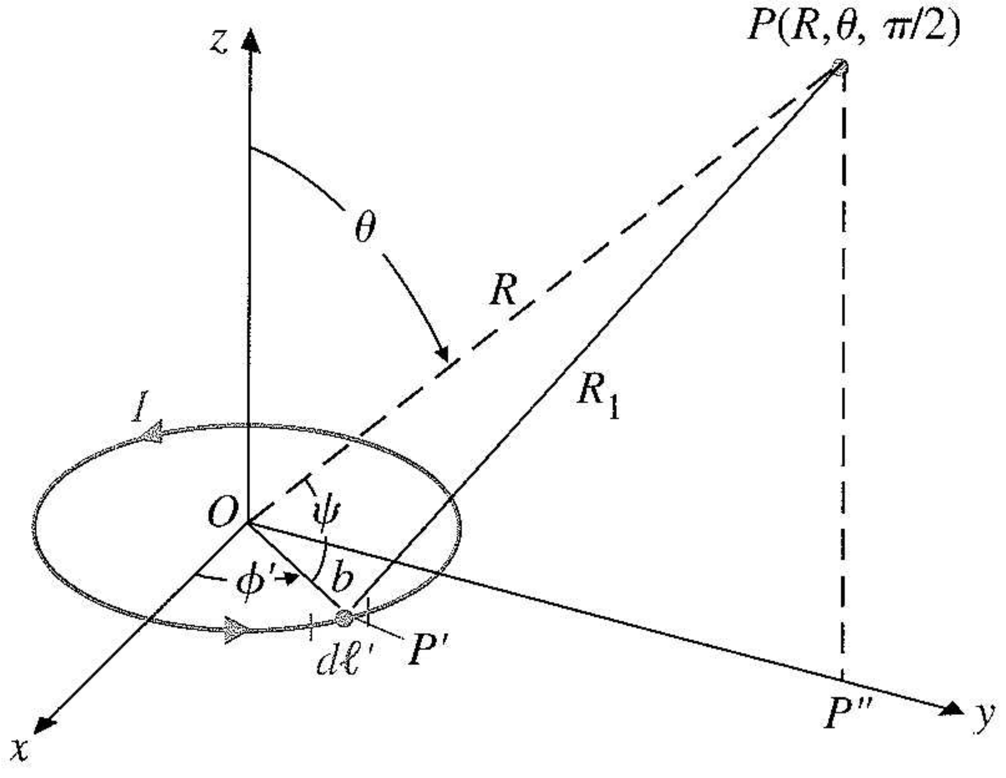
- Solution to Vector Poisson's équation

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}}{R} dv' \quad \text{(Wb/m)}$$

c.f.) $V = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{V'} \frac{\rho}{R} dv$ for $-\nabla \cdot \boldsymbol{E} \underset{\boldsymbol{E} = -\nabla V}{=} \nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$

Magnetic dipole

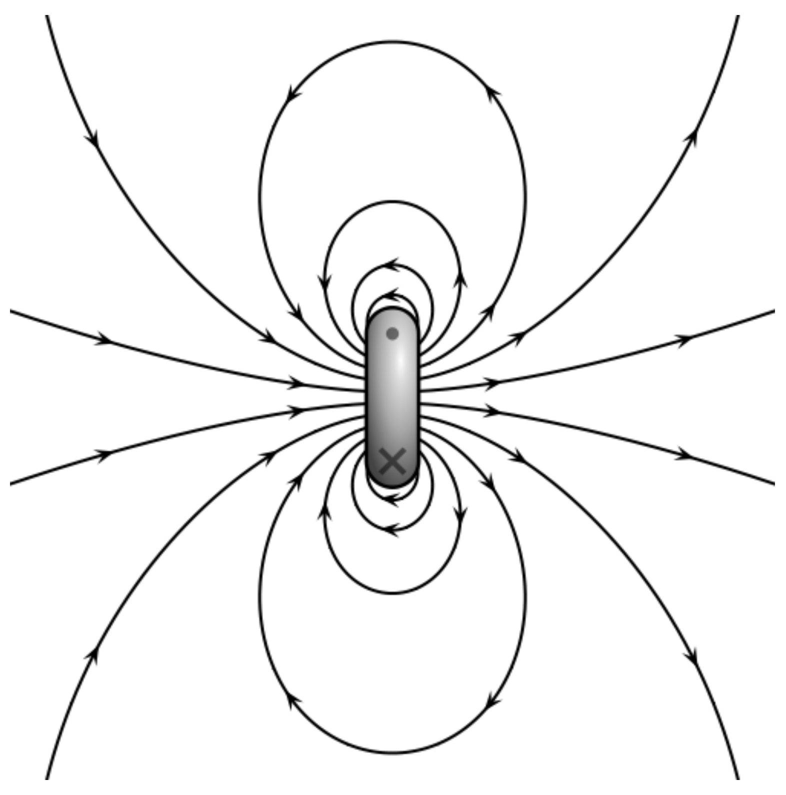
• Small current-carrying loop (with a radius *b*, and carrying the current *I*)



Refer to Ch. 6-5 for derivation

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{C'} \frac{I \cdot d\boldsymbol{l}'}{R} = \boldsymbol{a}_\phi \frac{\mu_0 I b^2}{4R^2}\sin\theta$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{\mu_0 I b^2}{4R^3}\left(\boldsymbol{a}_R 2\cos\theta + \boldsymbol{a}_\theta \sin\theta\right) \quad (\mathrm{Wb/m^2})$$



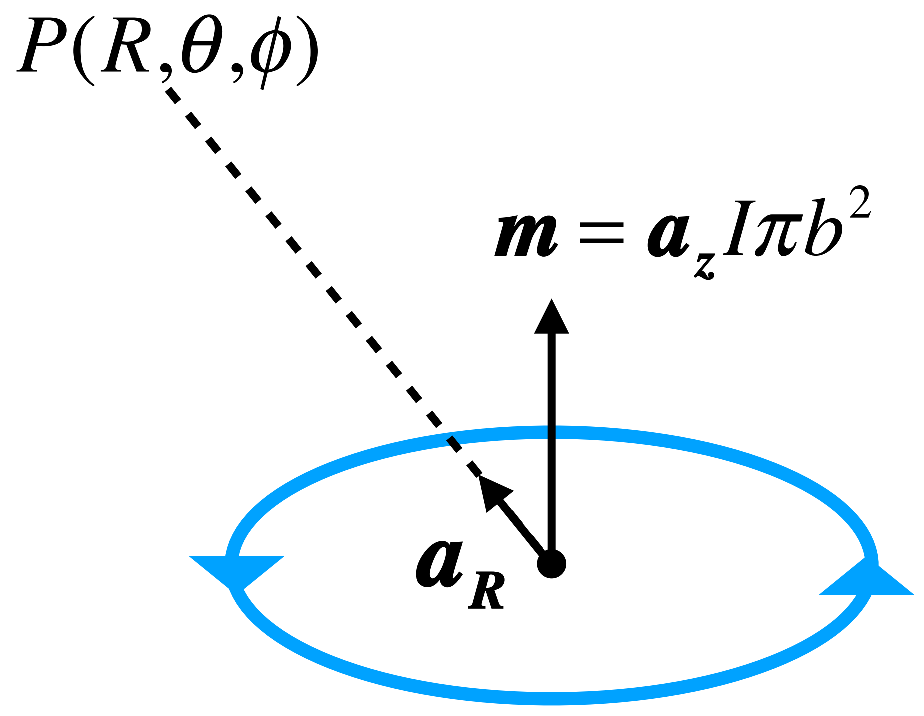
B by magnetic dipole

Magnetic dipole Moment, *m*

$$\boldsymbol{A} = \boldsymbol{a}_\phi \frac{\mu_0 I b^2}{4R^2}\sin\theta = \boldsymbol{a}_\phi \frac{\mu_0 \left(I\pi b^2\right)\sin\theta}{4\pi R^2} = \frac{\mu_0 \boldsymbol{m}\times\boldsymbol{a}_R}{4\pi R^2}$$

where $\boldsymbol{m} = \boldsymbol{a}_z I\pi b^2 = \boldsymbol{a}_z IS = \boldsymbol{a}_z m$

Also applicable to non-circular shape!

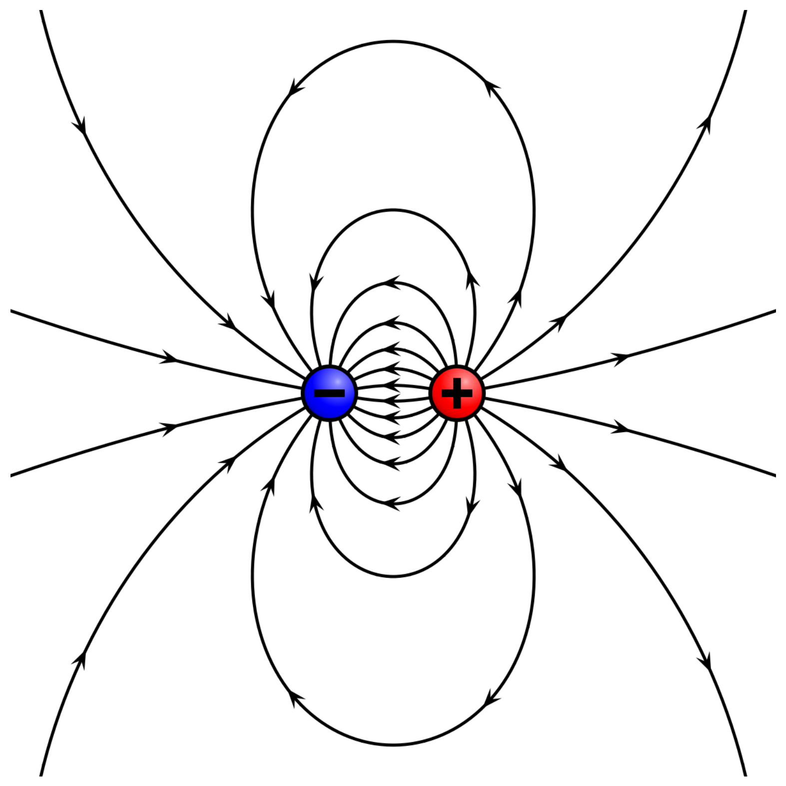


• ***B*** in terms of ***m***

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi R^3}\left(\boldsymbol{a}_R 2\cos\theta + \boldsymbol{a}_\theta \sin\theta\right) \quad (\mathrm{T})$$

c.f.)

$$\boldsymbol{E} = \frac{p}{4\pi\varepsilon_0 R^3}\left(\boldsymbol{a}_R 2\cos\theta + \boldsymbol{a}_\theta \sin\theta\right) \quad (\mathrm{V/m}) \text{ where } \boldsymbol{p} = q\boldsymbol{d}$$



E by electric dipole

Img src: Wikipedia

Course Intro | Magnetization

Magnetic dipoles *"in the medium"*

- Microscopic view: orbiting electrons around nucleus → circulating currents → microscopic magnetic dipoles
- Under no external magnetic field, magnetic dipoles of atom are in random orientations → No net magnetic moments
- External magnetic field applies to the medium → Induced magnetic moments due to changed electron orbiting motion

Magnetization vector

$$\boldsymbol{M} = \lim_{\nabla v \to 0} \frac{\sum_{k=1}^{n\nabla v} \boldsymbol{m}_k}{\nabla v} \quad \text{(A/m)}$$

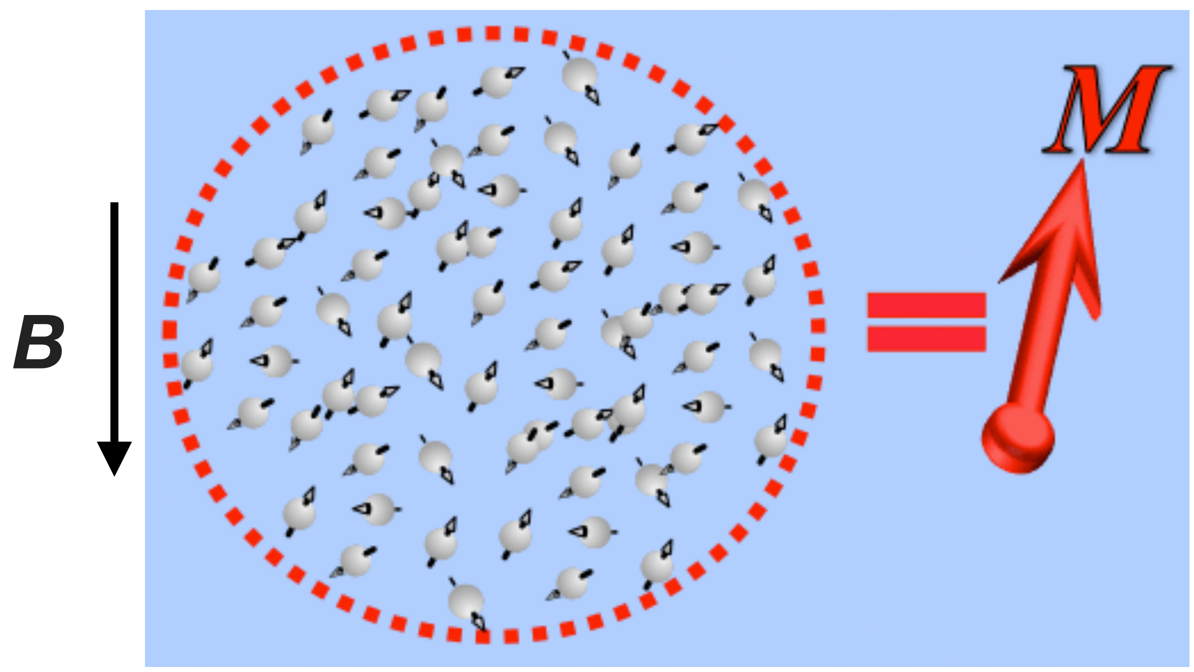
M: Volume density of magnetic dipole moment

→ Macroscopic effect of induced magnetic dipoles

where $\boldsymbol{m}_k$ is the magnetic dipole moment of *k*-th atom and

n is the number of atoms per unit volume

Img src: mriquestions.com



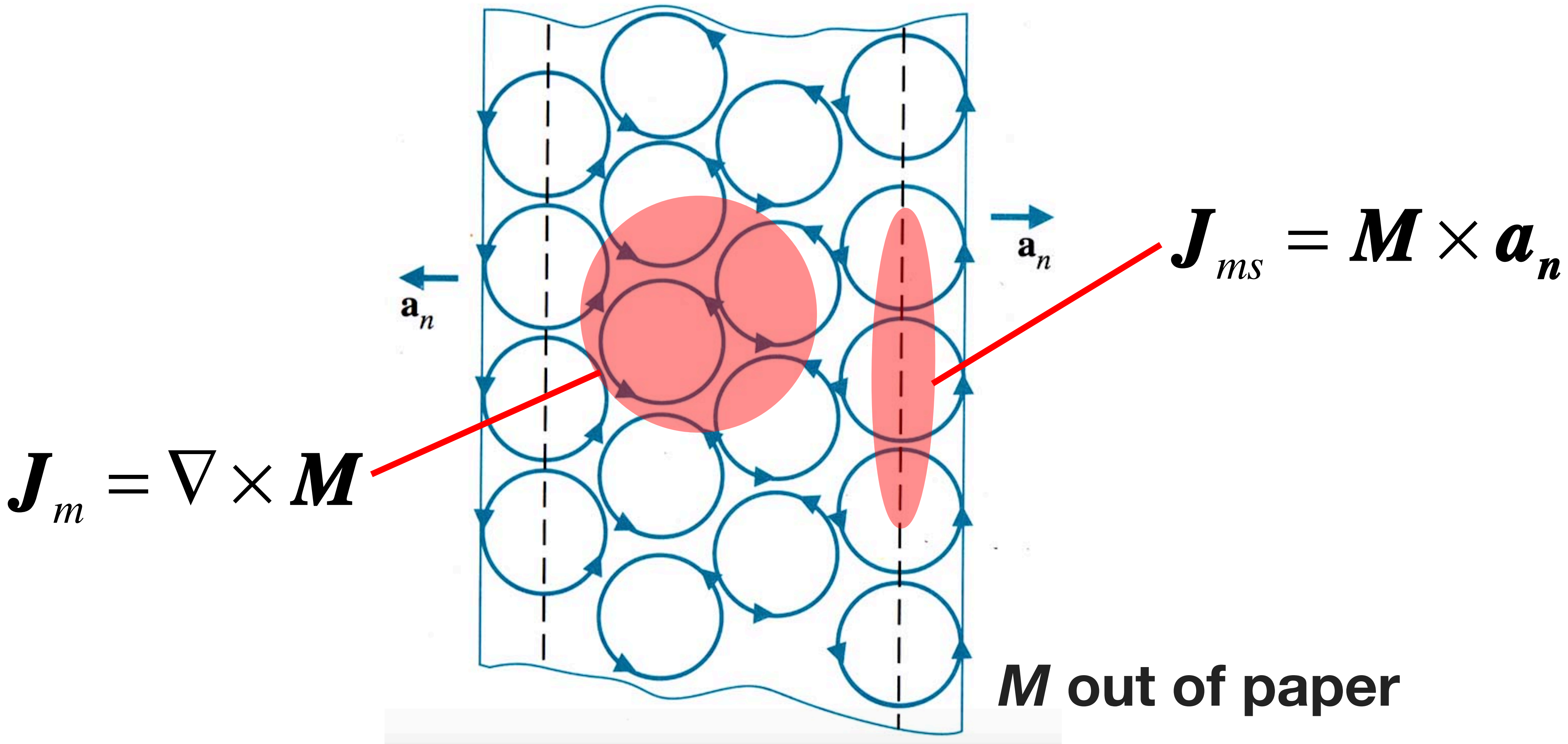
Vector magnetic potential caused by *magnetization*

$$\boldsymbol{A} = \int_{V'} \frac{\mu_0 \boldsymbol{M} \times \boldsymbol{a_R}}{4\pi R^2} dv' \qquad \left(\because \boldsymbol{A} = \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{a_R}}{4\pi R^2} \right)$$

$$= \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \boldsymbol{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\boldsymbol{M} \times \boldsymbol{a}'_n}{R} ds'$$

$$= \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J_m}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\boldsymbol{J_{ms}}}{R} ds'$$

(refer to Ch 6-6 for derivation)



Course Intro | Magnetic field intensity, *H*

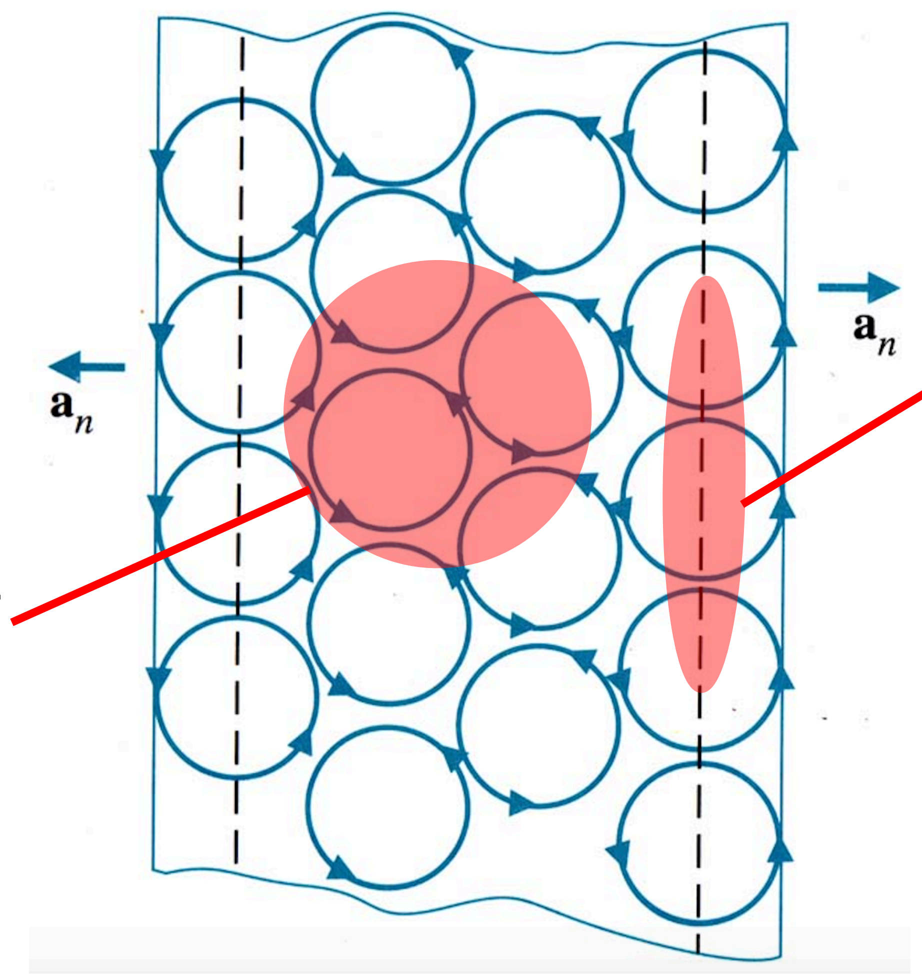
Curl postulate of magnetic field "in the medium"

$$\frac{1}{\mu_0}\nabla\times\boldsymbol{B}=\boldsymbol{J}+\boldsymbol{J}_m$$

B: magnetic flux density "in the medium"
J: free volume current density
***J**m*: Magnetized volume current density

$$\nabla\times\left(\frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M}\right)=\boldsymbol{J},\quad (\because \boldsymbol{J}_m=\nabla\times\boldsymbol{M})$$

M **out of paper**



$$\boldsymbol{J}_m=\nabla\times\boldsymbol{M}$$

$$\boldsymbol{J}_{ms}=\boldsymbol{M}\times\boldsymbol{a}_n$$

Magnetic field intensity, *H*

$$\boldsymbol{H}\triangleq\frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M}\quad(\text{A/m})$$

Curl postulate "in any medium"

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}\quad(\text{A/m}^2)$$

Permeability

For **linear and isotropic** medium,

$$\boldsymbol{M}=\chi_m\boldsymbol{H}$$ where χ_m is magnetic susceptibility

$$\rightarrow \boldsymbol{H}=\frac{1}{\mu_0}\boldsymbol{B}-\boldsymbol{M}=\frac{1}{\mu_0}\boldsymbol{B}-\chi_m\boldsymbol{H}$$

$$\rightarrow \boldsymbol{B}=\mu_0(1+\chi_m)\boldsymbol{H}=\mu_0\mu_r\boldsymbol{H}=\mu\boldsymbol{H}$$

(i.e. Material specific)

Constitutive relation

$$\boldsymbol{B}=\mu\boldsymbol{H}$$

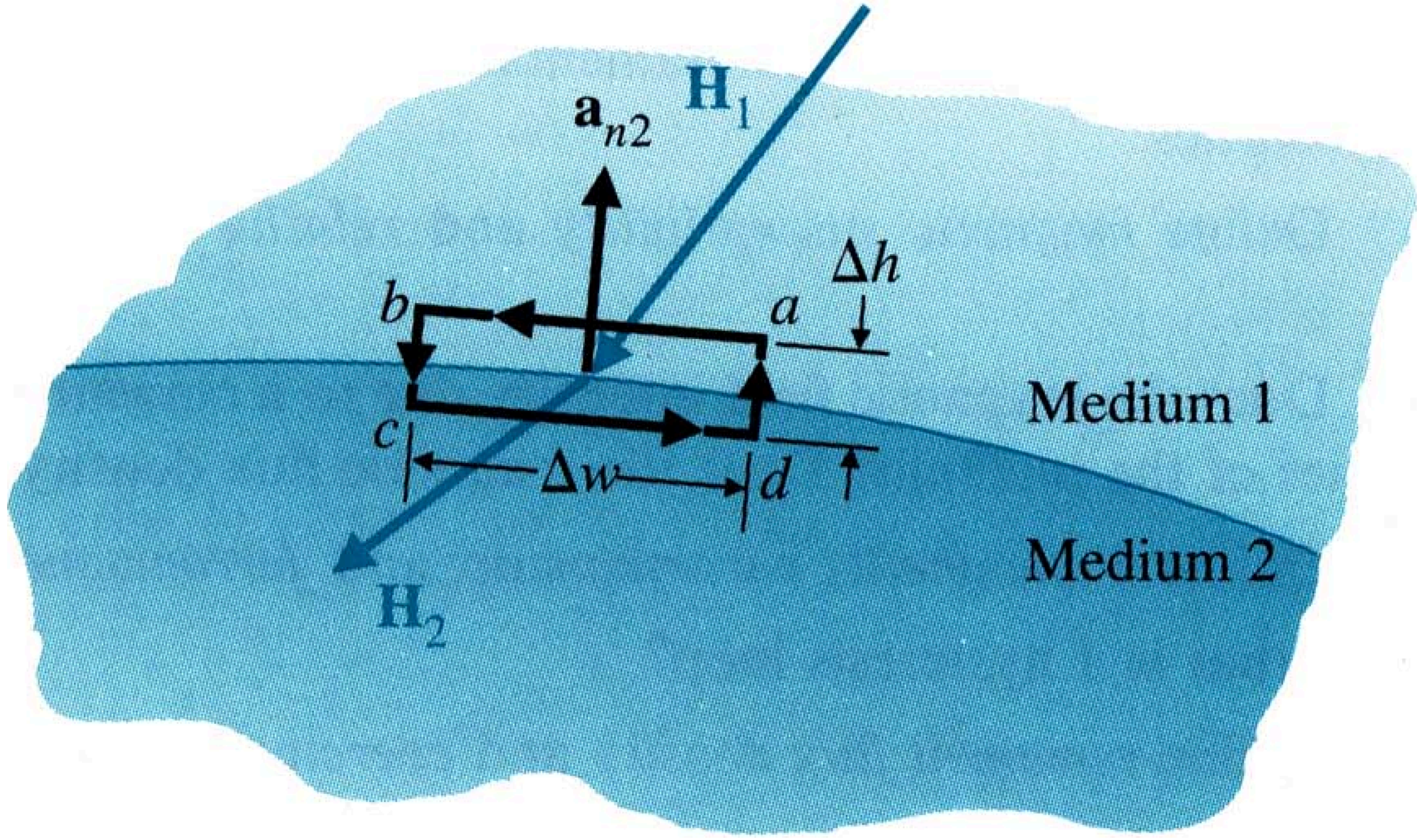
$\mu=\mu_0\mu_r$: Absolute permeability

$\mu_r=1+\chi_m$: relative permeability

Course Intro | Boundary Conditions

Normal component of *B*

$$B_{1n} = B_{2n} \quad (\because \nabla \cdot B = 0)$$ → *Continuous across the interface*



Tangential component of *B*

$$\boldsymbol{a}_{n2} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_S \quad \text{(A/m)}$$

where $\boldsymbol{a}_{n2}$ is outward unit normal from medium 2 at the interface

where $\boldsymbol{J}_S$ is the surface current density flowing at the interface

→ Tangential component is *discontinuous across the interface* where a surface current exists